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Derivation of Composition Formula for Evaluating Double Integrals Numerically BY Using Trapezoidal rule and Midpoint and And Compare The results Using Accelerated Methods.

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Abstract-*The main objective of this paper is to find the values of the two-dimensional integrals numerically, when their integrals are continuous when the partial periods on the two dimensions are not equal, where a composite method was derived for calculating the two-dimensional integrals by using the trapezoid rule on the inner dimension and the midpoint rule on the outer dimension) when The numbers of partial periods on the two dimensions are not equal with the limits of correction, and then compare the results obtained after improving them using Romperk and Aitkin acceleration. The method was characterized with Romperk acceleration and the results were of high accuracy and speed in reaching the closest value to the true value with a number of relatively few partial periods.*

Keywords-:Region;integral;dimensions; subintervals;Romberk

1.Introduction

One of the most important features in the subject of numerical analysis is the creation of various methods for finding approximate solutions to mathematical problems. The efficiency of this method depends on the accuracy and ease with which it can be calculated. Modern numerical analysis is the numerical interface to the broad field of applied analysis.

The study of error is a central task in numerical analysis because the results obtained from applying most numerical methods are only an approximation of the real solution. It is important to know the resulting error and how to estimate it through a set of calculations.

It is known to students of the subject of numerical analysis that the numerical solutions of integrals constitute an important part of this topic (finding approximate values of integrals), as this importance is more evident in the practical applications practiced by engineers and physicists, as the value of the specific integral represents the area between the curve (f) x) $\geq$0) and the x-axis denoted by the lines x = a, x = b. The importance of binary integrals lies in finding the area of surfaces, finding the intermediate centers and the intrinsic momentities of flat surfaces, and finding the volume below the surface of the bilateral integration, which has prompted many researchers to work in the field of bilateral integrals, including akkar [1] in 2010, Frank Ayers [3]
, and Hilal [4]in٢٠١ 3and others .

In this paper we dealt with a numerical method to calculate the values of double integrals by applying the Rombrk acceleration method to the values resulting from the application of (the composite base from the base of the trapezoid on the inner dimension and the base of the midpoint on the outer dimension), when the number of partial periods is not equal, as well as the use of acceleration Aitken then compared the results between the two accelerations and the statement of the best, and we took a special case when) and our code for the base is B, where (M represents the base of the midpoint, T represents the base of the trapezoid, and R represents the Rombreck rule). The general formula for the limits of the correction (error formulas) was derived for the composite method in the event that the integral (continuous at each point of the integration region), as well as our symbol for Aitken acceleration in the symbol E with the base ETM.

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2- **Deriving the rules for calculating binary integrals with Continuous integrandsand error formulas using the trapezoid rule and the midpoint rule.**

We now review a numerical method for calculating binary integrals and then apply the Rombrk acceleration method to the values resulting from applying the composite rule (the trapezoid rule on the inner dimension and the midpoint rule on the outer dimension. The deflector is on the inner dimension and the base of the midpoint on the outer dimension and to derive the base:

let I the integral is defined as follows: -

$$I = \int_c^d \int_a^b f\left(x, y\right) dx dy$$

whereas $f\left(x, y\right)$ Continuous integral at every point of the integration region $[a,b]\times[c,d]$

In general, the integration can be written I as follows:

$$I = \int_c^d \int_a^b f\left(x, y\right) dx dy = RTRM\left(h_1, h_2\right) + E\left(h_1, h_2\right) \qquad ...(1)$$

Where $RTRM\left(h_1 h_2\right)$ the integral value is numerically represented using the trapezoid rule on both dimensions, and $E\left(h\right)$ it is a series of correctio term limits that can be added to values $RTRM\left(h_1 h_2\right)$ ،and $h_1 = \dfrac{(b-a)}{n}, h_2 = \dfrac{(d-c)}{m}, n = 2m$

We know that the value of the unitary integral $\int_{x_0}^{x_n} f(x)dx$ Using the base of the trapezoid

$$\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_0 + h) + 2f(x_0 + 2h) + \cdots + f(x_n)\right]$$

Where $i = 1, 2, \cdots n-1 \quad x_i = a + ih_1, \; j = 0, 1, 2, \cdots n-1 \quad y_j = c + jh_2 \quad , h_1 = \dfrac{x_n - x_0}{n}$

And the error form for unary integrals with continuous integrals using the trapezoid rule is

$$E_T(h) = -\frac{1}{12}h^2(f_n^{(1)} - f_0^{(1)}) + \frac{1}{720}h^4(f_n^{(3)} - f_0^{(3)}) - \cdots \qquad \cdots(2)$$

As for when using the mid-point rule, it is: -

$$E_M(h)=\frac{1}{6}h^2(f_n'-f_0')-\frac{7}{360}h^4(f_n^{(3)}-f_0^{(3)})+\frac{31}{15120}h^6(f_n^{(5)}-f_0^{(5)})-\ldots \qquad \ldots(3)$$

Fox [2]

Using the mean-value theorem for derivatives of the formula (2) and (3), we obtain

$$E_T(h)=\frac{-(x_n-x_0)}{12}h^2 f^{(2)}(\mu_1)+\frac{(x_n-x_0)}{720}h^4 f^{(4)}(\mu_2)+\cdots \qquad \cdots(4)$$

$$E_M(h)=\frac{(x_n-x_0)}{6}h^2 f^{(2)}(\eta_1)+\frac{7(x_n-x_0)}{360}h^4 f^{(4)}(\eta_2)+\ldots \qquad (5)$$

Frank Ayers [3]

As for the internal integral $\int_a^b f(x,y)dx$, it can be computed numerically by the rule of the trapezoid on the dimension and x (dealing with y a constant) from the formula: -

$$T=\int_a^b f(x,y)dx=\frac{h}{2}\left(f(a,y)+f(b,y)+2\sum_{i=1}^{n-1}f(x_i,y)\right)-$$

$$\frac{(b-a)}{12}h^2\frac{\partial^2 f(\mu_1,y)}{\partial x^2}+\frac{(b-a)}{720}h^4\frac{\partial^4 f(\mu_2,y)}{\partial x^4}-\frac{(b-a)}{30240}h^6\frac{\partial^6 f(\mu_3,y)}{\partial x^6}+\cdots \qquad \cdots(6)$$

Theorem

For the function $f(x,y)$ to be continuous and derivable at each point in the integration region $[a,b]\times[c,d]$, the approximate value of the double integration is

$$I=\int_c^d\int_a^b f(x,y)dxdy \qquad \ldots(6)$$

whears $a=x_0<x_1<\cdots<x_n=b,\ c=y_0<y_1<\cdots<y_m=d$

It can be calculated by applying the rule TM from the following formula:

$$TM=\frac{h_1h_2}{2}\sum_{j=0}^{n-1}\ f\left(x_0,y_j+.5h_2\right)+f\left(x_n,y_j+.5h_2\right)+2\sum_{i=1}^{n-1}f\left(x_0+ih_1,y_j+.5h_2\right)$$

$$E_{RTM}(h_1h_2)=I-RTM(h_1h_2)=A_1h_1^2+A_2h_1^4+\ldots+B_1h_2^2+B_2h_2^4+\ldots$$

whear $j=0,1,2,3,\cdots,n-1\ ,\ y_j=c+jh_2\ ,\ i=1,2,\ldots,n-1\ ,\ x_i=a+ih_1$

proof:

Calculates the value of the internal integral according to the base of the trapezoid

$\mu_i\in(a,b)$, $i=1,2,3,\cdots$, $x_i=a+ih_1$ i=1,2,3,$\cdots$n-1 , $h_1=\frac{b-a}{n}$, $h_2=\frac{d-c}{m}$

To calculate the integral over the external dimension by applying the midpoint rule to each term of equation (7), we obtain

$$i)\ \int_c^d \frac{h_1}{2} f(a,y)dy = \frac{h_1 h_2}{2}\sum_{j=0}^{n-1} f(a, y_j + \tfrac{h}{2}) +$$
$$\frac{h_1}{2}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(a,\lambda_{11})}{\partial y^2} - \frac{7(d-c)}{360}h_2^{\,4}\frac{\partial^4 f(a,\lambda_{12})}{\partial y^4} + \frac{31(d-c)}{15120}h_2^{\,6}\frac{\partial^6 f(a,\lambda_{13})}{\partial y^6} - \cdots\right] + \qquad \cdots(8)$$

$$ii)\ \int_c^d \frac{h_1}{2} f(b,y)dy = \frac{h_1 h_2}{2}\sum_{j=0}^{n-1} f(b, y_j + \tfrac{h}{2}) +$$
$$\frac{h_1}{2}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(b,\lambda_{21})}{\partial y^2} - \frac{7(d-c)}{360}h_2^{\,4}\frac{\partial^4 f(b,\lambda_{22})}{\partial y^4} + \frac{31(d-c)}{15120}h_2^{\,6}\frac{\partial^6 f(b,\lambda_{23})}{\partial y^6} - \cdots\right] + \qquad \cdots(8)$$

$$iii)\ \int_c^d h_1 \sum_{i=1}^{n-1} f(x_i,y)dy = h_1 h_2 \sum_{j=0}^{n-1}\sum_{i=1}^{n-1} f(x_i, y_j + \tfrac{h}{2}) +$$
$$h_1\sum_{i=1}^{n-1}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(x_i,\lambda_{2+i1})}{\partial y^2} - \frac{7(d-c)}{360}h_2^{\,4}\frac{\partial^4 f(x_i,\lambda_{2+i2})}{\partial y^4} + \frac{31(d-c)}{15120}h_2^{\,6}\frac{\partial^6 f(x_i,\lambda_{2+i3})}{\partial y^6} + \cdots\right] \ (9)$$

whears $y_j = c + .5h_2 \quad j = 0,1,2,3,\cdots,n-1 \qquad \mathrm{x_i} = a + ih_1 \quad \mathrm{i}=1,2,3,\cdots \mathrm{n}\text{-}1$

And by combining equations (7), (8) and, (9) with the integral of the error formula in equation (7), we get

$$MT = \int_c^d\int_a^b f(x,y)dxdy = \frac{h_1 h_2}{2}\sum_{j=0}^{n-1}\left(f(a,y_j+\tfrac{h_2}{2}) + f(b,y_j+\tfrac{h_2}{2}) + 2\sum_{i=1}^{n-1} f(x_i,y_j+\tfrac{h_2}{2})\right) +$$
$$\int_c^d\left[\frac{(b-a)}{-12}h_1^{\,2}\frac{\partial^2 f(\mu_1,y)}{\partial x^2} + \frac{(b-a)}{720}h_1^{\,4}\frac{\partial^4 f(\mu_2,y)}{\partial x^4} - \frac{(b-a)}{30240}h_1^{\,6}\frac{\partial^6 f(\mu_3,y)}{\partial x^6} + \cdots\right]dy +$$
$$\frac{h_1}{2}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(a,\lambda_{11})}{\partial y^2} - \frac{7(d-c)}{360}h_2^{\,4}\frac{\partial^4 f(a,\lambda_{12})}{\partial y^4} - + \frac{31(d-c)}{15120}h_2^{\,6}\frac{\partial^6 f(a,\lambda_{13})}{\partial y^6} - \cdots\right] +$$
$$\frac{h_1}{2}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(b,\lambda_{21})}{\partial y^2} - \frac{7(d-c)}{360}h_2^{\,4}\frac{\partial^4 f(b,\lambda_{22})}{\partial y^4} + \frac{31(d-c)}{15120}h_2^{\,6}\frac{\partial^6 f(b,\lambda_{23})}{\partial y^6} - \cdots\right] +$$
$$h_1\sum_{i=1}^{n-1}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(x_i,\lambda_{2+i1})}{\partial y^2} - \frac{7(d-c)}{360}h_2^{\,4}\frac{\partial^4 f(x_i,\lambda_{2+i2})}{\partial y^4} + \frac{31(d-c)}{15120}h_2^{\,6}\frac{\partial^6 f(x_i,\lambda_{2+i3})}{\partial y^6} - \right] \qquad \cdots(10)$$

whears $\mu_1,\mu_2,\mu_3,\cdots \in (a,b)$, $\lambda_{kl} \in (c,d)$, $k = 1,2,3,\cdots,n+1$ ، $l = 1,2,3,\cdots$

And since the partial derivatives of the function f with respect to the variable x are continuous in its region of integration, using the median value theorem of integration (10) we obtain

$$RTM\left(h_1,h_2\right)=\frac{h_1h_2}{2}\sum_{j=0}^{n-1}\left(f(a,y_j+\tfrac{h_2}{2})+f(b,y_j+\tfrac{h_2}{2})+2\sum_{i=1}^{n-1}f(x_i,y_j+\tfrac{h_2}{2})\right)+$$

$$(b-a)(d-c)\left[\frac{h_1^2}{-12}\frac{\partial^2 f(\mu_1,\theta_1)}{\partial x^2}+\frac{h_1^4}{720}\frac{\partial^4 f(\mu_2,\theta_2)}{\partial x^4}-\frac{h_1^6}{30240}\frac{\partial^6 f(\mu_3,\theta_3)}{\partial x^3}+\cdots\right]+$$

$$\frac{h_1}{2}\left[\frac{(d-c)}{6}h_2^{\,2}\frac{\partial^2 f(a,\lambda_{11})}{\partial y^2}+h_2^{\,2}\frac{(d-c)}{6}\frac{\partial^2 f(b,\lambda_{21})}{\partial y^2}+h_1\sum_{i=1}^{n-1}\frac{(d-c)}{6}\frac{\partial^2 f(x_i,\lambda_{2+i1})}{\partial y^2}\right]+$$

$$\frac{h_1}{2}\left[\frac{7(d-c)}{-360}h_2^{\,4}\frac{\partial^4 f(a,\lambda_{12})}{\partial y^4}+h_2^{\,4}\frac{7(d-c)}{-360}\frac{\partial^4 f(b,\lambda_{22})}{\partial y^4}+h_1\sum_{i=1}^{n-1}\frac{7(d-c)}{-360}\frac{\partial^4 f(x_i,\lambda_{2+i2})}{\partial y^4}\right]+\cdots \qquad \cdots(11)$$

$\theta_i\in(c,d)$, $i=1,2,3,\cdots$

Since the $\frac{\partial^2 f}{\partial x^2},\frac{\partial^4 f}{\partial x^4},\ldots$, $\frac{\partial^2 f}{\partial y^2},\frac{\partial^4 f}{\partial y^4},\ldots$ Continuous functions at every point in the region $[a,b]\times[c,d]$ The error formula (limits of correction) for the integration with the base of the trapezoid on the inner dimension x and the base of the middle point on the outer dimension y becomes:

$$E_{RTRM}(h_Ih_2)=(d-c)(b-a)\left(\frac{-h_1^2}{12}\frac{\partial^2 f\left(\bar{\lambda}_1,\bar{\eta}_1\right)}{\partial x^2}+\frac{h_1^4}{720}\frac{\partial^4 f\left(\bar{\lambda}_2,\bar{\eta}_2\right)}{\partial x^4}-\ldots\right)+(d-c)(b-a)$$

$$\left(\frac{h_2^2}{6}\frac{\partial^2 f\left(\hat{\lambda}_1,\hat{\eta}_1\right)}{\partial y^2}-\frac{7h_2^4}{360}\frac{\partial^4 f\left(\hat{\lambda}_2,\hat{\eta}_2\right)}{\partial y^4}-\ldots\right) \qquad \ldots(12)$$

$$\left(\hat{\lambda}_1,\hat{\eta}_1\right),\left(\hat{\lambda}_2,\hat{\eta}_2\right),\ldots,\left(\bar{\lambda}_1,\bar{\eta}_1\right),\left(\bar{\lambda}_2,\bar{\eta}_2\right),\ldots\in[a,b]\times[c,d]$$

Hence, we note that the integral value of (6) using the rule with the terms of correction becomes

$$RTM\left(h_1,h_2\right)=\frac{h_1h_2}{2}\sum_{j=0}^{n-1}\ f\left(x_0,y_j+.5h_2\right)+f\left(x_n,y_j+.5h_2\right)+2\sum_{i=1}^{n-1}f\left(x_0+ih_1,y_j+.5h_2\right)$$

$$E_{RTM}\left(h_I,h_2\right)=I-RTM\left(h_1h_2\right)=A_1h_1^2+A_2h_1^4+\ldots+B_1h_2^2+B_2h_2^4+\ldots$$

3.Examples and results

1- $I=\int_0^1\int_0^1\sin(\frac{\pi}{2}(x+y))dxdy$ And its analytical value is) 0.810569469138702 Rounded to fifteen decimal places

2- $I=\int_0^1\int_0^1 xe^{-(x+y)}dxdy$ And its analytical value is(0.167032242958898) Rounded to fifteen decimal places

3 . $\int_1^2\int_1^2 \ln(x+y)dxdy$ And its analytical value is (1.08913865206603 Rounded to fourteen decimal places

1. That integral $I = \int_0^1\int_0^1 \sin(\frac{\pi}{2}(x+y))dxdy$ An identifier and a derivative of each $(x, y) \in [0,1]X[0,1]$

When applying the TM rule, we obtained the approximate value rounded to three decimal places when m = 2n = 32,64 as in Table (1), and when applying the rule with Romberk acceleration, we obtained an identical value to the real value at h = 10 and by using Aitken acceleration we obtained a value rounded to eight decimal places.

2. That integral $I = \int_0^1\int_0^1 xe^{-(x+y)}dxdy$ An identifier and a derivative of each $(x, y) \in [0,1]X[0,1]$

When applying the TM rule, we obtained the approximate value rounded to two decimal places when n = 32,64 as in Table 2, and when applying the rule with Rombrk acceleration, we obtained a value identical to the real value at h = 10 and by using Aitken acceleration we obtained a value rounded to ten decimal places.

3. That in**tegral** $\int_1^2\int_1^2 \ln(x+y)dxdy$ *An identifier and a derivative of each* $(x, y) \in [1,2]X[1,2]$

When applying TM, we obtained the approximate value rounded to five decimal places when n = 32 as in Table 3, and when applying the rule with Rombrk acceleration, we obtained a value identical to the real value at h = 10 and by using Aitken acceleration we obtained a value rounded to nine decimal places

N	M	TM	**RTM**	ETM
١	٢	0.707106781186547	0.900316316157106	
٢	٤	0.788580507474737	0.808939650992312	
٤	٨	0.805290377838142	0.810576105256015	0.810940430839578
٨	١٦	0.809262985907292	0.810569462585778	0.810592835671573
١٦	٣٢	0.810243673661090	0.810569469140307	0.810569463117329
٣٢	٦٤	0.810488071770553	0.810569469138702	0.810569467978309

Table (1)

Calculate Double integration $I = \int_0^1\int_0^1 \sin(\frac{\pi}{2}(x+y))dxdy$

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N	M	TM	**RTM**	ETM
١	٢	0.1115650800742	0.160270339415774	
٢	٤	0.1523937612390	0.166983123655508	
٤	٨	0.1633220453380	0.167032162086274	0.165304927429723
٨	١٦	0.1661014942391	0.167032242926547	0.167044986431567
١٦	٣٢	0.1667993552179	0.167032242958895	0.167033036596949
٣٢	٦٤	0.1669740084791	0.167032242958898	0.167032242934620

Table (٢)

Calculate Double integration $\int_1^2 \int_0^1 xe^{-(x+y)}$

N	M	TM	**RTM**	ETM
١	٢	1.09394356120769	1.09394356004840	
٢	٤	1.09035882204032	1.08916390961153	
٤	٨	1.08913226057701	1.08913875666004	1.08883320145205
٨	١٦	1.08913823176279	1.08913865229665	1.08906205060568
١٦	٣٢	1.08913865346258	1.08913865206625	1.08911948666992
٣٢	٦٤	1.08913865212987	1.08913865206603	1.08913385977589

Table (٣)

Calculate Double integration $\int_1^2 \int_1^2 \text{Ln}(x+y)$

4 .The reasoning

In this paper, we dealt with calculating the approximate values of binary integrals with continuous integrals using the midpoint rules and using the trapezoid and midpoint rules on the inner and outer dimensions x, y when the partial periods are not equal on both dimensions, which is denoted by TM, and then comparing the results obtained and improving them using expedited Romper and Waitkin

And when using Romperc acceleration with several devices symbolized by RTM and with AITKEN as ETM.

We conclude from the results and tables of this research that this method gives good results with high accuracy as this rule gave correct results for three decimal places compared to the analytical value and after using Romperk acceleration, I got a value close to fifteen decimal places as in example (1) and in a short time as well as when Their use in the third and second examples gave correct values (to several decimal places). In the second integral, we obtained an approximate value close to fifteen decimal places.

When we did the third integral, we got an approximate value, not a quarter of a decimal place. From observing the results of the aforementioned examples, we conclude that the method with Romperk T RM acceleration is the best as it has high accuracy, excellent speed, short time and can be relied upon in the calculation of continuous binary integrals.

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