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Non-Local Problem With Shift For Mixed-Type Equation With Strong Degeneration

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Abstract—For mixed-type equation of the second kind with two lines of degeneracy are formulated three problems with non-local conditions on the lines of degeneracy and on the lateral characteristics. The tasks are reduced to equivalent problems in the field of elliptic equations. The theorems about the existence of a unique solution of the problems are studied.

Keywords—Second kind mixed type equation; Frankl type problem; singular integral equation.

1. Introduction

For the first time, mixed type equations in two variables were investigated by Italian mathematicians F.Tricomi [1] and M.Cibrario [2]. Simplest example of the mixed type equations is so called The Tricomi equation $yu_{xx} + u_{yy} = 0$, which is hyperbolic in the domain $y < 0$ and elliptic in the domain $y > 0$.

Mixed type equations have numerous applications, for instance, in problems related with transonic gas dynamics [3].

In the works by Bitsadze [4], the Tricomi problem [1] for a mixed type equation was investigated by various methods and new boundary problems were formulated.

In the present work we formulate and investigate for the unique solvability, Frankl type non-local problems for the mixed type equation

$$xu_{xx} + yu_{yy} + \alpha\, u_x + \alpha u_y = 0, \quad 0 < \alpha = const < (1/2) \tag{1}$$

2. Formulation of the problems and main results

Let Ω be finite simple-connected domain of the plane xOy, bounded by segments $\overline{AB} = \{(x,y): x + y = 1,\ x \geq 0, y \geq 0\}$, $\overline{A^*O} = \{(x,y): y = 0, -1 \leq x \leq 0\}$, $\overline{B^*O} = \{(x,y): x = 0, -1 \leq y \leq 0\}$ and by arcs $A^*B = \left\{(x,y): \sqrt{-x} + \sqrt{y} = 1,\ x < 0,\ y > 0\right\}$, $B^*A = \left\{(x,y): \sqrt{x} + \sqrt{-y} = 1,\ x > 0,\ y < 0\right\}$, and $\Omega_0 = \Omega \cap \{(x,y):\ x > 0,\ y > 0\}$, $\Omega_1 = \Omega \cap \{(x,y):\ x + y > 0,\ x > 0,\ y < 0\}$, $\Omega_2 = \Omega \cap \{(x,y):\ x + y > 0,\ x < 0,\ y > 0\}$, $\Omega_1^* = \Omega \cap \{(x,y):\ x + y < 0,\ x < 0,\ y > 0\}$, $\Omega_2^* = \Omega \cap \{(x,y):\ x + y < 0,\ x > 0,\ y < 0\}$, where $O(0,0)$, $A(1,0)$, $B(0,1)$, $A^*(0,-1)$, $B^*(-1,0)$.

Problem $F^{(3)}$. To find a function $u(x,y) \in C(\overline{\Omega})$, satisfying the following conditions:

1) it is a regular solution of (1) in the domain Ω_0;

2) it is a generalized solution of (1) from the class R_2 [6,8] in the domains Ω_1, Ω_2, Ω_1^*, Ω_2^*;

3) gluing conditions are valid

$$\lim_{y\to -0} (-y)^\alpha u_y(x,y) = -\lim_{y\to +0} y^\alpha u_y(x,y), \quad 0 < x < 1; \tag{2}$$

$$\lim_{x\to -0} (-x)^\alpha u_x(x,y) = -\lim_{x\to +0} x^\alpha u_x(x,y), \quad 0 < y < 1; \tag{3}$$

4) satisfies boundary conditions

$$u(x,y) = \varphi(x,y), \qquad (x,y) \in \overline{AB}; \tag{4}$$

$$u(x,0) + m \cdot u(-x,0) = f_1(x), \qquad 0 \leq x \leq 1; \tag{5}$$

$$u(0,y) - m \cdot u(0,-y) = f_2(y), \qquad 0 \leq y \leq 1; \tag{6}$$

$$u\left[x, -\left(1-\sqrt{x}\right)^2\right] - u\left[\left(1-\sqrt{x}\right)^2, -x\right] = p_1(x), \qquad (1/4) \leq x \leq 1; \tag{7}$$

$$u\left[-\left(1-\sqrt{y}\right)^2, y\right] - u\left[-y, \left(1-\sqrt{y}\right)^2\right] = p_2(y), \qquad (1/4) \leq y \leq 1; \tag{8}$$

Problem $F^{(4)}$. To find a function $u(x,y) \in C(\overline{\Omega})$, satisfying all the conditions of the problem $F^{(3)}$, in which conditions (5) and (6) are replaced by the conditions

$$\lim_{y\to +0} y^\alpha u_y(x,y) + m \cdot \lim_{y\to +0} y^\alpha u_y(-x,y) = g_1(x), \qquad 0 < x < 1; \tag{9}$$

$$\lim_{x\to +0} x^\alpha u_x(x,y) - m \cdot \lim_{x\to +0} x^\alpha u_x(x,-y) = g_2(y), \qquad 0 < y < 1; \tag{10}$$

Problem $F^{(5)}$. To find a function $u(x,y)\in C(\overline{\Omega})$, satisfying all the conditions of the problem $F^{(3)}$, in which condition (6) is replaced by condition (10), where $\varphi(x,y)$, $f_j(t)$, $g_j(t)$, $p_j(t)$ - are the given functions, $m=\pm 1$ and $p_j(1/4)=0$, $j=\overline{1,2}$.

Note that equation (1) in the domain belong to the mixed type, precisely: in the domain Ω_0 belongs to the elliptic type, in the domains Ω_1, Ω_2, Ω_1^* and Ω_2^*, to the hyperbolic type, and the segments $OA=\{(x,0):\ 0<x<1\}$ and $OB=\{(0,y):\ 0<y<1\}$ are type changing lines. Conditions (5) and (6) are analogs of the Frankly conditions [3], in case when mixed type equation has two lines of type changing.

Let $u(x,y)$ be a solution of the problem $F^{(3)}$. Introduce the following designations:

$$\left.\begin{aligned} &\tau_1(x)=u(x,0),\quad (x,0)\in\overline{OA}; && \nu_1(x)=\lim_{y\to -0}(-y)^{\alpha}u_y(x,y),\quad (x,0)\in OA;\\ &\tau_1^*(x)=u(x,0),\quad (x,0)\in\overline{A^*O}; && \nu_1^*(x)=\lim_{y\to +0}y^{\alpha}u_y(x,y),\quad (x,0)\in A^*O;\\ &\tau_2(y)=u(0,y),\quad (0,y)\in\overline{OB}; && \nu_2(y)=\lim_{x\to -0}(-x)^{\alpha}u_x(x,y),\quad (0,y)\in OB;\\ &\tau_2^*(y)=u(0,y),\quad (0,y)\in\overline{B^*O}; && \nu_2^*(y)=\lim_{x\to +0}x^{\alpha}u_x(x,y),\quad (0,y)\in B^*O; \end{aligned}\right\} \tag{11}$$

$$u_j(x,y)=u(x,y),\quad (x,y)\in\Omega_j;\quad u_j^*(x,y)=u(x,y),\quad (x,y)\in\Omega_j^*,\quad j=\overline{1,2}.$$

As is known [8], if $u_j(x,y)$ and $u_j^*(x,y)$ are generalized solutions of equation (1) from the classR_2 in the domains Ω_j and Ω_j^*, $j=\overline{1,2}$ respectively, then the functions $\tau_j(t)$, $\nu_j(t)$, $\tau_j^*(t)$ and $\nu_j^*(t)$, $j=\overline{1,2}$ can be represented as

$$\tau_j(t)=\tau_j(0)+\int_0^t(t-z)^{2\beta}T_j(z)dz,\qquad 0\le t\le 1;\qquad \nu_j(t)=t^{1-\alpha}\tilde{\nu}_j(t),\qquad 0<t<1; \tag{12}$$

$$\tau_j^*(t)=\tau_j^*(0)+\int_t^0(z-t)^{2\beta}T_j^*(z)dz,\qquad -1\le t\le 0;\qquad \nu_j^*(t)=t^{1-\alpha}\tilde{\nu}_j^*(t),\qquad -1<t<0, \tag{13}$$

in total, the solutions $u_j(x,y)$ and $u_j^*(x,y)$, $j=\overline{1,2}$ are rewritten as

$$u_j(x,y)=\int_0^{\xi}\sigma^{-\beta}T_j(t)dt+\frac{1}{2cos(\beta\pi)}\int_{\xi}^{\eta}(-\sigma)^{-\beta}N_j(t)dt,\qquad (x,y)\in\Omega_j; \tag{14}$$

$$u_j^*(x,y)=\int_0^{\xi}\sigma^{-\beta}T_j^*(-t)dt+\frac{1}{2cos(\beta\pi)}\int_{\xi}^{\eta}(-\sigma)^{-\beta}N_j^*(-t)dt,\qquad (x,y)\in\Omega_j^*. \tag{15}$$

Here $\xi=\left[\sqrt{|x|}-\sqrt{|y|}\right]^2$, $\eta=\left[\sqrt{|x|}+\sqrt{|y|}\right]^2$, $\sigma=(\xi-t)(\eta-t)$, $N_j(t)=T_j(t)-k\,t^{\alpha-1}\nu_j(t)$, $N_j^*(t)=T_j^*(t)+k\,t^{\alpha-1}\nu_j^*(t)$, and besides $T_j(t)$,$T_j^*(-t)$,$\tilde{\nu}_j(t)$, $\tilde{\nu}_j^*(-t)\in C(0,1)\cap L(0,1)$, $k=2^{4\beta}cos(\beta\pi)\Gamma(1-2\beta)\Gamma^{-2}(1-\beta)$, $\beta=\alpha-(1/2)$, $\Gamma(z)$ - is the Euler gamma-function.

Moreover, by virtues of $u(x,y)\in C(\overline{\Omega})$ of the problem $F^{(3)}$, equalities

$$\lim_{y\to -x+0}u_1(x,y)=\lim_{y\to -x-0}u_2^*(x,y),\qquad 0<x<(1/4); \tag{16}$$

$$\lim_{x\to -y+0}u_2(x,y)=\lim_{x\to -y-0}u_1^*(x,y),\qquad 0<y<(1/4) \tag{17}$$

are valid.

Substituting the functions $u_j(x,y)$ and $u_j^*(x,y)$, $j=\overline{1,2}$, defined by formulas (14) and (15), into equalities (16) and (17), after some transformations, we obtain

$$T_j(t)=T_i^*(-t)+k\ t^{\alpha-1}\left[\nu_j(t)+\nu_i^*(-t)\right],\quad 0<t<1,\ \ j,i=\overline{1,2},\ \ j\ne i. \tag{18}$$

Let us substitute the functions $T_1(t)$ and $T_2(t)$, defined by equalities (18), into the first of equalities (12) and assuming $\tau_j(0)=\tau_j^*(0)=0$, $j=\overline{1,2}$, have

$$\tau_j(t)-\tau_i^*(-t)=k\ \int_0^t(t-z)^{2\beta}\left[\nu_j(z)+\nu_i^*(-z)\right]dz,\quad 0<t<1,\ \ j,i=\overline{1,2},\ \ j\ne i. \tag{19}$$

Further, using formulas (14) and (15), we find $u_1\left[x,-\left(1-\sqrt{x}\right)^2\right]$, $u_2\left[-\left(1-\sqrt{y}\right)^2,y\right]$, $u_1^*\left[-y,\left(1-\sqrt{y}\right)^2\right]$, $u_2^*\left[\left(1-\sqrt{x}\right)^2,-x\right]$ and substitute it into conditions (7) and (8).

Then, taking into account conditions (5), (6) and using equalities (18), (19), taking into account $\tau_j(0)=\tau_j^*(0)=0$, $j=\overline{1,2}$, after some transformation , we get

$$\tau_1(t)=\frac{1}{2}\left[f_1(t)-m\cdot f_2(t)\right]+\frac{1}{2}\Xi\left[q_1(t)+m\cdot q_2(t)\right],\qquad 0\le t\le 1; \tag{20}$$

$$\tau_2(t)=\frac{1}{2}\left[f_2(t)+m\cdot f_1(t)\right]+\frac{1}{2}\Xi\left[q_2(t)-m\cdot q_1(t)\right],\qquad 0\le t\le 1, \tag{21}$$

where $\Xi[\psi(t)]=\frac{sin(\beta\pi)}{\beta\pi}\int_0^t\frac{(t-z)^{-2\beta}}{(1-z)^{-\beta}}\left[\frac{d^2}{dz^2}\int_0^z(z-\varsigma)^{\beta}\psi(\varsigma)d\varsigma\right]dz$, $q_j(t)=p_j\left[\left(1+\sqrt{t}\right)^2/4\right]$.

This reduces the problem $F^{(3)}$ to the Dirichlet problem for equation (1) in the domain Ω_0 with boundary conditions (4), $u(x,0)=\tau_1(x)$, $u(0,y)=\tau_2(y)$, where $\tau_j(t)$ - are functions defined by equalities (20) and (21).

By a similar method, problem $F^{(4)}$ is equivalently reduced to problem N [6] for equation (1) in the domain Ω_0 with the boundary conditions $u(0,0)=0$, (4) and

$$\lim_{y\to -0}(-y)^{\alpha}u_y(x,y) = -\frac{1}{2}[g_1(x) - m\cdot g_2(x)] + \frac{1}{2}\Phi[q_1(x) + m\cdot q_2(x)],\ 0<x<1;$$
$$\lim_{x\to -0}(-x)^{\alpha}u_x(x,y) = -\frac{1}{2}[g_2(y) + m\cdot g_1(y)] + \frac{1}{2}\Phi[q_2(y) - m\cdot q_1(y)],\ 0<y<1,$$

here $\Phi[\psi(t)] = \frac{sin(\beta\pi)}{k\beta\pi}t^{1-\alpha}(1-t)^{\beta}\int_0^t (t-z)^{\beta}\psi(z)dz$, and problem $F^{(5)}$ to the following problem $\tilde{F}^{(5)}$: find a regular solution $u(x,y)\in C(\overline{\Omega_0})$ of equation (1) in the domain Ω_0, satisfying the conditions $u(0,0)=0$, (4) and

$$u(t,0) + m\cdot u(0,t) = f_1(t) + m\cdot \Xi[q_2(t)], \qquad 0\le t\le 1;$$
$$\lim_{y\to -0}(-y)^{\alpha}u_y(t,y) - m\cdot \lim_{x\to -0}(-x)^{\alpha}u_x(x,t) = m\cdot g_2(t) + \Phi[q_1(t)], \qquad 0<t<1.$$

Theorem 1. Problems $F^{(3)}$, $F^{(4)}$ and $F^{(5)}$ have no more than one solution.

This theorem is proved using the following lemma.

Lemma. Let $u(x,y)\in C(\overline{\Omega_0})$ be a regular solution to equation (1) in the domain Ω_0, satisfying the conditions $u(x,y) = \varphi(x,y)$, $(x,y)\in\overline{AB}$, and

$$\lim_{\varepsilon\to 0}\int_{l_\varepsilon}(xy)^{\alpha-1}u(x,y)\left[xu_x(x,t)dy - yu_y(x,y)dx\right] \equiv 0, \qquad (x,y)\in\Omega_0,$$

where $l_\varepsilon : x+y=\varepsilon, \varepsilon>0$. Then the equality

$$\iint_{\Omega_0}(xy)^{\alpha-1}\left(x\cdot u_x^2 + y\cdot u_y^2\right)dxdy - \int_0^1 x^{\alpha-1}\tau_1(x)\nu_1(x)dx - \int_0^1 y^{\alpha-1}\tau_2(y)\nu_2(y)dy = 0.$$

The following theorems are true.

Theorem 2. Let the given functions satisfy the following conditions:

1) $\varphi(x,y) = [x(1-x)]^{\varepsilon}\tilde{\varphi}_1(x), \quad \tilde{\varphi}_1(x)\in C[0,1],\ \varepsilon>1+\alpha;$

2) $f_j(t) = \tilde{f}_j(0)\left(t^{-2\beta}-1\right) + \tilde{f}_j(t), \qquad \tilde{f}_j(t)\in C^{(2,\lambda)}[-1,0], \qquad \lambda>0, \quad j=\overline{1,2},$

$$\tilde{f}_j(0) = 4^{2\beta}\beta\Gamma^2(\beta)[2\pi\Gamma(2\beta)]^{-1}\int_0^1 \tilde{\varphi}_1(t)\,[t(1-t)]^{\alpha+\varepsilon-1}dt$$

3) $p_j(t) = [t-(1/4)]^{\gamma_1}[1-t]^{\gamma_2}\,\tilde{p}_j(t), \quad \tilde{p}_j(t)\in C[(1/4),1], \qquad \gamma_1\ge 2-4\beta, \qquad \gamma_2\ge 4,\ j=\overline{1,2}.$

Then the problem $F^{(3)}$ has a unique solution.

Theorem 3. Let the functions $\varphi(x,y)$ and $p_j(t)$, $j=\overline{1,2}$ satisfy the conditions of Theorem 2, and the functions $g_j(t)$, $j=\overline{1,2}$ - conditions $g_j(t) = |t|^{\delta}\tilde{g}_j(t)$, where $\tilde{g}_j(t)\in C(0,1)\cap L(0,1)$, $\delta\ge 1+\alpha$, $j=\overline{1,2}$. Then the problem $F^{(4)}$ has a unique solution.

Theorem 4. Let the functions $\varphi(x,y)$, $f_1(t)$ and $p_j(t)$, $j=\overline{1,2}$ satisfy the conditions of Theorem 2, and the functions $g_2(t)$ - conditions of Theorem 3.Then the problem $F^{(5)}$ has a unique solution.

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