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Anti-Fuzzy Filters of $\boldsymbol{\rho}$-algebra

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Abstract:- *Here we introduce the concept of anti-fuzzy filter and prime anti-fuzzy filter on ρ-algebra. Also, we give some theorems and relevance between them. We study the spectrum of a prime anti-fuzzy filter on ρ-algebra.*

Keyword: ρ-algebra, filter, prime filter, fuzzy filter, -antifuzzy filter, prime anti-fuzzy filter, spectrum of anti-fuzzy.

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INTRODUCTION

In 1996, J.Meng introduced BCK –filter,[12]. In 1999, J. Neggers and el ct, gave some results of d-algebra and introduced d-ideals in d-algebra,[14,15]. In 1999, W.K.Jeong gave Anti Fuzzy Prime Ideals in BCK-algebra,[8].

In 2006, Y.B. Junand and E.H. Row introduced Nil subset in BCH-algebra,[9].In 2002, Lele C.gave Fuzzy filter in BCI –algebra,[10]. In 2012, M.B. Ahamed and A. Ibrahim gave anti fuzzy implicative filter in lattice W –algebras,[4]. In 2013, S.M. Mostafa and A. T. Hameed studied Anti-fuzzy KUS-ideals of KUS-algebras,[13]. In 2015, A.T. Hameed introduced Fuzzy ideal of some algebras,[7]. In 2017, S. Khalil and M.Alradha studied Characterizations of ρ-algebra and Generation Permutation Topological ρ-algebra Using Permutation in Symmetric Group,[11]. In 2018, A.T. Hameed and B.H.Hadi introduced Anti-Fuzzy AT-Ideals on AT-algebras,[6]. In 2020, A.T. Hameed and N.J. Raheem and A.H. Abed gave Anti-Fuzzy SA-Ideals on SA -algebras,[5]. In 2020, H .K. Abdullah and A.K.Mohammad gave the Fuzzy $\boldsymbol{\rho}$-filter and fuzzy c-$\boldsymbol{\rho}$-filter in $\boldsymbol{\rho}$-algebra and Some Types filter of $\boldsymbol{\rho}$-algebra,[2,3]. In 2021, H.K. Abdullah and et cl, introduced the concept the Spectrum of Prime filter on $\boldsymbol{\rho}$-algebra,[1]. The purpose of this article is to introduce some kind anti-fuzzy filter, prime anti-fuzzy filter and spectrum of them in ρ –algebra, also we study the relation between them.

1. PRELIMINARIES

Paraphrased text will appear here useful our result. We present some definitions and properties:

<u>Definition 1.1 [11]:</u> Let X non empty set and the constant 0∈ X with binary operation $(*)$ satisfying the following:
ρ_1) $\varepsilon * \varepsilon = 0 \quad \forall \varepsilon \in X$;
ρ_2) $0 * \varepsilon = 0 \quad \forall \varepsilon \in X$;
ρ_3) If $\varepsilon * \omega = 0 = \omega * \varepsilon$ imply $\varepsilon = \omega$, $\forall \varepsilon, \omega \in X$;
ρ_4) For all $\varepsilon \neq \omega$ and $\varepsilon . \omega \in X$-{0} imply $\varepsilon * \omega = \omega * \varepsilon \neq 0$.
Then $(X;*,0)$ is called ρ-algebra.
<u>Remark 1.2 [11]</u>: In ρ-algebra if $= \emptyset$, then it called **trivial $\boldsymbol{\rho}$-algebra**.
<u>Definition 1.3 [3]</u>: Let $(Y;*.0)$ is ρ-algebra, $\emptyset \neq A \subseteq Y$. Then A **is a subalgebra of $\boldsymbol{Y}$** if
$n * m \in A \; \forall n.m \in A$.
<u>Lemma 1.4 [2]:</u> If $(A;*.0)$ is a subalgebra of ρ-algebra $(Y;*.0)$, then $0 \in A$.
<u>Remark 1.5 [2]:</u> Let $(X;*.0)$ is ρ-algebra and $b \in X$. b^* is denotes to (b $*$0) and if $(b^*)^* = b$, then b called **an involuntary element of $\boldsymbol{X}$**. X is called **an involuntary $\boldsymbol{\rho}$-algebra**, when all elements on X is involuntary.
<u>Proposition 1.6 [2]:</u> In ρ-algebra $(X;*.0)$ an constant element 0 is an involuntary element.
<u>Definition 1.7 [11]</u>: The set X is **supper commutative,** if $\forall \theta.\delta \in X.\theta \neq \delta \neq 0,\; \theta * \delta = \delta * \theta \neq 0$.
<u>Definition 1.8 [11]</u>: Let $(X;*.0)$ be ρ-algebra, we define the two binary operations $\wedge$ and $\vee$ as follow:
1. $\hbar \wedge e = e * (e * \hbar)$ *and* $e \wedge \hbar = \hbar * (\hbar * e)$;
2. $\hbar \vee e = (\hbar^* \wedge e^*)^* = \left(e^* * (e^* * \hbar^*)\right) * 0$.

<u>Remark 1.9 [1]</u>: In ρ-algebra $(X;*.0)$, if we assume $g * 0 = g \; (g^* = g). \forall g \in X$, then we get:
1. $(g \vee y)^* = g \wedge y$;(since $((g^* \wedge y^*)^*) * 0 = \left((g \wedge y) * 0\right) * 0 = (g \wedge y)$

2. $(g \wedge y)^* = (g \vee y)$;(since $(g^* \wedge y^*)^* = g \vee y$).

Definition 1.10 [1]: Impose $(W;* .0)$is ρ-algebra. An element $\mathcal{V} of\ W$ is said to be ***ρ*- unit** if $\mathcal{V} * W = W. where\ \mathcal{V} * W = \{\mathcal{V} * y\ | y \in W\}\ and\ U(W) = \{\mathcal{V} \in W: \mathcal{V} * W = W\}$.

Definition 1.11[2]: Let $(B;* .0)$be ρ-algebra and $\{\sigma_i$:i∈ I}is a collection of fuzzy subsets of B, then

1- $\bigcap_{i \in I} \sigma_i(\complement) = inf\ \{\sigma_i(\complement). i \in I\}. \forall \complement \in B$, and $\bigcup_{i \in I} \sigma_i(\complement) = sup\ \{\sigma_i(\complement). i \in I\}. \forall \complement \in B$. Which are represented fuzzy subset of B.

2- If $\sigma. \varepsilon$ be two fuzzy subsets of B, then $\sigma \leq \varepsilon$. Mean $\sigma(\complement) \leq \varepsilon(\complement). \forall \complement \in B$.

Lemma 1.12 [3]: Let $\beta: (\mathcal{J};* .0) \to (\mathcal{S}; \Delta.0)$be mapping from ρ-algebra $(\mathcal{J}; *,0)$ to another ρ-algebra $(\mathcal{S};\Delta,0')$ is homomorphism, then:

1. $\beta(0) = 0'$;
2. $(\beta(f^*)) = (\beta(f))^*$;
3. $\forall f. y \in \mathcal{J} - \{0\}. f \neq y.\ \beta(f * y) = \beta(y) \Delta\, \beta(f)$;
4. If $\leq y$, then $\beta(f) \leq \beta(y). \forall f. y \in X$ **.**

Definition 1. 13 : Let $\beta: (\mathcal{J};*. 0) \to (\mathcal{J};* .0)$, $\mathcal{J}$ is ρ-algebra. **The set *Ker* (*β*)** defined: $Ker\ (\beta) = \{\mathcal{S}: \mathcal{S} \in \mathcal{J}. \beta(\mathcal{S}) = 0\}$.

Proposition 1.14 [2]: Let $\beta: (\mathcal{J};* .0) \to (\mathcal{S}; \Delta.0')$be isomorphism ρ-algebras. Then $\forall j. y \in \mathcal{S}$:

1. $\beta^{-1}(j) = (\beta^{-1}(j))'$;
2. $\beta^{-1}(j \Delta y) = \beta^{-1}(j) \Delta \beta^{-1}(y)$.

Definition 1.15 [11]: Let$(X;* .0)$be ρ-algebra and nonempty subset of X .Then $\mathcal{K}$ called **a filter in *X*** if these conditions are met:

1. $\forall\ \mathbb{v}. \mathbb{h} \in \mathcal{K}. \mathbb{v} \wedge \mathbb{h} \in \mathcal{K}$ ($\mathcal{K}$ be closed under $\wedge$);
2. If $(\mathbb{v}^* * \mathbb{h}^*)^* \in \mathcal{K}. \mathbb{h} \in \mathcal{K}$ imply $\mathbb{v} \in \mathcal{K}$.

Proposition 1.16 [1]: In an involuntary ρ-algebra $(G;* .0)$, $\mathbb{m} \in \mathcal{K} \Leftrightarrow \mathbb{m}^* \in \mathcal{K}$.

Proposition 1.17[1]: Put $(X;* .0)$ is ρ-algebra with $\mathbb{a}^* = \mathbb{a}, \mathcal{K}$ a nonempty subset of X, then $\mathcal{K}$ be a filter of X if and only if achieved :

1. For all $\mathbb{d}. \mathbb{g} \in \mathcal{K}, \mathbb{d} \vee \mathbb{g} \in \mathcal{K}$;
2. If $(\mathbb{d} * \mathbb{g}) \in \mathcal{K}, for\ all\ \mathbb{g} \in \mathcal{K}$ imply$\mathbb{d} \in \mathcal{K}$.

Definition 1.18 [1]: Let $(X;* .0)$ be ρ-algebra. **Prime filter of** X is proper filter $\mathcal{K}$ subject to requirement for any $n. m \in X. n \vee m \in \mathcal{K}$ lead $n \in \mathcal{K}$ or $m \in \mathcal{K}$ and denoted $\mathcal{K} <$ pk X.

Definition 1.19 [2]: Let $f: (X;* .0) \to (Y;*',0')$ be a mapping from a nonempty set $X$to anonempty set Y.If β is fuzzy subset of X,then the fuzzy subset μ of Y defined by:

$$f(\mu(x)) = \beta(y) = \begin{cases} inf_{x \in f^{-1}(y)} \mu(x) & if\ f^{-1}(y) = \{x \in X; f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases}$$

Is called to the image of μ under f.

Similarly, if μ is a fuzzy subset of Y.then the fuzzy subset μ=$(\beta o f)$in X(i.e.,the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$)is said the preimage of β under f.

Definition 1.20: Let $X \neq \emptyset$ a fuzzy set μ is amapping from X to [0,1], $(\mu: X \to [0.1])$.the value of $\mu(x)$ describes a degree of membership of x in μ.

Definition 1.21[1]: Let $(X;* ,0)$ be ρ –algebra and μ fuzzy subset of X.We defined $L(\mu; r) = \{x \in X: \mu(x) \leq r\}$, $r \in [0,1]$ is called a lower r-level cut of X, $U(\mu; k)=\{x \in X: \mu(x) \geq k\}$,k∈ [0,1] is called an upper k-level cut of X.

Definition 1.22: The complement of fuzzy set μ of X, denoted by $\overline{\mu}$, is fuzzy set in Xgiveb by $\overline{\mu}$(x)=1-μ(x),for all x∈ X.

Definition 1.23[1]: Let$(X;* ,0)$be ρ-algebra and $\emptyset \neq \Gamma \subseteq X$. Γ called a **fuzzy subalgebra of *X***, if $\Gamma(\ell * k) \geq min\ \{\Gamma(\ell), \Gamma(k)\}$, for all $\ell, k \in X$.

Definition 1.25[1]: fuzzy filter of *ρ*-algebra $(X;* .0)$ is anon constant fuzzy subset ϱ of X which checks conditions ,$\forall \mathbb{e}. \mathbb{d} \in X$:

(FF_1) $\varrho(\mathbb{e} \wedge \mathbb{d}) \geq \min\{\varrho(\mathbb{e}), \varrho(\mathbb{d})\}$;

(FF_2) $\varrho(\mathbb{e}) \geq \min\{\varrho((\mathbb{e}^* * \mathbb{d}^*)^*), \varrho(\mathbb{d})\}$.

Definition 1.25[1]: Let ζ non constant fuzzy filter of ρ-algebra $(X;* .0)$ is called **prime fuzzy filter** if $\zeta_P(\eta \vee \gamma) \geq max\ \{\zeta_P(\eta), \zeta_P(\gamma)\}$. for all $\eta. \gamma \in X$.

2. Anti-Fuzzy Filters on *ρ*-algebra

In this section, we provide definitions of anti-fuzzy filter and prime anti-fuzzy filter on ρ-algebra, and study its relationship with them on ρ-algebra.

Definition 2.1: Let$(X;*\,.0)$be ρ-algebra and $\emptyset \neq \Gamma \subseteq X$. Γ called **an anti-fuzzy subalgebra of *X***, if $\Gamma(\ell * \hbar) \leq max\,\{\Gamma(\ell), \Gamma(\hbar)\}$, for all $\ell, \hbar \in X$.

Example 2.2: Let $X = \{0, \mathbb{q}, \mathbb{m}\}$ with $(*)$ binary knowledge as :

$*$	0	$\mathbb{q}$	$\mathbb{m}$
0	0	0	0
$\mathbb{q}$	$\mathbb{q}$	0	$\mathbb{q}$
$\mathbb{m}$	$\mathbb{m}$	$\mathbb{q}$	0

Clear $(X;*\,.0)$ is ρ-algebra, let Γ fuzzy subset of X defined as: $\Gamma(\ell) = \begin{cases} 0.2 & \ell = 0 \\ 0.4 & \ell = \mathbb{q} \\ 0.9 & \ell = \mathbb{m} \end{cases}$.

Then Γ is an anti-fuzzy subalgebra of .

Proposition 2.3: Let $(X;*\,.0)$ be ρ-algebra and Γ is anti-fuzzy subalgebra of X, then $\Gamma(0) \leq \Gamma(\mathfrak{u})$. $\forall \mathfrak{u} \in X$.

Proof: $\Gamma(0) = \Gamma(\mathfrak{u} * \mathfrak{u}) \leq max\,\{\Gamma(\mathfrak{u}), \Gamma(\mathfrak{u})\} = \Gamma(\mathfrak{u})$,$\forall\, \mathfrak{u} \in X$.

Proposition 2.4: In an anti-fuzzy subalgebra μ of ρ-algebra X, then $\mu(x^*) \leq \mu(x), \forall x \in X$.

Proof: Let μ be an anti-fuzzy subalgebra of X, $\mu(0) \leq \mu(x)$,for all x$\in X$, by Proposition (2.3). Then $\mu(x^*) = \mu(x * 0) \leq$ max$\{\mu(x).\mu(0)\} = \mu(x)$. Thus $\mu(x^*) \leq \mu(x)$.

Definition 2.5: Anti-fuzzy filter of *ρ*-algebra $(X;*\,,0)$ is anon constant fuzzy subset ϱ of X which checks conditions ,$\forall \mathbb{e}.\mathbb{d} \in X$:

(AFF_1) $\varrho(\mathbb{e} \wedge \mathbb{d}) \leq max\{\varrho(\mathbb{e}).\varrho(\mathbb{d})\}$;

$(AFF_2)$$\varrho(\mathbb{e}) \leq max\{\varrho((\mathbb{e}^* * \mathbb{d}^*)^*).\varrho(\mathbb{d})\}$.

Example 2.6: Impose X ={0,1,2,3}, And $(*)$ defined:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	1
2	2	3	0	2
3	3	1	2	0

Then $(X;*\,,0)$ is a ρ-algebra

$\mu : X \rightarrow$[0,1] fuzzy subset defined as: $\mu(x)$=$\begin{cases} 0.2 & if\ \ x \in \{3\} \\ 0.5 & if\ \ x \in \{0.1.2\} \end{cases}$

Clear $(X;*\,,0)$ is $\boldsymbol{\rho}$-algebra and μ(x)is an anti fuzzy filter of X, since

(x ∧ y)	0	1	2	3
0	0	0	0	0
1	0	1	2	1
2	0	1	2	2
3	0	0	0	3

And (AFF_1), μ(x ∧ y)≤ max {μ(x),μ(y)},∀x, y ∈X, as

μ(x ∧ y)	0	1	2	3
0	0.5	0.5	0.5	0.5
1	0.5	0.5	0.5	0.5
2	0.5	0.5	0.5	0.5
3	0.5	0.5	0.5	0.2

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(AFF_2) $\mu(0) = 0.5 \leq max\{\mu(0^* * 3^*)^*.\mu((3)\} = 0.5;$
$\mu(1) = 0.5 \leq max\{\mu(1^* * 3^*)^*, \mu(3)\} = 0.5;$
$\mu(2) = 0.5 \leq max\{\mu(2^* * 3^*)^*, \mu(3)\} = 0.5.$
Then μ represented an anti-fuzzy filter.

Remark 2.7: not necessary all fuzzy subset of X is AFF as assume that X={0,1,2,3},($*$) binary operation like in following table represented ρ-algebra:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	1	0	2
3	3	1	2	0

Note that if we defined μ which is fuzzy subset of X as that: $\mu(x) = \begin{cases} 0.5 & if\ x \in \{0.1\} \\ 0.6 & if\ x \in \{2.3\} \end{cases}$

μ(2)=0.6≰max {μ (2^**1^*)^*,μ(1)}=max{μ(1).μ(1)}=0.5.

Proposition 2.8: Let $(X;*.0)$ be ρ-algebra, μ is AFF of X. then $\mu(x^*) \leq \mu(x)$,∀x∈ X.
Proof: By Proposition (x ∧ x)=$x^*$and since μ is AFF , then
$\mu(x^*) = \mu(x \wedge x) \leq \max\{\mu(x), \mu(x)\} = \mu(x).$
Proposition 2.9 : Let $(X;*,0)$be ρ-algebra with $x^* \leq y^*$and μ be AFF of X, then either
μ(x)≤ μ(y) or $\mu(x) = \mu(0)$, for all $x, y \in X$.
Proof: Let μ be AFF of X, then
μ(x)≤max{$\mu(x^* * y^*)^*$,μ(y)}
≤max{$\mu(0), \mu(y)\} = \mu(0)$ or $\mu(y)$.
that's mean $\mu(x) \leq \mu(0)$ or $\mu(x) \leq \mu(y)$,$\forall x, y \in X$.
Remark 2.9: In ρ-algebra $(X;*,0)$ since $0^* \leq x^*$ for all x∈ X. $\mu(0) \leq \mu(x)$, where μ is an AFF of X.
Proposition 2.10: Assume that $(X;*,0)$ be ρ-algebra with $x \leq y$, for all $x, y \in X$ and μ be AFF of X, then μ is anti-fuzzy subalgebra of X.
Proof: Since $\mu(0) = \mu(x * y) \leq \max\{\mu(x), \mu(y)\}$.
Remark 2.11: An anti-fuzzy subalgebra of ρ-algebra $(X;*,0)$ need not be an AFF of X, in general.
Proposition 2.12: Let $(X;*,0)$ be ρ-algebra with y ≤ x and μ be AFF of X, then
$\mu(y^*) \leq \mu(x^*)$ or $\mu(y^*) \leq \mu$(y).
Proof: Since y ≤ x, μ is AFF of X,then
μ(x∧y) =$\mu(y^*)$
≤ max{ μ(x), μ(y)}, this indicates that μ(y^*) ≤μ(x^*) or μ(y^*)≤ μ(y).

Proposition 2.13: Let $(X;*,0)$ be ρ-algebra with x=x^*,and μ is AFF of X, then

$\mu(0) \leq \mu(x)$, for all x∈ X.
Proof: Since $0^* = 0$, and 0≤x, for all x∈ X([$by\ \rho_2$] and by Proposition (2.12),
$\mu(0) \leq \mu(x)$.
Proposition 2.14: Let $(X;*,0)$ be ρ-algebra with x^*=x for all x∈ X, and μ is an AFF of X, then for all x, y ∈X ,μ(x) ≤ max{μ(x∗y),(y)}.
Proof: Let μ be an AFF of ρ-filter $X$and by $(AFF_2($
μ(x) ≤ max{$\mu(x^* * y^*)^*, \mu(y)\} \Longrightarrow \mu(x) \leq \max\{\mu(x * y), \mu(y)\}$.
Theorem 2.15: Assume that {$\mu_i, i \in I$ } be collection of AFF of $\boldsymbol{\rho}$-algebra X, then $\cap\mu_i$is AFF of X.
Proof: since $\mu_{i\in I}$ is AFF ∀i ∈I, then
$\mu_i(x \wedge y)$ ≤max{μ_i(x),μ_i(y)∀x ,y∈ X}, then
inf{$\mu_{i\in I}(x \wedge y)$ ≤ inf{$max\{\mu_{i\in I}(x), \mu_{i\in I}(y), \forall x, y \in X\}\}$.
≤ max{inf $\mu_{i\in I}$(x), inf $\mu_{i\in I}$(y), ∀x ,y∈ X}.
That's mean $\cap_{i\in I}\mu_i(x \wedge y) \leq max\{\cap_{i\in I}\mu_i(x), \cap_{i\in I}\mu_i$(y)}.
$\mu_{i\in I}(x)$ ≤ max{$\mu_{\ i\in I}(x^* * y^*\ )^*$,$\mu_{i\in I}$(y), ∀x,y∈ X} ,then

Inf$\mu_{i\in I}(x) \le \inf\{\max\{\mu_{i\in I}(x^* * y^*)^*,\ \mu_{i\in I}(y), \forall\, x, y \in X\}$
$\le \max\{\inf \mu_{i\in I}(x^* * y^*)^*, \inf \mu_{i\in I}(y), \forall\, x, y \in X\}$
That's mean $\bigcap_{i\in I}\mu_i(x) \le \max\{\bigcap_{i\in I}\mu_i(x^* * y^*)^*, \bigcap_{i\in I}\mu_{i(y)}\}$. So $\bigcap_{i\in I}\mu_i$ is AFF.

Remark 2.16: clear union of two anti-fuzzy filters un necessary represented AFF of X as in this example:

Example 2.17: Impose X={0, a, b, c,}, with $(*)$ defined as:

*	0	a	b	c
0	0	0	0	0
a	a	0	c	c
b	b	c	0	a
c	c	c	a	0

And let we put defined to $\mu(x).\vartheta(x)$ which are anti-fuzzy filters of ρ-algebra $(X;*,0)$

$\mu(x) = \begin{cases} 0.2 & if\ x \in \{0, a\} \\ 0.9 & if\ x \in \{b, c\} \end{cases}$ and $(x) = \begin{cases} 0.5 & if\ x \in \{0, b\} \\ 0.6 & if x \in \{a, c\} \end{cases}$.

But $(\mu \cup \vartheta(x) = \begin{cases} 0.5 & if\ x \in \{0\} \\ 0.6 & if\ x \in \{a, c\} \\ 0.9 & if\ x \in \{b\} \end{cases}$. is not AFF since

$(\mu \cup \vartheta)(b) = 0.9 \nleq \sup\{(\mu \cup \vartheta)(b^* * a^*)^*, \mu \cup \vartheta(a)\}\} = \sup\{(\mu \cup \vartheta(c), (\mu \cup \vartheta)(a)\} = 0.6.$

Proposition 2.18 : In ρ-algebra $(X;*,0)$, μ is AFF of X if and only if $\overline{\mu}$ is a fuzzy filter of X.

Proof: Suppose μ *is AFF of* X, then we can see

Since $\overline{\mu}$(x∧y) =1- μ (x ∧y),∀x ,y∈ X , by Definition (2.5), but

$\overline{\mu}$ (x ∧y) =1- μ(x ∧y) ≤ 1-max{μ (x), μ (y)} ≥ 1-max{1-$\overline{\mu}$(x),1-$\overline{\mu}$(y)}= min{$\overline{\mu}$(x), $\overline{\mu}$(y)}. And we see

$\overline{\mu}$(x)=1- μ(x)≥1-max{μ $(x^* * y^*)^*$, μ(y)} = min {1-$\mu(x^* * y^*)^*$,1-μ (y)}= min {$\overline{\mu}$ $(x^* * y^*)^*$, $\overline{\mu}$(y)}. So $\overline{\mu}$is a fuzzy filter of X.

Conversely, assume that $\overline{\mu}$ is fuzzy filter of X and x, y ∈X, then

(FF_1) 1-μ(x∧ y) ≥ $\overline{\mu}$(x∧ y) ≥ min ($\overline{\mu}$(x), $\overline{\mu}$(y)} ≥ min {1-μ(x),1-μ(y)}=1-max{$\mu(x)$, μ(y),so

μ(x∧y) ≤ $max\{\mu$(x), μ(y)},and

(FF_2) $\overline{\mu}(x)$ ≥min{$\overline{\mu}(x^* * y^*)^*$, $\overline{\mu}$(y)},

$\mu(x)$ = 1-$\overline{\mu}(x)$ ≤1-min{$\overline{\mu}(x^* * y^*)^*$,$\overline{\mu}(y)$} = max{1-$\overline{\mu}(x^* * y^*)^*$,1-$\overline{\mu}(y)$},

=max{$\mu(x^* * y^*)^*$,$\mu(y)$}, so$\mu(x) \le$ max$(x^* * y^*)^*.\mu(y)$}.Which mean μ is represented an AFF.

Remark 2.19: Let $\mathcal{F}$be filter of ρ-algebra$(X;*,0)$and $\alpha,\beta \in [0,1]$such that $\alpha > \beta$.The complement $\overline{\mu}_{\mathcal{F}}$of μ is given by $\overline{\mu}_{\mathcal{F}} = \begin{cases} 1-\alpha & if\ x \in \mathcal{F} \\ 1-\beta & otherwise \end{cases}$.

By Proposition (2.6), it is AFF of X.

Theorem 1.20: Let μ be an AFF ρ -algebra $(X;*,0)$,with $x^* = x$.Then the set defined as:

X_μ ={x∈ X:$\mu(x) = \mu(0)$} is a filter.

Proof: Let x, y∈ X and $x, y \in X_\mu$.Then $\mu(x) = \mu(0)$,$\mu(y) = \mu(0)$. Hence, by Definition (2.5),

(AFF_1) $\mu(x \wedge y) \le \max\{\mu(x), \mu(y)\}$=$\mu(0)$, we get $\mu(x \wedge y) \ge \min\{\mu(x), \mu(y)\}$.

Now, x, y ∈ X,such that $(x^* * y^*)^*.y \in X_\mu$, then $\mu(x^* * y^*)^*$=$\mu(0).\mu(y) = \mu(0)$.

Since μ is an AFF, then

$\mu(x) \le \max\{\mu(x^* * y^*)^*, \mu(y)\} = \mu(0), since\ \mu(0) \le \mu(x)$, so $\mu(0) = \mu(x)$.

$\mu(x) \ge \min\{\mu(x^* * y^*)^*, \mu(y)\}$, then we concluding $X_\mu$is a filter.

Remark 2.21: The conversely Theorem (2.20) does not true as in this example:

Example 2.22: Let $(X;*.0)$be ρ –algebra with $x^* = x$,$X = \{0,\ r,\ f,\ z,\ s\}$ a ,$(*)$ binary operation as:

*	0	r	f	z	s
0	0	0	0	0	0
r	r	0	r	s	s

f	f	r	0	z	s
z	z	s	z	o	s
s	s	s	s	s	0

$K_1 = \{0, r, f\}$is a filter and μ be fuzzy subset of X such that: $\mu(x)=\begin{cases}0.7 & if\ x \in \{0,\ r, f\}\\ 0.3 & if\ x \in \{z, s\}\end{cases}$

μ is not AFF since $\mu(f)= 0.7 \nleq \max\{\mu(f^* * z^*)^*. \mu(z)\} = 0.3$.

Theorem 2.23: Let $(X;* ,0)$ be ρ-algebra with x*=x , μ is fuzzy subset of X. Then μ is an AFF of if and only if for each r ∈ [0,1], L$(\mu; r)$ is a filters of X or L$(\mu; r) = \emptyset$.

Proof: Assume that μ is an AFF of r, k∈[0,1] such that L(μ; r),L(μ; k) are nonempty sets and let x, y∈ L(μ; r).Then $\mu(x) \leq r. and\ \ \mu(y) \leq r$.That mean

$\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\} \leq r$. So by Definition (2.5), thus (x∧ y)∈ L(μ; r)

Now, let we suppose that x, y∈ X,such that $(x^* * y^*)^* \in L(\mu; k)$ and y∈ L(μ; r), then

$\mu(x^* * y^*)^* \leq r$ and μ(y) $\leq r$,so $\mu(x) \leq \max\{\mu(x^* * y^*)^*. \mu(y) \leq r$, by Definition (2.5), so x∈ L(μ; r), then L(μ; r) is filter in X.

Conversely, let L(μ; r) be a filter of X or L(μ; r)=∅, for any r∈ [0.1]. Assume that (AFF_1) does not valied.

Then there are c, d∈ Xsuch that $\mu(c \wedge d) > \max\{\mu(c), \mu(d)$.Let us defined

$n = \frac{1}{2}(\mu(c \wedge d) + \max\{\mu(c), \mu(d)\}$.

We get $\mu(c \wedge d) > n > \max\{\mu(c), \mu(d)\ , n\mu(c)$ and $n > \mu(d)$.Hence

c, d∈ $L(\mu; n)$ and (c∧ d)∉ $L(\mu; n)$.This is a $contradiction$ (AFF_1) of definition of a filter. Hence $(AFF_1\ )holds$.

Now, suppose $(c^* * d^*)^*$,d∈ X,such that $(c^* * d^*)^* \leq d$ and $\mu(c^* * d^*)^* \leq \mu(d)$.

Taking $\varepsilon = \frac{1}{2}(\mu(c^* * d^*)^* + \mu(d))$.

We have $\mu(c^* * d^*)^* < \varepsilon < \mu(d)$, hence$(c^* * d^*)^* \in L(\mu; \varepsilon)$ and y∉ $L(\mu; \varepsilon)$. It $contradiction$ with (AFF_2) of definition of a filter. Hence $(AFF_1\ )holds$.

Then μ is AFF of X.

Corollary 2.24: If μ is AFF of ρ-algebra $(X;* ,0)$, then the set $X_c = \{x \in X: \mu(x) \leq \mu(c)\}$ is a filter of X,∀ $c \in X$.

Definition 2.25: Let $(X;* ,0)$ be ρ-algebra and μ be an AFF of X , then the set $\langle\mu\rangle$ can be defined as $\langle\mu\rangle = \bigcap\{\delta: \delta\ is\ an\ AFF\ of\ X. \mu \subseteq \delta\}$and it called **generated by μ**.

Remark 2.26: Suppose that $(X;* ,0)$ be ρ-algebra and μ be an AFF of X.Then

1. $\langle\mu\rangle$ is an AFF of X continaing μ.
2. If μ is AFF of X , then $\langle\mu\rangle = \mu$.
3. $\langle\mu\rangle$ is smallest AFF containing μ.

Theorem 2.27: Let f :(X; ∗ ,0) → (Y; ∗ `,0`) be an epimophism from ρ-algebra X to another ρ-algebra Y, if α be an AFF of Y . Then the pre-image of α under f (μ) is also an AFF of X.

Proof: Let μ be pre-image of α under f. Then $\mu(x) = \alpha(f(x))$ for all x∈ X. Since α is an AFF of , then

$$\begin{aligned}\mu(x \wedge y) &= f^{-1}(\alpha(x \wedge y) = \alpha(f(x \wedge y))\\ &\leq \max\{\alpha(f(x)), \alpha(f(y))\}\\ &= \max\{f^{-1}(\alpha(x), f^{-1}(\alpha(y))\}\\ &= \max\{(f^{-1}(\alpha))(x), (f^{-1}(\alpha))(y)\}\\ &= \max\{\alpha(f((x)), \alpha(f(y))\}\\ &= \max\{\mu(x), \mu(y)\}.\end{aligned}$$

That's mean $\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\}$.

$$\begin{aligned}\mu(x) &= f^{-1}(\alpha(y)) = \alpha(f(y))\\ &\leq \max\{\alpha(f((x^* * y^*)^*). \alpha(f(y)\}\\ &= \max\{f^{-1}(\alpha(x^* * y^*)^*). f^{-1}(\alpha(y))\}\\ &= \max\{(f^{-1}(\alpha))(x^* * y^*)^*, (f^{-1}(\alpha))(y)\}\\ &= \max\{\mu(x^* * y^*)^*, \mu(y)\}.\end{aligned}$$

Then $\mu(x) \leq \max(\mu(x^* * y^*)^*, \mu(y)\}$.

Therefor $\mu(x) \leq \max\{\mu(x^* * y^*)^*,\mu(y)\}$, since $f(x) \in Y$is arbitrary and f is onto, x∈ X ,therefore $\mu(x) \leq$max$\{\mu(x^* * y^*)^*, \mu(y)\}$ is true for all x, y∈ X.

Hence μ=$f^{-1}(\alpha)$ is an AFF of X.

Definition 2.28: An anti-fuzzy subset μ of a set X has inf property if for any subset T of X, there exist t_0∈ T such that μ (t) = $\inf_{t \in T} \mu(t)$.

Theorem 2.29: Let f :(X; ∗ ,0) → (Y; ∗ `,0`) be a homomorphism between ρ-algebras X and Y respectively has inf property. For every β an AFF μ of X, f (β) is an AFF of Y.

Proof: By definition β(y') = f (μ)(y')= $\inf_{x\in f^{-1}(y')} \mu(x)$, for all y'∈Y and ∅ = 0.

We have to prove that β(x') ≤ max{β($(y'^{*} * x'^{*})^{*}$),β(y')}, for all x', y'∈ Y.

Let f : X → Y be an onto homomorphism of ρ-algebrsa, μ is an AFF of X with inf property and β the image of μ under f , since μ is an AFF of X,

For any x', y'∈ Y, let $x_0 \in f^{-1}$ (x'), $y_0 \in f^{-1}$ (y') be such that μ $((y_0{}^{*} * x_0{}^{*})^{*}) = inf_{t\in f^{-1}(x' * y'))} \mu\,(t)$,μ $(y_0) = \inf_{t\in f^{-1}(y')} \mu(t)$ and

$$\mu(x \wedge y) = inf_{t\in f^{-1}(x' * y')} \mu\,(t)$$
$$\leq max\ \{inf_{t\in f^{-1}(x')} \mu\,(t)\ , inf_{t\in f^{-1}(y')} \mu\,(t)\}$$
$$= \max\{\mu(x), \mu(y)\}.$$

That's mean $\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\}$.

μ $(x_0) = inf_{t\in f^{-1}(x')} \mu\,(t)$. Then

$$\beta\,(x') = inf_{t\in f^{-1}(x')} \mu\,(t) = \mu\,(x_0)$$
$$\leq \max\{\mu\,((y_0{}^{*} * x_0{}^{*})^{*})\ ,\mu\,(y_0)\}$$
$$= \max[inf_{t\in f^{-1}(x' * y')} \mu\,(t), inf_{t\in f^{-1}(y')} \mu\,(t)]$$
$$= \max\{\beta((y'^{*} * x'^{*})^{*}), \beta(y')\}.$$

Hence β is an AFF of Y. ⌂

3. Prime Anti-fuzzy Filter of ρ-algebra.

In this part, we study prime anti-fuzzy filter of ρ-algebra $(X; * .0)$. Having been thought about it

Definition 3.1: Let ζ non constant fuzzy filter of ρ-algebra $(X; * .0)$ is called **prime anti-fuzzy filter** if $\zeta_P(\eta \vee \gamma) \leq min\,\{\zeta_P(\eta), \zeta_P(\gamma)\}$, for all $\eta, \gamma \in X$, denoted by (PAFF).

Example 3.2: Let $(X; *, 0)$ be ρ –algebra, F is prime filter of X and $\beta \in (0,1]$ and defined a fuzzy subset μ by :

$$\mu(x) = \begin{cases} 0 & if\ x \in F \\ \beta & otherwise \end{cases}.$$

We show that μ is an PAFF of X, by Remark, μ is an AFF.

Let $x, y \in X$ such that $x \vee y \in F$. Then $x \in F$or $y \in F$, by Definition (3.1), then

$$\mu(x \vee y) = 0 = \min\{\mu(x), \mu(y)\}$$

Now, suppose that $x \vee y \notin F$,(x∨ y) = (x∧ y) , since F is AFF,

$\mu(x) \leq \min\ \{\mu(x^{*} * y^{*})^{*}. \mu(y)\}=\beta$. Imply $\mu(x \vee y) = \min\{\ \mu(x), \mu(y)\} = \beta$.

Proposition 3.3: Every AFPF $\mu$of ρ- algebra $(X; * .0)$is an AFF of X.

Proof: Direct by Definition (3.1).

Theorem 3.4: Let $(X; * .0)$ be ρ-algebra ,μ a nonconstant AFF of X.Then the following are equivalents:

(1) μ is an PAFF of X.

(2) For all x, y∈ X, if $\mu(x \vee y) = \mu(0)$, $then\ \mu(x) = \mu(0)\ or\ \mu(y) = (0)$.

(3) For all $x, y \in X$,$\mu(y^{*}) = \mu(0)\ or\ \mu(y^{*} * x^{*}) = \mu(y^{*})$.

(4) For all $x, y \in X$, if $x^{*} = x\ and\ y^{*} = y. then\ \mu(y) = \mu(0) or\ \mu(y * x) = \mu(0)$.

(5) For all $x, y \in X$, if $x = x^{*}$and $y = y^{*}$, then $\mu(x \wedge y) = \mu(0)$.

Proof:

(1)⟹ (2) Let μ be an AFPF of $X$and $x. y \in X$ such that $\mu(x \vee y) = \mu(0)$, then $\mu(x \vee y)$= min$\{\mu(x). \mu(y)\} = \mu(0)$ and hence $\mu(x) = \mu(0) or\ \mu(y) = \mu(0)$.

(2)⟹ (3) Let $x, y \in X$. Suppose that x∨ y=0 ,then by proposition $(y^{*} = y^{*} * x^{*})$or $(y^{*} = 0)$.So $\mu(y^{*}) = \mu(y^{*} * x^{*})$or $\mu(y^{*}) = \mu(0)$.

(3)⟹ (4) Direct by Definition (3.1).

(5)⟹ (1) Since $\mu(0) \leq \mu(x)$ and $(x \vee y) = (x \wedge y) \leq \min\{\mu(x), \mu(y)\}$. Therefore $\mu(x \vee y) = \min\{\mu(x), \mu(y)\}$, hence μ is a PAFF.

Theorem 3.5: Let μ be AFF of ρ-algebra $(X; * .0)$ with $x^{*} = x$. Then μ is a PAFF of X if and only if $X_{\mu}=\{x \in X; \mu(x) = \mu(0)\}$ is prime filter.

Proof: Let μ be AFF of ρ –algebra X. By Theorem (), $X_\mu$is filter of X, to prove $X_\mu$is prime. Let $x \vee y \in X_\mu$.Then by hypothesis and Definition)

$\mu(0) = \mu(x\vee y)$ = min $\{\mu(x),\mu(y)\}$.

Therefore $\mu(0) = \mu(x)$ or $\mu(0) = \mu(y)$ that's represented $x \in X_\mu$or $y \in X_\mu$. So $X_\mu$is prime filter.

Conversely, let $X_\mu$be a prime filter of X. And $x, y \in X - \{0\}. x \neq y$. Then $(x * y)\vee(y * x)$=0$\in X_\mu$ and by Definition () , $(x * y) \in X_\mu$ or $(y * x) \in X_\mu$.

Hence $\mu(x * y) = \mu(0)$ or $(y * x) = \mu(0)$. By Theorem () , μ is PAFF.

Corollary 3.6: Let μ be an PAFF of ρ-algebra $(X;*,0)$, then F={x∈ X: $\mu(x) = 0$} is either empty set or a prime filter .

Proof: Direct by Theorem () .

Theorem 3.7: Let μ be a non-constant AFF of ρ-algebra $(X;*,0)$, with $x^* = x$. Then the following are equivalent:

(i) μ is a PAFF of X.

(ii) For every $\alpha \in [0,1]$, if L$(\mu.\alpha) \neq \emptyset$ and L$(\mu,\alpha) \neq X$.then L$(\mu,\alpha)$is prime filter of X.

Proof:

(i)⟹ (ii) Suppose that μ is a PAFF of X and let L$(\mu,\alpha) \notin \emptyset, X$. From Theorem () , L$(\mu.\emptyset)$ is a filter .

Now, let show L(μ,α) is prime .Since L$(\mu,\alpha) \neq X$, it is proper. Let $x\vee y \in L(\mu.\alpha)$, then $\mu(x\vee y)$ =min$\{\mu(x).\mu(y)\} \leq \alpha$.

Hence $\mu(x) \leq \alpha$ or $\mu(y) \leq \alpha$, then $x \in L(\mu.\alpha)$ or y∈ $L(\mu.\alpha)$.So ,L$(\mu.\alpha)$ is prime.

(ii)⟹ (i) Suppose that μ is not a PAFF. Then $\mu(x\vee y) <$ min$\{\mu(x).\mu(y)\}$, for some $x, y \in X$. Let we defined $\beta = \frac{1}{2}(\mu(x\vee y) +$ (min$\{\mu(x).\mu(y)\}$. Then we have $\mu(x \vee y) < \beta <$ min$\{\mu(x).\mu(y)\}$.

Then we get $x\vee y) \in L(\mu,\beta)$ and $x \notin L(\mu,\beta)$ and y$\notin L(\mu,\beta)$. Hence L$(\mu,\beta) \neq \emptyset$, but L(μ,β) is not prime .It contradicts an assumption. Then μ is a PAFF of X.

Theorem 3.8: Let f :(X; ∗ ,0) → (Y; ∗ `,0`) be an epimophism from ρ-algebra X to another ρ-algebra Y, if α be a PAFF of Y . Then the pre-image of α under f(μ) is also a PAFF of X.

Proof: Let μ be pre-image of α under f. Then $\mu(x) = \alpha(f(x))$ for all x∈ X. Since α is an AFF of , then

$$\begin{aligned}\mu(x \vee y) &= f^{-1}(\alpha(x \vee y) = \alpha(f(x \vee y)) \\ &\leq \min\{\alpha(f(x)), \alpha(f(y))\} \\ &= \min\{f^{-1}(\alpha(x), f^{-1}(\alpha(y))\} \\ &= \min\{(f^{-1}(\alpha))(x), (f^{-1}(\alpha))(y)\} \\ &= \min\{\alpha(f((x)), \alpha(f(y))\} \\ &= \min\{\mu(x), \mu(y)\}.\end{aligned}$$

That's mean $\mu(x \vee y) \leq$ min$\{\mu(x), \mu(y)\}$.

Hence μ=$f^{-1}(\alpha)$ *is* a PAFF of X.

Theorem 3.9: Let f :(X; ∗ ,0) → (Y; ∗ `,0`) be a homomorphism between ρ-algebras X and Y respectively has inf property. For every β a PAFF μ of X, f (β) is a PAFF of Y.

Proof: By definition β(y') = f (μ)(y')= $\inf_{x\in f^{-1}(y')} \mu(x)$, for all y'∈Y and ∅ = 0.

We have to prove that β(x') ≤ max{β($(y'^* * x'^*)^*$),β(y')}, for all x', y'∈ Y.

Let f : X → Y be an onto homomorphism of ρ-algebrsa, μ is a PAFF of X with inf property and β the image of μ under f .

For any x', y'∈ Y, let $x_0 \in f^{-1}$(x'), $y_0 \in f^{-1}$(y') be such that $\mu((y_0{}^* * x_0{}^*)^*) = inf_{t\in f^{-1}(x' * y'))} \mu(t)$,$\mu(y_0) = \inf_{t\in f^{-1}(y')} \mu(t)$ and

$\mu(y \vee y) = \beta(f(x \vee y)) \leq \min\{\beta(f(x)), \beta(f(y))\} = \min\{\mu(x), \mu(y)\}$.

Hence β is a PAFF of Y. ⌂

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