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On Local Irregularity Vertex Coloring of Corona Product Graph ($P_m \odot E_{3,n}$)

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Abstract: Let $G = (V, E)$ be a graph with vertex set V and edge set E. The graph G is a said to be a local irregular vertex coloring if there is a function f is a called a local irregularity vertex coloring if : $(i)\ l : (V(G)) \rightarrow \{1,2,3, \dots k\}$ as a vertex irregular k-labeling and $w : (V(G)) \rightarrow N$, for every $uv \in: E(G), w(u) \neq w(v)$ where $w(u) \sum_{v \in N(u)} l(i)$ and $(ii)\ opt(l) = \min\{\max l(i) ; l(i) vertex\ irregular\ labeling\}$. The chormatic number of local irregularity vertex coloring of G denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all vertex coloring. In this paper, we study local irregular vertex coloring of path graph corona product E graph ($P_m \odot E_{3,n}$).

Keywords: local irregularity vertex coloring, corona product,

E graph

1. INTRODUCTION

Definition 1 *[4] suppose* $l : (V(G)) \rightarrow \{1,2,3, \dots k\}$ is called vertex irregular k-labeling and $w : (V(G)) \rightarrow N$, where $w(u) \sum_{v \in N(u)} l(i)$, is called local irregularity vertex coloring, if

$(i)\ opt(l) = \min\{\max l(i) ; l(i) vertex\ irregular\ labeling\}$
(ii) for every $uv \in: E(G), w(u) \neq w(v)$.

Definition 2 *[3]* The chromatic number local irregular denoted by $\boldsymbol{\chi_{lis}(G)}$; is minimum of cardinality local irregularity vertex coloring.

Lemma 1 *[2]* Let G, *simple and connected graph* $\boldsymbol{\chi_{lis}(G)} \geq \boldsymbol{\chi}\ \ (\boldsymbol{G})$.

Propotion 1 *[3]* Let G, be a graph with each two vertices adjencent have a different degree of the vertex then the $opt\ (l) = 1$.

Propotion 2 *[3]* Let G, be a graph with each two vertices adjencent have a same degree of the vertex then the $opt\ (l) \geq 2$.

In this paper, we will analyze the new result of the chromatic number of local irregular vertex coloring of corona product by path grapth and E graph ($P_m \odot E_{3,n}$). Here is the definition of corona product.

Definition 3 *[1]* Let G and H be two connected graphs. Let be a vertex of H. The corona product of combination of two graphs G and H is defined as the graph obtained by taking a duplicate of graph G and $|V(G)|$ a duplicate of graph H, namely H_i; $i = 1,2,3, \dots |V(G)|$ then connects each vertex to i in G to each vertex in H_i. The corona product of the graph G and H is denoted by ($P_m \odot E_{3,n}$).

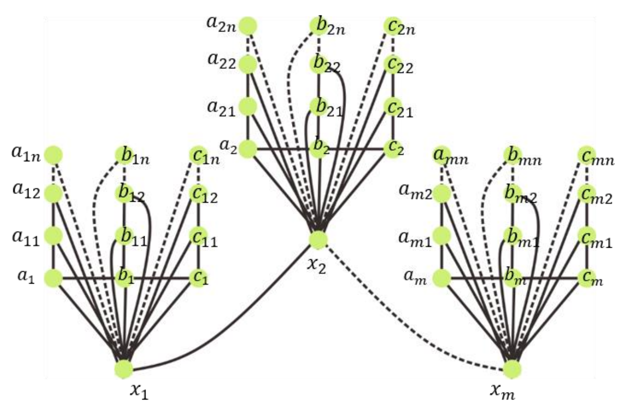

Figure 1. The illustration of ($P_m \odot E_{3,n}$)

For an example of corona product between P_m and $E_{3,n}$ provide in Figure 1. Based on Definition 3. ($P_m \odot E_{3,n}$) is a graph obtained by taking a duplicate of graph P_m and $|V(P_m)|$ a duplikat graph $E_{3,n}$, then connects each vertex to i in $E_{3,n}$.

2. RESULT

In this paper, we discuss some new results of the chromatic number local irregular of corona product graph

Theorem Let $G = P_m \odot E_{3,n}$ be a path graph order m coro product E graph order n for $n;\ m \geq 2$, the chromatic number local irregular is

$$\chi_{lis}(P_m \odot E_{3,n}) = \begin{cases} 5, & for\ m = 3\ and\ n = 2 \\ 6, & for\ m = 2\ and\ n = 2 \\ & for\ m = 3\ and\ n \geq 3 \\ 7, & for\ m = 2\ and\ n \geq 3 \\ & for\ m \geq 4\ and\ n = 2 \\ 8, & for\ m \geq 4\ and\ n \geq 3 \end{cases}$$

with $opt(l)$ is

$$opt(l)(P_m \odot E_{3,n}) = 1{,}2\ for\ m \geq 2\ and\ n \geq 2$$

Proof. Vertex set is $V(P_m \odot E_{3,n}) = \{x_i; 1 \leq i \leq m\} \cup \{a_i; 1 \leq i \leq m\} \cup \{b_i; 1 \leq i \leq m\} \cup \{c_i; 1 \leq i \leq m\} \cup \{a_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{b_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{c_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\}$ and the edge set is $E(P_m \odot E_{3,n}) = \{x_i x_{i+1}; 1 \leq i \leq m-1\} \cup \{a_i b_i; 1 \leq i \leq m\} \cup \{b_i c_i; 1 \leq i \leq m\} \cup \{a_i a_{i1}; 1 \leq i \leq m\} \cup \{b_i b_{i1}; 1 \leq i \leq m\} \cup \{c_i c_{i1}; 1 \leq i \leq m\} \cup \{x_i b_i; 1 \leq i \leq m\} \cup \{x_i b_i; 1 \leq i \leq m\} \cup \{x_i c_i; 1 \leq i \leq m\} \cup \{x_i a_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{x_i b_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{x_i c_{ij}; 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{a_{ij} a_{ij+1}; 1 \leq i \leq m; 1 \leq j \leq n-1\} \cup \{b_{ij} b_{ij+1}; 1 \leq i \leq m; 1 \leq j \leq n-1\} \cup \{c_{ij} c_{ij+1}; 1 \leq i \leq m; 1 \leq j \leq n-1\}$. The order and size respectively are $3mn + 4m$ and $6mn + 6m - 1$. This proof can be divided into 16 cases in the following.

Case 1 for $m = 3$ and $n = 2$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\chi_{lis}(P_m \odot E_{3,n}) \geq$ - $\chi\ (P_m \odot E_{3,n}).= 3$. Assume $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{4}$, let $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{4}$, **if** $l(x_i) = l(a_i) = l(c_i) = 1;\ l(b_i) = 2;\ l(a_{i1}) = l(b_{ij}) = l(c_{i1}) = 1;\ l(a_{i2}) = l(c_{i2}) = 2$ then $w(b_i) = w(c_i)$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If $l(x_i) = l(a_i) = l(c_i) = 1;\ l(b_i) = 1;\ l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq 3; j = 1; l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq 3; j = 2;\ l(c_{i1}) = 1;\ l(b_{ij}) = 1$ then $w(b_i) \neq w(c_i); w(a_i) \neq w(b_i);\ w(x_1) \neq w(x_2)$ then $\chi_{lis}(P_m \odot E_{3,n}) \geq 5$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 5$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1{,}2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j = 1 \\ 2, & for\ 1 \leq i \leq 3; and\ j = 2 \end{cases}$$

$$l(b_{ij}) = 1$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j = 1 \\ 2, & for\ 1 \leq i \leq 3; and\ j = 2 \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & for\ i = 1{,}3 \\ 13, & for\ i = 2 \end{cases}$$

$$w(a_i) = 3$$

$$w(b_i) = 4$$

$$w(c_i) = 3$$

$$w(a_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = 2 \\ 4, & for\ 1 \leq i \leq 3; and\ j = 1 \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = 2 \\ 3, & for\ 1 \leq i \leq 3; and\ j = 1 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = 2 \\ 4, & for\ 1 \leq i \leq 3; and\ j = 1 \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \leq i \leq 3$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \leq i \leq 3; j = 1{,}2$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \leq i \leq 3; j = 1{,}2$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq 3; j = 1{,}2$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \leq i \leq 3; j = 1{,}2$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$;

$1 \leq i \leq 3; j = 1,2;$ $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \leq i \leq 3; j = 1,2;$ $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \leq i \leq 3; j = 1,2;$ $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq 3; j = 1,2;$ $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \leq i \leq 3; j = 1,2;$ $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq 3; j = 1,2;$ $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \leq i \leq 3; j = 1,2;$ $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \leq i \leq 3; j = 1,2;$ $w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 5$. We have $5 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 5$, so $\chi_{lis}(P_m \odot E_{3,n}) = 5$.

Case 2 for $m = 2$ and $n = 2$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n})$. $= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{5}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{5}$, **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{i1}) = 2$; $l(b_{i2}) = 1$; $l(c_{i1}) = 1$- ; $l(c_{i2}) = 2$ then $w(b_i) = w(c_i)$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If $l(x_1) = 1; l(x_2) = 2$; $l(a_i) = l(c_i) = 1$; $l(b_i) = 1$; $l(b_{ij}) = 1$; $l(a_{ij}) = l(c_{ij}) = 1; i = 1,2; j = 1; l(a_{ij}) = l(c_{ij}) = 2; i = 1,2;$ $j = 2$, then $w(b_i) \neq w(b_{i1}); w(a_i) \neq w(a_{i1})$; $w(x_1) \neq w(x_2)$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq 6$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 6$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i = 1 \\ 2, & for\ i = 2 \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ i = 1,2; and\ j = 1 \\ 2, & for\ i = 1,2; and\ j = 2 \end{cases}$$

$$l(b_{ij}) = 1$$

$$l(c_{ij}) = \begin{cases} 1, & for\ i = 1,2; and\ j = 1 \\ 2, & for\ i = 1,2; and\ j = 2 \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & for\ i = 2 \\ 13, & for\ i = 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i = 1 \\ 4, & for\ i = 2 \end{cases}$$

$$w(b_i) = \begin{cases} 4, & for\ i = 1 \\ 5, & for\ i = 2 \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i = 1 \\ 4, & for\ i = 2 \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i = 1\ and\ j = 2 \\ 3, & for\ i = 2\ and\ j = 2 \\ 4, & for\ i = 1\ and\ j = 1 \\ 5, & for\ i = 2\ and\ j = 1 \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i = 1\ and\ j = 2 \\ 3, & for\ i = 2\ and\ j = 1 \\ & for\ i = 2\ and\ j = 2 \\ 5, & for\ i = 2\ and\ j = 1 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i = 1\ and\ j = 2 \\ 3, & for\ i = 2\ and\ j = 2 \\ 4, & for\ i = 1\ and\ j = 1 \\ 5, & for\ i = 2\ and\ j = 1 \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}; i = 1,2;$ $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $i = 1,2; j = 1,2;$ $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i; i = 1,2; j = 1,2;$ $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq 3; j = 1,2;$ $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $i = 1,2; j = 1,2;$ $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $i = 1,2; j = 1,2;$ $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $i = 1,2; j = 1,2;$ $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $i = 1,2; j = 1,2;$ $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq 3; j = 1,2;$ $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $i = 1,2; j = 1,2;$ $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $i = 1,2; j = 1,2;$ $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $i = 1,2; j = 1,2;$ $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \leq i \leq 3; j = 1,2;$ $w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 6$. We have $6 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 6$.

Case 3 for $m = 3$ and $n \equiv 1,3(mod4)$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$**.** Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq$ - $\chi\ (P_m \odot E_{3,n}).= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{5}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{5}$, **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{ij}) = l(c_{ij}) = 1$, then $w(c_i) = w(c_{i1})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = l(b_{ij}) = 1$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq 3; j \equiv 1(mod4);\ j \equiv 0(mod2);\ l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq 3; j \equiv 3(mod4);\ l(b_{ij}) = 1; 1 \leq i \leq 3 j \equiv 1(mod4);\ j \equiv 0(mod2);\ j \neq 1;\ l(b_{ij}) = 2; 1 \leq i \leq 3;\ j \equiv 3(mod4);\ j = 1$, , then $w(b_i) \neq w(b_{i1}); w(a_i) \neq w(a_{i1});\ w(x_1) \neq w(x_2)$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq 6$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 6$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1$$

$$l(a_i) = 1$$

$$l(b_i) = 2$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j \equiv 1(mod4)\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod2) \\ 2 & for\ 1 \leq i \leq 3; and\ j \equiv 3(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod2)\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 1(mod4); j \neq 1 \\ 2, & for\ 1 \leq i \leq 3; and\ j = 1\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j \equiv 1(mod4)\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod2) \\ 2 & for\ 1 \leq i \leq 3; and\ j \equiv 3(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1,3; and\ n \equiv 3(mod4) \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 2; and\ n \equiv 3(mod4) \\ 9 + 15 \left\lceil \frac{n}{5} \right\rceil, & for\ i = 1,3; and\ n \equiv 1(mod4) \\ 10 + 15 \left\lceil \frac{n}{5} \right\rceil, & for\ i = 2; and\ n \equiv 1(mod4) \end{cases}$$

$$w(a_i) = 4$$

$$w(b_i) = 5$$

$$w(c_i) = 4$$

$$w(a_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = n \\ 3, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod4); j \neq n \\ 4, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod2) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = n \\ 3, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod4); j \neq n \\ 4, & for\ 1 \leq i \leq 3; and\ j = 1\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod2); j \neq 2 \\ 5, & for\ 1 \leq i \leq 3; and\ j = 2 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = n \\ 3, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod4); j \neq n \\ 4, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod2) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \leq i \leq 3$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}; 1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 6$. We have $6 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 6$.

Case 4 for $m = 3$ and $n \equiv 0(mod4)$

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First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\chi_{lis}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n}). = 3$. Assume $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{5}$, let $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{5}$, **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{ij}) = l(c_{ij}) = 1$, then $w(c_i) = w(c_{i1})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i)$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq 3; j \equiv 1,3(mod 4)$; $j \equiv 0(mod 4)$; $l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq 3; j \equiv 2(mod 4)$; $l(b_{ij}) = 1; 1 \leq i \leq 3$; $j \equiv 3(mod 4)$; $j \equiv 0(mod 2)$; $l(b_{ij}) = 2; 1 \leq i \leq 3$; $j \equiv 1(mod 4)$; then $w(b_{ij}) \neq w(b_{ij+1})$; $w(a_{ij}) \neq w(a_{ij+1})$; $w(x_1) \neq w(x_2)$ then $\chi_{lis}(P_m \odot E_{3,n}) \geq 6$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 6$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1$$

$$l(a_i) = 1$$

$$l(b_i) = 2$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod 4)\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod 4) \\ 2 & for\ 1 \leq i \leq 3; and\ j \equiv 2(mod 4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod 2)\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 3(mod 4) \\ 2 & for\ 1 \leq i \leq 3; and\ j = 1(mod 4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod 4)\ or \\ & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod 4) \\ 2 & for\ 1 \leq i \leq 3; and\ j \equiv 2(mod 4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1,3 \\ 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 2 \end{cases}$$

$$w(a_i) = 3$$

$$w(b_i) = 5$$

$$w(c_i) = 3$$

$$w(a_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = n \\ 3, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod 2); j \neq n \\ 4, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod 4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = n \\ 3, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod 4) \\ 4, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod 2); j \neq n \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ 1 \leq i \leq 3; and\ j = n \\ 3, & for\ 1 \leq i \leq 3; and\ j \equiv 0(mod 2); j \neq n \\ 4, & for\ 1 \leq i \leq 3; and\ j \equiv 1,3(mod 4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \leq i \leq 3$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}; 1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \leq i \leq 3; 1 \leq j \leq n$; $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}; 1 \leq i \leq 3; 1 \leq j \leq n$; $w(c_{ij}) \neq w(c_{ij+1})$. The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 6$. We have $6 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 6$.

Case 5 for $m = 3$ and $n \equiv 2(mod 4)$
First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\chi_{lis}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n}). = 3$. Assume $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{5}$, let $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{5}$, **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{ij}) = l(c_{ij}) = 1$, then $w(c_i) = w(c_{i1})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i)$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq 3; j \equiv 1,3(mod 4)$; $j \equiv 0(mod 4)$; $l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq$

$3; j \equiv 2(mod4);\ l(b_{ij}) = 1; 1 \le i \le 3;\ j \equiv 1(mod4); j \ne 1;\ \ j \equiv 0(mod2);\ \ l(b_{ij}) = 2; 1 \le i \le 3;\ j \equiv 3(mod4); j = 1$ then $w(b_{ij}) \ne w(b_{ij+1}); w(a_{ij}) \ne w(a_{ij+1});\ w(x_1) \ne w(x_2)$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge 6$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \ge 6$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1$$

$$l(a_i) = 1$$

$$l(b_i) = 2$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \le i \le 3; and\ j \equiv 1,3(mod4)\ or \\ & for\ 1 \le i \le 3; and\ j \equiv 0(mod4) \\ 2 & for\ 1 \le i \le 3; and\ j \equiv 2(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \le i \le 3; and\ j \equiv 0(mod2)\ or \\ & for\ 1 \le i \le 3; and\ j \equiv 1(mod4)\ ; j \ne 1 \\ 2, & for\ 1 \le i \le 3; and\ j = 1\ atau \\ & for\ 1 \le i \le 3; and\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \le i \le 3; and\ j \equiv 1,3(mod4)\ or \\ & for\ 1 \le i \le 3; and\ j \equiv 0(mod4) \\ 2 & for\ 1 \le i \le 3; and\ j \equiv 2(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1,3 \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 2 \end{cases}$$

$$w(a_i) = 3$$

$$w(b_i) = 5$$

$$w(c_i) = 3$$

$$w(a_{ij}) = \begin{cases} 2, & for\ 1 \le i \le 3; and\ j = n \\ 3, & for\ 1 \le i \le 3; and\ j \equiv 0(mod2); j \ne n \\ 4, & for\ 1 \le i \le 3; andj \equiv 1,3(mod4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ 1 \le i \le 3; and\ j = n \\ 3, & for\ 1 \le i \le 3; and\ j \equiv 1,3(mod4) \\ 4, & for\ 1 \le i \le 3; and\ j \equiv 0(mod2); j \ne 2, n \\ 5, & for\ 1 \le i \le 3; and\ j = 2 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ 1 \le i \le 3; and\ j = n \\ 3, & for\ 1 \le i \le 3; and\ j \equiv 0(mod2); j \ne n \\ 4, & for\ 1 \le i \le 3; andj \equiv 1,3(mod4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \le i \le 3$; $w(x_i) \ne w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \le i \le 3; 1 \le j \le n$; $w(x_i) \ne w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \le i \le 3; 1 \le j \le n$; $w(x_i) \ne w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \le i \le 3; 1 \le j \le n$; $w(x_i) \ne w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \le i \le 3; 1 \le j \le n$; $w(x_i) \ne w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \le i \le 3; 1 \le j \le n$; $w(x_i) \ne w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \le i \le 3; 1 \le j \le n$; $w(x_i) \ne w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \le i \le 3; 1 \le j \le n$; $w(a_i) \ne w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \le i \le 3; 1 \le j \le n$; $w(b_i) \ne w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \le i \le 3; 1 \le j \le n$; $w(c_i) \ne w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \le i \le 3; 1 \le j \le n$; $w(a_{ij}) \ne w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \le i \le 3; 1 \le j \le n$; $w(b_{ij}) \ne w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \le i \le 3; 1 \le j \le n$; $w(c_{ij}) \ne w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \le 6$. We have $6 \le \chi_{lis}(P_m \odot E_{3,n}) \le 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 6$.

Case 6 for $m \equiv 0(mod2); m \ge 4$ and $n = 2$
First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge$ - $\boldsymbol{\chi}\ (P_m \odot E_{3,n}) .= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = 1;\ l(a_{ij}) = 1;\ l(b_{i1}) = 2; l(b_{i2}) = 1;\ l(c_{i1}) = 1;\ l(c_{i2}) = 2$then $w(x_{i+1}) = w(x_{i+2})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 1,3(mod4); i \equiv 2(mod4); l(x_i) = 2; i \equiv 0(mod4);\ l(a_i) = l(c_i) = l(b_i) = 1;\ l(a_{ij}) = l(c_{ij}) = 1; 1 \le i \le m; j \equiv 1;\ l(a_{ij}) = l(c_{ij}) = 2; 1 \le i \le m; j = 2;\ l(b_{ij}) = 1$ then $w(b_{ij}) \ne w(b_{ij+1}); w(a_{ij}) \ne w(a_{ij+1});\ w(x_{i+1}) \ne w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge 7$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \ge 7$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local

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irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1,3(mod4)\ or \\ & i \equiv 2(mod4) \\ 2 & i \equiv 0(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m\ and\ j = 1 \\ 2, & for\ 1 \le i \le m\ and\ j = 2 \end{cases}$$

$$l(b_{ij}) = 1$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m\ and\ j = 1 \\ 2, & for\ 1 \le i \le m\ and\ j = 2 \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & for\ i = 1, m \\ 13, & for\ i \equiv 0(mod2); i \neq m \\ 14, & for\ i \equiv 1,3(mod4); i \neq 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 4, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 4, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 5, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 4, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 2 \\ 3, & for\ i \equiv 0(mod4)\ and\ j = 2 \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j = 1\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 1 \\ 5, & for\ i \equiv 0(mod4)\ and\ j = 1 \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 2 \\ 3, & for\ i \equiv 0(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 1,3(mod4)\ and\ j = 1\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 1 \\ 4, & for\ i \equiv 0(mod4)\ and\ j = 1 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 2 \\ 3, & for\ i \equiv 0(mod4)\ and\ j = 2 \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j = 1\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 1 \\ 5, & for\ i \equiv 0(mod4)\ and\ j = 1 \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \le i \le m$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \le i \le m; j = 1,2$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \le i \le m; j = 1,2$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \le i \le m; j = 1,2$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \le i \le m; j = 1,2$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \le i \le m; j = 1,2$; $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \le i \le m; j = 1,2$; $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \le i \le m; j = 1,2$; $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \le i \le m; j = 1,2$; $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \le i \le m; j = 1,2$; $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \le i \le m; j = 1,2$; $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \le i \le m; j = 1,2$; $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \le i \le m; j = 1,2; w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \le 7$. We have $6 \le \chi_{lis}(P_m \odot E_{3,n}) \le 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 7$.

Case 7 for $m \equiv 1,3(mod4); m \ge 5$ and $n = 2$
First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge$ - $\chi\ (P_m \odot E_{3,n}).= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, **if** $l(x_i) = l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{i1}) = 2; l(b_{i2}) = 1$; $l(c_{i1}) = 1$; $l(c_{i2}) = 2$then $w(x_{i+1}) = w(x_{i+2})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 1(mod4); i \equiv 0(mod2); l(x_i) = 2; i \equiv 3(mod4)$; $l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \le i \le m; j \equiv 1$; $l(a_{ij}) = l(c_{ij}) = 2; 1 \le i \le m; j = 2$; $l(b_{ij}) = 1$ then $w(b_{ij}) \neq$

$w(b_{ij+1}); w(a_{ij}) \neq w(a_{ij+1}); w(x_{i+1}) \neq w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq 7$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 7$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 2 & for\ i \equiv 3(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq m\ and\ j = 1 \\ 2, & for\ 1 \leq i \leq m\ and\ j = 2 \end{cases}$$

$$l(b_{ij}) = 1$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq m\ and\ j = 1 \\ 2, & for\ 1 \leq i \leq m\ and\ j = 2 \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & for\ i = 1, m \\ 13, & for\ i \equiv 1,3(mod4); i \neq m \\ 14, & for\ i \equiv 0(mod2); i \neq 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i \equiv 1(mod4)\ \ or \\ & for\ i \equiv 0(mod2) \\ 4, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 4, & for\ i \equiv 1(mod4)\ \ or \\ & for\ i \equiv 0(mod2) \\ 5, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i \equiv 1(mod4)\ \ or \\ & for\ i \equiv 0(mod2) \\ 4, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 0(mod2)\ and\ j = 2 \\ 3, & for\ i \equiv 3(mod4)\ and\ j = 2 \\ 4, & for\ i \equiv 1(mod4)\ and\ j = 1\ \ or \\ & for\ i \equiv 0(mod2)\ and\ j = 1 \\ 5, & for\ \ i \equiv 3(mod4)\ and\ j = 1 \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = 2\ \ or \\ & for\ i \equiv 0(mod2)\ and\ j = 2 \\ 3, & for\ i \equiv 3(mod4)\ and\ j = 2\ \ or \\ & for\ i \equiv 1(mod4)\ and\ j = 1\ \ or \\ & for\ i \equiv 0(mod2)\ and\ j = 1 \\ 4, & for\ i \equiv 3(mod4)\ and\ j = 1 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 0(mod2)\ and\ j = 2 \\ 3, & for\ i \equiv 3(mod4)\ and\ j = 2 \\ 4, & for\ i \equiv 1(mod4)\ and\ j = 1\ \ or \\ & for\ i \equiv 0(mod2)\ and\ j = 1 \\ 5, & for\ \ i \equiv 3(mod4)\ and\ j = 1 \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \leq i \leq m$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \leq i \leq m; j = 1,2;$ $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \leq i \leq m; j = 1,2;$ $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq m; j = 1,2;$ $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \leq i \leq m; j = 1,2;$ $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \leq i \leq m; j = 1,2;$ $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \leq i \leq m; j = 1,2;$ $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \leq i \leq m; j = 1,2;$ $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq m; j = 1,2;$ $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \leq i \leq m; j = 1,2;$ $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq m; j = 1,2;$ $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \leq i \leq m; j = 1,2;$ $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \leq i \leq m; j = 1,2; w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 7$. We have $6 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 7$.

Case 8 for $m = 2$ and $n \equiv 1,3 mod4$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$**.** Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n}).= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, **if** $l(x_i) = l(a_i) = l(b_i) = 1; l(a_{ij}) = 1; l(b_{ij}) = 1; i = 1,2; 2 \leq j \leq n;$ $l(b_{i1}) = 2; l(c_i) = 2;$ $l(c_{i1}) = 1;$ $l(c_{i2}) = 2;$ $l(c_{ij}) = 1; i = 1,2; 2 \leq j \leq n$ then $w(x_1) = w(x_2)$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_2)=2;\ l(x_1)=1; l(a_i)=\ l(c_i)=l(b_i)=1;\ l(a_{ij})=l(c_{ij})=1; i=1,2;\ j\equiv 1(mod4); j\equiv 0(mod2);\ l(a_{ij})=l(c_{ij})=2; i=1,2;\ j\equiv 1(mod4);\ l(b_{ij})=1;\ i=1,2;\ j\equiv 1(mod4);\ j\neq 1;\ j\equiv 0(mod2);\ l(b_{ij})=2;\ i=1,2;\ j\equiv 3(mod4);\ j=1$ then $w(b_{ij})\neq w(b_{ij+1}); w(a_{ij})\neq w(a_{ij+1});\ w(x_{i+1})\neq w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m\odot E_{3,n})\geq 7$. Based on that we have the lower bound of $\chi_{lis}(P_m\odot E_{3,n})\geq 7$.

After that, we will find the upper bound of $(P_m\odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m\odot E_{3,n})$, we have define $l: V\left(P_m\odot E_{3,n}\right)\to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i)=\begin{cases}1, & untuk\ i=1\\ 2, & untuk\ i=2\end{cases}$$

$$l(a_i)=1$$

$$l(b_i)=2$$

$$l(c_i)=1$$

$$l(a_{ij})=\begin{cases}1, & for\ i=1,2\ and\ j\equiv 1(mod4)\ or\\ & for\ i=1,2\ and\ j\equiv 0(mod2)\\ 2, & for\ i\equiv 3(mod4)\end{cases}$$

$$l(b_{ij})=\begin{cases}1, & for\ i=1,2; and\ j\equiv 0(mod2)\ or\\ & for\ i=1,2; and\ j\equiv 1(mod4); j\neq 1\\ 2, & for\ i=1,2; and\ j=1\ atau\\ & for\ i=1,2; and\ j\equiv 3(mod4)\end{cases}$$

$$l(c_{ij})=\begin{cases}1, & for\ i=1,2\ and\ j\equiv 1(mod4)\ or\\ & for\ i=1,2\ and\ j\equiv 0(mod2)\\ 2, & for\ i\equiv 3(mod4)\end{cases}$$

Hence $opt(l)=2$ and the labelling provides vertex-weight as follows:

$$w(x_i)=\begin{cases}3n+7+\left\lceil\frac{2n}{3}\right\rceil, & for\ i=2; and\ n\equiv 3(mod4)\\ 3n+7+\left\lceil\frac{2n}{3}\right\rceil, & for\ i=1; and\ n\equiv 3(mod4)\\ 9+15\left\lceil\frac{n}{5}\right\rceil, & for\ i=2; and\ n\equiv 1(mod4)\\ 10+15\left\lceil\frac{n}{5}\right\rceil, & for\ i=1; and\ n\equiv 1(mod4)\end{cases}$$

$$w(a_i)=\begin{cases}4, & for\ i=1\\ 5, & for\ i=2\end{cases}$$

$$w(b_i)=\begin{cases}5, & for\ i=1\\ 6, & for\ i=2\end{cases}$$

$$w(c_i)=\begin{cases}4, & for\ i=1\\ 5, & for\ i=2\end{cases}$$

$$w(a_{ij})=\begin{cases}2, & for\ i=1,2\ and\ j=2\\ 3, & for\ i=1\ and\ j\equiv 1,3(mod4); j\neq n\ or\\ & for\ i=2\ and\ j=n\\ 4, & for\ i=1\ and\ j\equiv 0(mod2)\ or\\ & for\ i=2\ and\ j\equiv 1,3(mod4);\ j\neq n\\ 5, & for\ i=2\ and\ j=\equiv 0(mod2)\end{cases}$$

$$w(b_{ij})=\begin{cases}2, & for\ i=1\ and\ j=n\\ 3, & for\ i=1\ and\ j\equiv 1,3\ mod\ 4; j\neq 1,n\ or\\ & for\ i=2\ and\ j=n\\ 4, & for\ i=1\ and\ j=1\ or\\ & for\ i=1\ and\ j\equiv 0(mod2); j\neq 2; or\\ & for\ i=2\ and\ j\equiv 1,3(mod4); j\neq 1\\ 5, & for\ i=1\ and\ j=2; or\\ & for\ i=2\ and\ j=1\ or\\ & for\ i=2\ and\ j\equiv 0(mod2); j\neq 2\\ 6, & for\ i=2\ and\ j=2\end{cases}$$

$$w(c_{ij})=\begin{cases}2, & for\ i=1,2\ and\ j=2\\ 3, & for\ i=1\ and\ j\equiv 1,3(mod4); j\neq n\ or\\ & for\ i=2\ and\ j=n\\ 4, & for\ i=1\ and\ j\equiv 0(mod2)\ or\\ & for\ i=2\ and\ j\equiv 1,3(mod4);\ j\neq n\\ 5, & for\ i=2\ and\ j=\equiv 0(mod2)\end{cases}$$

For every $uv\in V\left(P_m\odot E_{3,n}\right)$, take any $u=x_i$ and $v=x_{i+1}$; $i=1,2;\ w(x_i)\neq w(x_{i+1})$, then $u=x_i$ and $v=a_i;\ ; i=1,2; 1\leq j\leq n;\ w(x_i)\neq w(a_i)$, then $u=x_i$ and $v=b_i;\ i=1,2; 1\leq j\leq n;\ w(x_i)\neq w(b_i)$, then $u=x_i$ and $v=c_i;\ i=1,2; 1\leq j\leq n; w(x_i)\neq w(c_i)$, then $u=x_i$ and $v=a_{ij};\ i=1,2; 1\leq j\leq n;\ w(x_i)\neq w\left(a_{ij}\right)$, then $u=x_i$ and $v=b_{ij}$; $i=1,2; 1\leq j\leq n;\ w(x_i)\neq w\left(b_{ij}\right)$, then $u=x_i$ and $v=c_{ij};\ \ i=1,2; 1\leq j\leq n;\ w(x_i)\neq w\left(c_{ij}\right)$, then $u=a_i$ and $v=a_{i1};\ \ i=1,2; 1\leq j\leq n; w(a_i)\neq w(a_{i1})$, then $u=b_i$ and $v=b_{i1};\ 1\leq i\leq m; j=1,2;\ w(b_i)\neq w(b_{i1})$, then $u=c_i$ and $v=c_{i1};\ \ i=1,2; 1\leq j\leq n;\ w(c_i)\neq w(c_{i1})$, then $u=a_{ij}$ and $v=a_{ij+1};\ \ 1\leq i\leq m; j=1,2;\ w\left(a_{ij}\right)\neq w\left(a_{ij+1}\right)$, then $u=b_{ij}$ and $v=b_{ij+1};\ \ i=1,2; 1\leq j\leq n;$ $w\left(b_{ij}\right)\neq w\left(b_{ij+1}\right)$, then $u=c_{ij}$ and $v=c_{ij+1};\ i=1,2; 1\leq j\leq n; w\left(c_{ij}\right)\neq w\left(c_{ij+1}\right)$.

The upper bound $\chi_{lis}(P_m\odot E_{3,n})\leq 7$. We have $6\leq\chi_{lis}(P_m\odot E_{3,n})\leq 6$, so $\chi_{lis}\left(P_m\odot E_{3,n}\right)=7$.

Case 9 for $m = 2$ and $n \equiv 0 mod 4$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\chi_{lis}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n}) = 3$. Assume $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{6}$, let $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{6}$, **if** $l(x_i) = l(a_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{ij}) = 1$; $i = 1,2$; $2 \leq j \leq n$; $l(b_{i1}) = 2$; $l(c_i) = 2$; $l(c_{i1}) = 1$; $l(c_{i2}) = 2$; $l(c_{ij}) = 1$; $i = 1,2$; $2 \leq j \leq n$ then $w(x_1) = w(x_2)$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_2) = 2$; $l(x_1) = 1$; $l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = l(c_{ij}) = 1$; $i = 1,2$; $j \equiv 1,3(mod 4)$; $j \equiv 0(mod 4)$; $l(a_{ij}) = l(c_{ij}) = 2$; $i = 1,2$; $j \equiv 2(mod 4)$; $l(b_{ij}) = 1$; $i = 1,2$; $j \equiv 3(mod 4)$; $j \neq 1$; $j \equiv 0(mod 2)$; $l(b_{ij}) = 2$; $i = 1,2$; $j \equiv 1(mod 4)$; $j = 1$ then $w(b_{ij}) \neq w(b_{ij+1})$; $w(a_{ij}) \neq w(a_{ij+1})$; $w(x_{i+1}) \neq w(x_{i+2})$ then $\chi_{lis}(P_m \odot E_{3,n}) \geq 7$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 7$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \rightarrow 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i = 1 \\ 2, & for\ i = 2 \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ i = 1,2\ and\ j \equiv 1,3(mod 4)\ or \\ & for\ i = 1,2\ and\ j \equiv 0(mod 4) \\ 2, & for\ i \equiv 2(mod 4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ i = 1,2; for\ j \equiv 3(mod 4)\ or \\ & for\ i = 1,2; for\ j \equiv 0(mod 2) \\ 2, & for\ i = 1,2; for\ j = 1\ or \\ & for\ i = 1,2; for\ j \equiv 1(mod 4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ i = 1,2\ and\ j \equiv 1,3(mod 4)\ or \\ & for\ i = 1,2\ and\ j \equiv 0(mod 4) \\ 2, & for\ i \equiv 2(mod 4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 2 \\ 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i = 1 \\ 4, & for\ i = 2 \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i = 1 \\ 6, & for\ i = 2 \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i = 1 \\ 4, & for\ i = 2 \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ \ i = 1\ and\ j = n \\ 3, & for\ \ i = 2\ and\ j = n\ or \\ & for\ \ i = 1\ and\ j \equiv 0(mod 2); j \neq n \\ 4, & for\ \ i = 1\ and\ j \equiv 1,3(mod 4)\ or \\ & for\ \ i = 2\ and\ j \equiv 0(mod 2);\ j \neq n \\ 5, & for\ \ i = 2\ and\ j \equiv 1,3(mod 4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ \ i = 1\ and\ j = n \\ 3, & for\ \ i = 2\ and\ j = n\ or \\ & for\ \ i = 1\ and\ j \equiv 1,3(mod 4) \\ 4, & for\ \ i = 1\ and\ j \equiv 1,3(mod 4)\ or \\ & for\ \ i = 2\ and\ j \equiv 0(mod 2);\ j \neq n \\ 5, & for\ \ i = 2\ and\ j \equiv 0(mod 2);\ j \neq n \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ \ i = 1\ and\ j = n \\ 3, & for\ \ i = 2\ and\ j = n\ or \\ & for\ \ i = 1\ and\ j \equiv 0(mod 2); j \neq n \\ 4, & for\ \ i = 1\ and\ j \equiv 1,3(mod 4)\ or \\ & for\ \ i = 2\ and\ j \equiv 0(mod 2);\ j \neq n \\ 5, & for\ \ i = 2\ and\ j \equiv 1,3(mod 4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $i = 1,2$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; ; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $i = 1,2; 1 \leq j \leq n; w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq m; j = 1,2$; $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $i = 1,2; 1 \leq j \leq n$; $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq m; j = 1,2$; $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $i = 1,2; 1 \leq j \leq n$;

$w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $i = 1,2; 1 \leq j \leq n; w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 7$. We have $6 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 7$.

Case 10 for $m = 2$ and $n \equiv 2mod4$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$**.** Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n})$.$= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{6}$**, if** $l(x_i) = l(a_i) = l(b_i) = 1$; $l(a_{ij}) = 1$; $l(b_{ij}) = 1$; $i = 1,2$; $2 \leq j \leq n$; $l(b_{i1}) = 2$; $l(c_i) = 2$; $l(c_{i1}) = 1$; $l(c_{i2}) = 2$; $l(c_{ij}) = 1$; $i = 1,2$; $2 \leq j \leq n$ then $w(x_1) = w(x_2)$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_2) = 2$; $l(x_1) = 1$; $l(a_i) = l(c_i) = l(b_i) = 1$; $l(a_{ij}) = l(c_{ij}) = 1$; $i = 1,2$; $j \equiv 1,3(mod4)$; $j \equiv 0(mod4)$; $l(a_{ij}) = l(c_{ij}) = 2$; $i = 1,2$; $j \equiv 2(mod4)$; $l(b_{ij}) = 1$; $i = 1,2$; $j \equiv 1(mod4)$; $j \neq 1$; $j \equiv 0(mod2)$; $l(b_{ij}) = 2$; $i = 1,2$; $j \equiv 3(mod4)$; $j = 1$ then $w(b_{ij}) \neq w(b_{ij+1})$; $w(a_{ij}) \neq w(a_{ij+1})$; $w(x_{i+1}) \neq w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq 7$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 7$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \rightarrow 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i = 1 \\ 2, & for\ i = 2 \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ i = 1,2\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i = 1,2\ and\ j \equiv 0(mod4) \\ 2, & for\ i = 1,2\ and\ j \equiv 2(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ i = 1,2; for\ j \equiv 1(mod4); j \neq 1\ or \\ & for\ i = 1,2; for\ j \equiv 0(mod2) \\ 2, & for\ i = 1,2; for\ j = 1\ or \\ & for\ i = 1,2; for\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ i = 1,2\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i = 1,2\ and\ j \equiv 0(mod4) \\ 2, & for\ i = 1,2\ and\ j \equiv 2(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 2 \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i = 1 \\ 4, & for\ i = 2 \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i = 1 \\ 6, & for\ i = 2 \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i = 1 \\ 4, & for\ i = 2 \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ \ i = 1\ and\ j = n \\ 3, & for\ \ i = 2\ and\ j = n\ or \\ & for\ \ i = 1\ and\ j \equiv 0(mod2); j \neq n \\ 4, & for\ \ i = 1\ and\ j \equiv 1,3(mod4)\ or \\ & for\ \ i = 2\ and\ j \equiv 0(mod2);\ j \neq n \\ 5, & for\ \ i = 2\ and\ j \equiv 1,3(mod4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i = 1\ and\ j = n \\ 3, & for\ i = 2\ and\ j = n\ or \\ & for\ i = 1\ and\ j \equiv 1,3(mod4) \\ 4, & for\ i = 1\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i = 2\ and\ j \equiv 0(mod2);\ j \neq 2, n \\ 5, & for\ i = 2\ and\ j \equiv 0(mod2);\ j \neq 2, n\ o \\ & for\ i = 1\ and\ j = 2 \\ 6, & for\ i = 2\ and\ j = 2 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ \ i = 1\ and\ j = n \\ 3, & for\ \ i = 2\ and\ j = n\ or \\ & for\ \ i = 1\ and\ j \equiv 0(mod2); j \neq n \\ 4, & for\ \ i = 1\ and\ j \equiv 1,3(mod4)\ or \\ & for\ \ i = 2\ and\ j \equiv 0(mod2);\ j \neq n \\ 5, & for\ \ i = 2\ and\ j \equiv 1,3(mod4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $i = 1,2$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; ;$i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $i = 1,2; 1 \leq j \leq n$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$;

$i = 1,2; 1 \le j \le n$; $w(x_i) \ne w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $i = 1,2; 1 \le j \le n$; $w(x_i) \ne w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $i = 1,2; 1 \le j \le n; w(a_i) \ne w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \le i \le m; j = 1,2$; $w(b_i) \ne w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $i = 1,2; 1 \le j \le n$; $w(c_i) \ne w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \le i \le m; j = 1,2$; $w(a_{ij}) \ne w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $i = 1,2; 1 \le j \le n$; $w(b_{ij}) \ne w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $i = 1,2; 1 \le j \le n; w(c_{ij}) \ne w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \le 7$. We have $6 \le \chi_{lis}(P_m \odot E_{3,n}) \le 6$, so $\chi_{lis}(P_m \odot E_{3,n}) = 7$.

Case 11 for $m \equiv 0(mod2); m \ge 4$and $n \equiv 1,3(mod4)$
First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge$ - χ $(P_m \odot E_{3,n})$. $= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{7}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{7}$, **if** $l(x_i) = 1; i \equiv 2(mod4)$; i =≡ $1,3(mod4); l(x_i) = 2; i \equiv 0(mod4)$; $l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1$, then $w(a_i) = w(a_{ij})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 2(mod4)$; i =≡ $1,3(mod4); l(x_i) = 2; i \equiv 0(mod4); l(a_i) = l(c_i) = 1; l(b_i) = 2$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \le i \le m$; $j \equiv 1(mod4); j \equiv 0(mod2)$; $l(a_{ij}) = l(c_{ij}) = 2; 1 \le i \le m$; $j \equiv 3(mod4)$; $l(b_{ij}) = 1$; $1 \le i \le m$; $j \equiv 1(mod4)$; $j \ne 1$; $j \equiv 0(mod2)$; $l(b_{ij}) = 2$; $1 \le i \le m$; $j \equiv 3(mod4)$; $j = 1$ then $w(b_{ij}) \ne w(b_{ij+1}); w(a_{ij}) \ne w(a_{ij+1})$; $w(x_{i+1}) \ne w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge 8$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \ge 8$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 2(mod4) or \\ & for\ i \equiv 1,3(mod4) \\ 2, & for\ i \equiv 0(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 2$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ \ 1 \le i \le m\ and\ j \equiv 1(mod4)\ or \\ & for\ \ 1 \le i \le m\ and\ j \equiv 0(mod2) \\ 2, & for\ \ 1 \le i \le m\ and\ j \equiv 3(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m; and\ j \equiv 1(mod4); j \ne 1\ or \\ & for\ 1 \le i \le m; and\ j \equiv 0(mod2) \\ 2, & for\ 1 \le i \le m; and\ j = 1\ or \\ & for\ 1 \le i \le m; and\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ \ 1 \le i \le m\ and\ j \equiv 1(mod4)\ or \\ & for\ \ 1 \le i \le m\ and\ j \equiv 0(mod2) \\ 2, & for\ \ 1 \le i \le m\ and\ j \equiv 3(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) \begin{cases} 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1, m\ and\ n \equiv 3(mod4) \\ 3n + 8 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 0(mod2); i \ne m\ and\ n \equiv 3(mod4) \\ 3n + 8 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1,3(mod4)\ and\ n \equiv 3(mod4) \\ 9 + 15 \left\lceil \frac{n}{5} \right\rceil, & for\ i = 1, m\ and\ n \equiv 1(mod4) \\ 10 + 15 \left\lceil \frac{n}{5} \right\rceil, & for\ i \equiv 0(mod2); i \ne m\ and\ n \equiv 1(mod4) \\ 11 + 15 \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1,3(mod4)\ and\ n \equiv 1(mod4) \end{cases}$$

$$w(a_i) = \begin{cases} 4, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 5, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 6, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 4, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 5, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4); j \ne n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4); j \ne n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2)\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 1,3(mod4)\ j \ne n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2) \end{cases}$$

$$w(b_{ij})\begin{cases} 2, & for\ i\equiv 1,3(mod4)\ and\ j=n\ or \\ & for\ i\equiv 2(mod4)\ and\ j=n \\ 3, & for\ i\equiv 1,3(mod4)\ and\ j\equiv 1,3(mod4); j\neq 1,n\ or \\ & for\ i\equiv 2(mod4)\ and\ j\equiv 1,3(mod4); j\neq 1,n\ or \\ & for\ i\equiv 0(mod4)\ and\ j=n \\ 4, & for\ i\equiv 1,3(mod4)\ and\ j\equiv 0(mod2); j\neq 2\ or \\ & for\ i\equiv 2(mod4)\ and\ j\equiv 0(mod2); j\neq 2\ or \\ & for\ i\equiv 0(mod4)\ and\ j\equiv 1,3(mod4); j\neq 1,n \\ & for\ i\equiv 1,3(mod4)\ and\ j=1 \\ & for\ i\equiv 2(mod4)\ and\ j=1 \\ 5, & for\ i\equiv 1,3(mod4)\ and\ j=2\ or \\ & for\ i\equiv 2(mod4)\ and\ j=2\ or \\ & for\ i\equiv 0(mod4)\ and\ j=1 \\ & for\ i\equiv 0(mod4)\ and\ j\equiv 0(mod2); j\neq 2 \\ 6, & for\ i\equiv 0(mod4)\ and\ j=2 \end{cases}$$

$$w(c_{ij})=\begin{cases} 2, & for\ i\equiv 1,3(mod4)\ and\ j=n\ or \\ & for\ i\equiv 2(mod4)\ and\ j=n \\ 3, & for\ i\equiv 1,3(mod4)\ and\ j\equiv 1,3(mod4); j\neq n\ or \\ & for\ i\equiv 2(mod4)\ and\ j\equiv 1,3(mod4); j\neq n\ or \\ & for\ i\equiv 0(mod4)\ and\ j=n \\ 4, & for\ i\equiv 1,3(mod4)\ and\ j\equiv 0(mod2)\ or \\ & for\ i\equiv 2(mod4)\ and\ j\equiv 0(mod2)\ or \\ & for\ i\equiv 0(mod4)\ and\ j\equiv 1,3(mod4)\ j\neq n \\ 5, & for\ i\equiv 0(mod4)\ and\ j\equiv 0(mod2) \end{cases}$$

For every $uv\in V(P_m\odot E_{3,n})$, take any $u=x_i$ and $v=x_{i+1}$; $1\le i\le m$; $w(x_i)\neq w(x_{i+1})$, then $u=x_i$ and $v=a_i$; $1\le i\le m; 1\le j\le n$; $w(x_i)\neq w(a_i)$, then $u=x_i$ and $v=b_i$; $1\le i\le m; 1\le j\le n$; $w(x_i)\neq w(b_i)$, then $u=x_i$ and $v=c_i$; $1\le i\le m; 1\le j\le n$; $w(x_i)\neq w(c_i)$, then $u=x_i$ and $v=a_{ij}$; $1\le i\le m; 1\le j\le n$; $w(x_i)\neq w(a_{ij})$, then $u=x_i$ and $v=b_{ij}$; $1\le i\le m; 1\le j\le n$; $w(x_i)\neq w(b_{ij})$, then $u=x_i$ and $v=c_{ij}$; $1\le i\le m; 1\le j\le n$; $w(x_i)\neq w(c_{ij})$, then $u=a_i$ and $v=a_{i1}$; $1\le i\le m; 1\le j\le n$; $w(a_i)\neq w(a_{i1})$, then $u=b_i$ and $v=b_{i1}$; $1\le i\le m; 1\le j\le n$; $w(b_i)\neq w(b_{i1})$, then $u=c_i$ and $v=c_{i1}$; $1\le i\le m; 1\le j\le n$; $w(c_i)\neq w(c_{i1})$, then $u=a_{ij}$ and $v=a_{ij+1}$; $1\le i\le m; 1\le j\le n$; $w(a_{ij})\neq w(a_{ij+1})$, then $u=b_{ij}$ and $v=b_{ij+1}$; $1\le i\le m; 1\le j\le n$; $w(b_{ij})\neq w(b_{ij+1})$, then $u=c_{ij}$ and $v=c_{ij+1}$; $1\le i\le m; 1\le j\le n$; $w(c_{ij})\neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m\odot E_{3,n})\le 8$. We have $8\le \chi_{lis}(P_m\odot E_{3,n})\le 8$, so $\chi_{lis}(P_m\odot E_{3,n})=8$

Case 12 for $m\equiv 0(mod2); m\ge 4$and $n\equiv 0(mod4)$

First step to prove this theorem is find the lower bound of $(P_m\odot E_{3,n})$**.** Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m\odot E_{3,n})\ge$ - χ $(P_m\odot E_{3,n})$.= 3. Assume $\boldsymbol{\chi_{lis}}(P_m\odot E_{3,n})=\mathbf{7}$, let $\boldsymbol{\chi_{lis}}(P_m\odot E_{3,n})=\mathbf{7}$, **if** $l(x_i)=1; i\equiv 2(mod4)$; i =≡ $1,3(mod4); l(x_i)=2; i\equiv 0(mod4)$; $l(a_i)=l(b_i)=l(c_i)=l(a_{ij})=l(b_{ij})=l(c_{ij})=1$, then $w(a_i)=w(a_{ij})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i)=1; i\equiv 2(mod4)$; i =≡ $1,3(mod4); l(x_i)=2; i\equiv 0(mod4); l(a_i)=l(c_i)=1; l(b_i)=1$; $l(a_{ij})=l(c_{ij})=1; 1\le i\le m$; $j\equiv 1,3(mod4); j\equiv 0(mod4)$; $l(a_{ij})=l(c_{ij})=2; 1\le i\le m$; $j\equiv 2(mod4)$; $l(b_{ij})=1$; $1\le i\le m$; $j\equiv 3(mod4)$; $j\equiv 0(mod2)$; $l(b_{ij})=2$; $1\le i\le m$; $j\equiv 1(mod4)$,then $w(b_{ij})\neq w(b_{ij+1}); w(a_{ij})\neq w(a_{ij+1})$; $w(x_{i+1})\neq w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m\odot E_{3,n})\ge 8$. Based on that we have the lower bound of $\chi_{lis}(P_m\odot E_{3,n})\ge 8$.

After that, we will find the upper bound of $(P_m\odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m\odot E_{3,n})$, we have define $l: V(P_m\odot E_{3,n})\to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i)=\begin{cases} 1, & for\ i\equiv 2(mod4) or \\ & for\ i\equiv 1,3(mod4) \\ 2, & for\ i\equiv 0(mod4) \end{cases}$$

$$l(a_i)=1$$

$$l(b_i)=1$$

$$l(c_i)=1$$

$$l(a_{ij})=\begin{cases} 1, & for\ \ 1\le i\le m\ and\ j\equiv 1,3(mod4)\ or \\ & for\ \ 1\le i\le m\ and\ j\equiv 0(mod4) \\ 2, & for\ \ 1\le i\le m\ and\ j\equiv 2(mod4) \end{cases}$$

$$l(b_{ij})=\begin{cases} 1, & for\ 1\le i\le m; and\ j\equiv 3(mod4)\ or \\ & for\ 1\le i\le m; and\ j\equiv 0(mod2) \\ 2, & for\ 1\le i\le m; and\ j\equiv 1(mod4) \end{cases}$$

$$l(c_{ij})=\begin{cases} 1, & for\ \ 1\le i\le m\ and\ j\equiv 1,3(mod4)\ or \\ & for\ \ 1\le i\le m\ and\ j\equiv 0(mod4) \\ 2, & for\ \ 1\le i\le m\ and\ j\equiv 2(mod4) \end{cases}$$

Hence $opt(l)=2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1, m \\ 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 0(mod2); i \neq m \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1,3(mod4)\ ; i \neq 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 4, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 6, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 4, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 1,3(mod4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4)\ \ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4)\ \ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2)\ j \neq n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2)\ j \neq n\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 1,3(mod4) \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 1,3(mod4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \leq i \leq m$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \leq i \leq m; 1 \leq j \leq n; w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \leq i \leq m; 1 \leq j \leq n;$ $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \leq i \leq m; 1 \leq j \leq n; w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 8$. We have $8 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 8$, so $\chi_{lis}(P_m \odot E_{3,n}) = 8$

Case 13 for $m \equiv 0(mod2); m \geq 4$and $n \equiv 2(mod4)$

First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$**.** Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq$ - χ $(P_m \odot E_{3,n})$.$= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{7}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{7}$, **if** $l(x_i) = 1; i \equiv 2(mod4);$ i =≡ $1,3(mod4); l(x_i) = 2; i \equiv 0(mod4);$ $l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1$, then $w(a_i) = w(a_{ij})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 2(mod4);$ i =≡ $1,3(mod4); l(x_i) = 2; i \equiv 0(mod4); l(a_i) = l(c_i) = 1; l(b_i) = 1;$ $l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq m;$ $j \equiv 1,3(mod4); j \equiv 0(mod4);$ $l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq m;$ $j \equiv 2(mod4);$ $l(b_{ij}) = 1;$ $1 \leq i \leq m;$ $j \equiv 1(mod4); j \neq 1$ $j \equiv 0(mod2);$ $l(b_{ij}) = 2;$ $1 \leq i \leq m;$ $j \equiv 1(mod4);$ $j = 1$,,then $w(b_{ij}) \neq w(b_{ij+1}); w(a_{ij}) \neq w(a_{ij+1});$ $w(x_{i+1}) \neq w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \geq 8$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 8$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \rightarrow 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 2(mod4) or \\ & for\ i \equiv 1,3(mod4) \\ 2, & for\ i \equiv 0(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m\ and\ j \equiv 1,3(mod4)\ or \\ & for\ 1 \le i \le m\ and\ j \equiv 0(mod4) \\ 2, & for\ 1 \le i \le m\ and\ j \equiv 2(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m; and\ j \equiv 1(mod4); j \ne 1\ or \\ & for\ 1 \le i \le m; and\ j \equiv 0(mod2) \\ 2, & for\ 1 \le i \le m; and\ j = 1\ or \\ & for\ 1 \le i \le m; and\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m\ and\ j \equiv 1,3(mod4)\ or \\ & for\ 1 \le i \le m\ and\ j \equiv 0(mod4) \\ 2, & for\ 1 \le i \le m\ and\ j \equiv 2(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1, m \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 0(mod2); i \ne m \\ 3n + 8 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1,3(mod4)\ ; i \ne 1 \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 4, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 6, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4) \\ 4, & for\ i \equiv 0(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2); j \ne n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2); j \ne n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \ne n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 1,3(mod4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2); j \ne n, n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2); j \ne n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2); j \ne n \\ 5, & for\ i \equiv 1,3(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 2(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2); j \ne 2\ or \\ 6, & for\ i \equiv 0(mod4)\ and\ j = 2 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1,3(mod4)\ and\ j = n\ or \\ & for\ i \equiv 2(mod4)\ and\ j = n \\ 3, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 0(mod2); j \ne n\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 0(mod2); j \ne n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1,3(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 2(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \ne n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 1,3(mod4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \le i \le m$; $w(x_i) \ne w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \le i \le m; 1 \le j \le n; w(a_i) \ne w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(b_i) \ne w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(c_i) \ne w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(a_{ij}) \ne w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(b_{ij}) \ne w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \le i \le m; 1 \le j \le n; w(c_{ij}) \ne w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \le 8$. We have $8 \le \chi_{lis}(P_m \odot E_{3,n}) \le 8$, so $\chi_{lis}(P_m \odot E_{3,n}) = 8$

Case 14 for $m \equiv 1,3(mod4); m \ge 5$and $n \equiv 0(mod4)$

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First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\chi_{lis}(P_m \odot E_{3,n}) \geq$ - $\chi\ (P_m \odot E_{3,n}).= 3$. Assume $\chi_{lis}(P_m \odot E_{3,n}) = 7$, let $\chi_{lis}(P_m \odot E_{3,n}) = 7$, **if** $l(x_i) = 1; i \equiv 3(mod4)$; i =≡ $3(mod4); l(x_i) = 2; i \equiv 0(mod2)$; $l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1$, then $w(a_i) = w(a_{ij})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 0(mod2)$; i =≡ $1(mod4); l(x_i) = 2; i \equiv 3(mod4); l(a_i) = l(c_i) = 1; l(b_i) = 1$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq m$; $j \equiv 1{,}3(mod4); j \equiv 0(mod4)$; $l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq m$; $j \equiv 2(mod4)$; $l(b_{ij}) = 1$; $1 \leq i \leq m$; $j \equiv 3(mod4)$; $j \equiv 0(mod2)$; $l(b_{ij}) = 2$; $1 \leq i \leq m$; $j \equiv 1(mod4)$,then $w(b_{ij}) \neq w(b_{ij+1}); w(a_{ij}) \neq w(a_{ij+1})$; $w(x_{i+1}) \neq w(x_{i+2})$ then $\chi_{lis}(P_m \odot E_{3,n}) \geq 8$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 8$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \rightarrow 1{,}2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1(mod4) or \\ & for\ i \equiv 0(mod2) \\ 2, & for\ i \equiv 3(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ \ 1 \leq i \leq m\ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ \ 1 \leq i \leq m\ and\ j \equiv 0(mod4) \\ 2, & for\ \ 1 \leq i \leq m\ and\ j \equiv 2(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq m; and\ j \equiv 3(mod4)\ or \\ & for\ 1 \leq i \leq m; and\ j \equiv 0(mod2) \\ 2, & for\ 1 \leq i \leq m; and\ j \equiv 1(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ \ 1 \leq i \leq m\ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ \ 1 \leq i \leq m\ and\ j \equiv 0(mod4) \\ 2, & for\ \ 1 \leq i \leq m\ and\ j \equiv 2(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1, m \\ 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1{,}3(mod4)\ ; i \neq 1 \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv\ 0(mod2); i \neq m \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 4, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 6, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 4, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod2)\ \ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ i \equiv 0(mod2)\ \ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 1{,}3(mod4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ i \equiv 0(mod2)\ \ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod2)\ \ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 1{,}3(mod4) \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2); j \neq n \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod2)\ \ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ i \equiv 0(mod2)\ \ and\ j \equiv 1{,}3(mod4)\ or \\ & for\ i \equiv 0(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \\ 5, & for\ i \equiv 0(mod4)\ and\ j \equiv 1{,}3(mod4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \leq i \leq m$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \leq$

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$i \leq m; 1 \leq j \leq n;\ w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \leq i \leq m; 1 \leq j \leq n; w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \leq i \leq m; 1 \leq j \leq n;\ w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \leq i \leq m; 1 \leq j \leq n; w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \leq 8$. We have $8 \leq \chi_{lis}(P_m \odot E_{3,n}) \leq 8$, so $\chi_{lis}(P_m \odot E_{3,n}) = 8$

Case 15 for $m \equiv 1,3(mod4); m \geq 5$and $n \equiv 2(mod4)$
First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$. Based on Lemma 1, $\chi_{lis}(P_m \odot E_{3,n}) \geq$ - $\chi\ (P_m \odot E_{3,n}). = 3$. Assume $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{7}$, let $\chi_{lis}(P_m \odot E_{3,n}) = \mathbf{7}$, **if** $l(x_i) = 1; i \equiv 3(mod4)$; i =≡ $3(mod4); l(x_i) = 2; i \equiv 0(mod2);\ l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1$, then $w(a_i) = w(a_{ij})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 0(mod2)$; i =≡ $1(mod4); l(x_i) = 2; i \equiv 3(mod4); l(a_i) = l(c_i) = 1; l(b_i) = 1;\ l(a_{ij}) = l(c_{ij}) = 1; 1 \leq i \leq m;\ j \equiv 1,3(mod4); j \equiv 0(mod4);\ l(a_{ij}) = l(c_{ij}) = 2; 1 \leq i \leq m;\ j \equiv 2(mod4);\ l(b_{ij}) = 1;\ 1 \leq i \leq m;\ j \equiv 1(mod4);\ j \neq 1;\ j \equiv 0(mod2);\ l(b_{ij}) = 2;\ 1 \leq i \leq m;\ j \equiv 3(mod4);\ j = 1$,then $w(b_{ij}) \neq w(b_{ij+1}); w(a_{ij}) \neq w(a_{ij+1});\ w(x_{i+1}) \neq w(x_{i+2})$ then $\chi_{lis}(P_m \odot E_{3,n}) \geq 8$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \geq 8$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1(mod4) or \\ & for\ i \equiv 0(mod2) \\ 2, & for\ i \equiv 3(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 1$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ \ 1 \leq i \leq m\ and\ j \equiv 1,3(mod4)\ or \\ & for\ \ 1 \leq i \leq m\ and\ j \equiv 0(mod4) \\ 2, & for\ \ 1 \leq i \leq m\ and\ j \equiv 2(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \leq i \leq m; and\ j \equiv 1(mod4); j \neq 1\ or \\ & for\ 1 \leq i \leq m; and\ j \equiv 0(mod2) \\ 2, & for\ 1 \leq i \leq m; and\ j = 1\ or \\ & for\ 1 \leq i \leq m; and\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ \ 1 \leq i \leq m\ and\ j \equiv 1,3(mod4)\ or \\ & for\ \ 1 \leq i \leq m\ and\ j \equiv 0(mod4) \\ 2, & for\ \ 1 \leq i \leq m\ and\ j \equiv 2(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 6 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1, m \\ 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1,3(mod4)\ ; i \neq 1 \\ 3n + 8 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv\ 0(mod2); i \neq m \end{cases}$$

$$w(a_i) = \begin{cases} 3, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 4, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 6, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 3, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 4, & for\ i \equiv 3(mod4) \end{cases}$$

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$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 3(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \\ 5, & for\ i \equiv 3(mod4)\ and\ j \equiv 1,3(mod4) \end{cases}$$

$$w(b_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 3(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2); j \neq 2, n\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 0(mod2); j \neq 2, n\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 1,3(mod4) \\ 5, & for\ i \equiv 1(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 0(mod2)\ and\ j = 2\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 0(mod2); j \neq 2, n\ or \\ 6, & for\ i \equiv 3(mod4)\ and\ j = 2\ or \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 0(mod2); j \neq n\ or \\ & for\ i \equiv 3(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 1,3(mod4)\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 0(mod2)\ j \neq n \\ 5, & for\ i \equiv 3(mod4)\ and\ j \equiv 1,3(mod4) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \le i \le m$; $w(x_i) \neq w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \neq w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \neq w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \neq w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \neq w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \neq w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \neq w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(a_i) \neq w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(b_i) \neq w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(c_i) \neq w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(a_{ij}) \neq w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(b_{ij}) \neq w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \le i \le m; 1 \le j \le n; w(c_{ij}) \neq w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \le 8$. We have $8 \le \chi_{lis}(P_m \odot E_{3,n}) \le 8$, so $\chi_{lis}(P_m \odot E_{3,n}) = 8$

Case 16 for $m \equiv 1,3(mod4); m \ge 5$and $n \equiv 1,3(mod4)$
First step to prove this theorem is find the lower bound of $(P_m \odot E_{3,n})$**.** Based on Lemma 1, $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge$ -$\chi\ (P_m \odot E_{3,n}).= 3$. Assume $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{7}$, let $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) = \mathbf{7}$, **if** $l(x_i) = 1; i \equiv 3(mod4)$; i =≡ $3(mod4); l(x_i) = 2; i \equiv 0(mod2)$; $l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1$, then $w(a_i) = w(a_{ij})$ then there are two adjacent vertices that have same color, it contradict with definition of vertex coloring. If **if** $l(x_i) = 1; i \equiv 0(mod2)$; i =≡ $1(mod4); l(x_i) = 2; i \equiv 3(mod4); l(a_i) = l(c_i) = 1; l(b_i) = 1$; $l(a_{ij}) = l(c_{ij}) = 1; 1 \le i \le m$; $j \equiv 1(mod4); j \equiv 0(mod2)$; $l(a_{ij}) = l(c_{ij}) = 2; 1 \le i \le m$; $j \equiv 3(mod4)$; $l(b_{ij}) = 1$; $1 \le i \le m$; $j \equiv 1(mod4)$; $j \neq 1$; $j \equiv 0(mod2)$; $l(b_{ij}) = 2$; $1 \le i \le m$; $j \equiv 3(mod4)$; $j = 1$,then $w(b_{ij}) \neq w(b_{ij+1}); w(a_{ij}) \neq w(a_{ij+1})$; $w(x_{i+1}) \neq w(x_{i+2})$ then $\boldsymbol{\chi_{lis}}(P_m \odot E_{3,n}) \ge 8$. Based on that we have the lower bound of $\chi_{lis}(P_m \odot E_{3,n}) \ge 8$.

After that, we will find the upper bound of $(P_m \odot E_{3,n})$. Furthermore the upper bound for the chromatic number local irregular $(P_m \odot E_{3,n})$, we have define $l : V(P_m \odot E_{3,n}) \to 1,2$ with vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & for\ i \equiv 1(mod4) or \\ & for\ i \equiv 0(mod2) \\ 2, & for\ i \equiv 3(mod4) \end{cases}$$

$$l(a_i) = 1$$

$$l(b_i) = 2$$

$$l(c_i) = 1$$

$$l(a_{ij}) = \begin{cases} 1, & for\ \ 1 \le i \le m\ and\ j \equiv 13(mod4)\ or \\ & for\ \ 1 \le i \le m\ and\ j \equiv 0(mod2) \\ 2, & for\ \ 1 \le i \le m\ and\ j \equiv 3(mod4) \end{cases}$$

$$l(b_{ij}) = \begin{cases} 1, & for\ 1 \le i \le m;\ and\ j \equiv 1(mod4);\ j \ne 1\ or \\ & for\ 1 \le i \le m;\ and\ j \equiv 0(mod2) \\ 2, & for\ 1 \le i \le m;\ and\ j = 1\ or \\ & for\ 1 \le i \le m;\ and\ j \equiv 3(mod4) \end{cases}$$

$$l(c_{ij}) = \begin{cases} 1, & for\ \ 1 \le i \le m\ and\ j \equiv 13(mod4)\ or \\ & for\ \ 1 \le i \le m\ and\ j \equiv 0(mod2) \\ 2, & for\ \ 1 \le i \le m\ and\ j \equiv 3(mod4) \end{cases}$$

Hence $opt(l) = 2$ and the labelling provides vertex-weight as follows:

$$w(x_i) \begin{cases} 3n + 7 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i = 1, m\ and\ n \equiv 3(mod4) \\ 3n + 8 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 1{,}3(mod4);\ i \ne m\ and\ n \equiv 3(mod4) \\ 3n + 9 + \left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 0(mod2)\ and\ n \equiv 3(mod4) \\ 9 + 15\left\lceil \frac{n}{5} \right\rceil, & for\ i = 1, m\ and\ n \equiv 1(mod4) \\ 10 + 15\left\lceil \frac{n}{5} \right\rceil, & for\ i \equiv 1{,}3(mod4);\ i \ne m\ and\ n \equiv 1(mod4) \\ 11 + 15\left\lceil \frac{2n}{3} \right\rceil, & for\ i \equiv 0(mod2)\ and\ n \equiv 1(mod4) \end{cases}$$

$$w(a_i) = \begin{cases} 4, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 5, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(b_i) = \begin{cases} 5, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 6, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(c_i) = \begin{cases} 4, & for\ i \equiv 1(mod4)\ or \\ & for\ i \equiv 0(mod2) \\ 5, & for\ i \equiv 3(mod4) \end{cases}$$

$$w(a_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 1{,}3(mod4);\ j \ne n\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 1{,}3(mod4);\ j \ne n\ or \\ & for\ i \equiv 3(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2)\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 0(mod2)\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 1{,}3(mod4)\ j \ne n \\ 5, & for\ i \equiv 3(mod4)\ and\ j \equiv 0(mod2) \end{cases}$$

$$w(b_{ij}) \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 1{,}3(mod4);\ j \ne 1, n\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 1{,}3(mod4);\ j \ne 1, n\ or \\ & for\ i \equiv 3(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2);\ j \ne 2\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 0(mod2);\ j \ne 2\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 1{,}3(mod4);\ j \ne 1, n \\ & for\ i \equiv 1(mod4)\ and\ j = 1 \\ & for\ i \equiv 0(mod2)\ and\ j = 1 \\ 5, & for\ i \equiv 1(mod4)\ and\ j = 2\ or \\ & for\ i \equiv 0(mod2)\ and\ j = 2\ or \\ & for\ i \equiv 3(mod4)\ and\ j = 1 \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 0(mod2);\ j \ne 2 \\ 6, & for\ i \equiv 3(mod4)\ and\ j = 2 \end{cases}$$

$$w(c_{ij}) = \begin{cases} 2, & for\ i \equiv 1(mod4)\ and\ j = n\ or \\ & for\ i \equiv 0(mod2)\ and\ j = n \\ 3, & for\ i \equiv 1(mod4)\ and\ j \equiv 1{,}3(mod4);\ j \ne n\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 1{,}3(mod4);\ j \ne n\ or \\ & for\ i \equiv 3(mod4)\ and\ j = n \\ 4, & for\ i \equiv 1(mod4)\ and\ j \equiv 0(mod2)\ or \\ & for\ i \equiv 0(mod2)\ and\ j \equiv 0(mod2)\ or \\ & for\ i \equiv 3(mod4)\ and\ j \equiv 1{,}3(mod4)\ j \ne n \\ 5, & for\ i \equiv 3(mod4)\ and\ j \equiv 0(mod2) \end{cases}$$

For every $uv \in V(P_m \odot E_{3,n})$, take any $u = x_i$ and $v = x_{i+1}$; $1 \le i \le m$; $w(x_i) \ne w(x_{i+1})$, then $u = x_i$ and $v = a_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(a_i)$, then $u = x_i$ and $v = b_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(b_i)$, then $u = x_i$ and $v = c_i$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(c_i)$, then $u = x_i$ and $v = a_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(a_{ij})$, then $u = x_i$ and $v = b_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(b_{ij})$, then $u = x_i$ and $v = c_{ij}$; $1 \le i \le m; 1 \le j \le n$; $w(x_i) \ne w(c_{ij})$, then $u = a_i$ and $v = a_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(a_i) \ne w(a_{i1})$, then $u = b_i$ and $v = b_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(b_i) \ne w(b_{i1})$, then $u = c_i$ and $v = c_{i1}$; $1 \le i \le m; 1 \le j \le n$; $w(c_i) \ne w(c_{i1})$, then $u = a_{ij}$ and $v = a_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(a_{ij}) \ne w(a_{ij+1})$, then $u = b_{ij}$ and $v = b_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(b_{ij}) \ne w(b_{ij+1})$, then $u = c_{ij}$ and $v = c_{ij+1}$; $1 \le i \le m; 1 \le j \le n$; $w(c_{ij}) \ne w(c_{ij+1})$.

The upper bound $\chi_{lis}(P_m \odot E_{3,n}) \le 8$. We have $8 \le \chi_{lis}(P_m \odot E_{3,n}) \le 8$, so $\chi_{lis}(P_m \odot E_{3,n}) = 8$

3. CONCLUTION

In this paper, we have studied local irregularity vertex coloring of corona product by path graph wtih E graph. We have determined the exact value of the chromatic number local irregular of corona product by path graph wtih E graph., namely $(P_m \odot E_{3,n})$

4. ACKNOWLEDGMENTS

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