International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 5 Issue 7, July - 2021, Pages: 48-58

The Soft Sub-space of Soft Tri-topological Space

Asmhan Flieh Hassan[1] and Ali Hussein Abaas[2]

[1,2]Department of Mathematics, Faculty of education for girls, University of Kufa, Iraq
[1]Corresponding author E-mail: asmhanf.alzuhairy@uokufa.edu.iq
[2]E-mail: ali.hussein.awad@gmail.com

Abstract— *The main aim of this paper is to present the concept of soft sub-space of a soft tri-topological space (w.r.t. soft $\boldsymbol{\delta^*-open}$ set) as an original study. Firstly, we introduce the new concept in soft tri-topological space, and we analyse whether a soft sub-space of soft tri-topological space is a soft topology or not. Secondly, we define another new concept which is the collection of soft sub-spaces of parametrized topologies on the initial universe set, i.e. corresponding to each parameter of a soft tri-topological space, we get a sub-spaces for the parametrized family of crisp topologies.*

Keywords— soft tri-topological space, soft $\boldsymbol{\delta^*-open}$ set, soft sub-space (w.r.t. soft $\boldsymbol{\delta^*-open}$ set).

1. Introduction

The soft set theory was initiated by Molodtsov [16] in 1999 as a mathematical tool for uncertain objects. In [16,21], the researcher applied the theory in several directions successfully, like operations research, probability, The theory of measurement, and others. And to develop soft set theory, the main operations of the soft sets are defined [22].
A large number of papers have been produced in order to generalize the topological concepts to bi-topological setting (see [1,17]). In 2004 Asmhan [2], was first initiated the concept of δ^*-open set by using three topologies. And she provide the tri-topological theory by all topological structures w.r.t. δ^*-open set in tri-topological space, such as; the base, connectedness, compactness, lindelofness, countability, separability , product space, quotient space, the reader can find that in [3 – 9].

A topology on a soft sets, called 'soft topology', and its related properties presented by Cagman et al [19] in 2011, and they introduced the foundations of the theory of soft topological spaces.
A soft set with one specific topological structure is not sufficient to develop the theory. In that case, it becomes necessary to introduce an additional structures on the soft set. To confirm this idea, soft bi-topological space by soft bi-topological theory (Ittanagi [15] in 2014), soft tri-topological space by soft tri-topological theory (Asmhan [10] in 2017) and soft N-topological theory (Giorgio [20] in 2019) were introduced. It makes it more flexible to develop the theory of soft topological spaces with its applications. Thus, in this paper, we make a new approach to the soft tri-topological space theory. the foundations and its related properties was presented in [12-14].
Also the 'fuzzy soft tri-topological theory' was first initiated by the same author (Asmhan) in 2019 [11].
Now, our motivation in this work is to state and continue the foundations of the theory of a soft tri-topological spaces. And more exactly, to define a soft sub-space of a soft tri-topological space (w.r.t. soft δ^*- open set) and study the main properties for these soft sub-spaces.

2. Preliminaries

In the following, some foundations concepts about soft sets, soft topological spaces and soft tri-topological spaces are given.
Definition 2.1 [16] Let X be an initial universe set and let $\mathbb{P}$ be a set of all parameters. And P(X) denotes the power set of initial universe X and $\mathbb{A} \neq \emptyset$, $\mathbb{A} \subseteq \mathbb{P}$. Then a pair $(\mathcal{S}, \mathbb{A})$ is said to be a soft set on X, $\mathcal{S}$ is a mapping defined as $\mathcal{S}: \mathbb{A} \to \mathrm{P(X)}$.
For more exactly, the soft set on an initial universe X is a parametrized family of sub-sets of it. For $\mathcal{P} \in \mathbb{A}$, $\mathcal{S}(\mathcal{P})$ may be as the set of $\mathcal{P}$ −approximate elements of $(\mathcal{S}, \mathbb{A})$, when $\mathcal{P} \notin \mathbb{A}$, then $\mathcal{S}(\mathcal{P}) = \emptyset$, i.e., $(\mathcal{S}, \mathbb{A}) = \{(\mathcal{P}, \mathcal{S}(\mathcal{P})): \mathcal{P} \in \mathbb{A} \subseteq \mathbb{P}, \mathcal{S}: \mathbb{A} \to \mathrm{P}(\mathcal{X})\}$, Clear that, a soft set is a function, not a set.
Definition 2.2 [22] The complement (relative complement) of a soft set $(\mathcal{S}, \mathbb{P})$ is denoted by $(\mathcal{S}, \mathbb{P})^c$ and is defined by $(\mathcal{S}, \mathbb{P})^c = (\mathcal{S}^c, \mathbb{P})$ where $\mathcal{S}^c: \mathbb{P} \to \mathcal{P}(\mathrm{X})$ is a mapping given by $\mathcal{S}^c(\mathcal{P}) = \mathrm{X} - \mathcal{S}(\mathcal{P})$ for all $\mathcal{P} \in \mathbb{P}$
Definition 2.3 [22] Let Y be a non-empty sub-set of X, then $\mathcal{Y}$ denotes the soft set $(\mathrm{Y}, \mathbb{P})$ over X for which $\mathrm{Y}(\mathcal{P}) = \mathrm{Y}$, for all $\mathcal{P} \in \mathbb{P}$. In particular, $(\mathrm{X}, \mathbb{P})$ will be denoted by $\mathcal{X}$.
Definition 2.4 [22] Let the ordinary point $\mathrm{x} \in \mathcal{X}$. Then the soft set on X for which $\mathrm{x}(\mathcal{P}) = \{\mathrm{x}\}$ denotes $(\mathrm{x}, \mathbb{P})$, for every $\mathcal{P} \in \mathbb{P}$.
Definition 2.5 [22] Let $(\mathcal{S}, \mathbb{P})$ be a soft set on $\mathcal{X}$ and $\mathrm{x} \in \mathrm{X}$. We can say that x belongs to $(\mathcal{S}, \mathbb{P})$ (i.e. $\mathrm{x} \in (\mathcal{S}, \mathbb{P})$), whenever $\mathrm{x} \in \mathcal{S}(\mathcal{P})$ for every $\mathcal{P} \in \mathbb{P}$, clear that $\mathrm{x} \notin (\mathcal{S}, \mathbb{P})$ to any $\mathrm{x} \in \mathcal{X}$, if $\mathrm{x} \notin \mathcal{S}(\mathcal{P})$ for some $\mathcal{P} \in \mathbb{P}$.
Definition 2.6 [22] The difference $(\mathrm{H}, \mathbb{P})$ of two soft sets $(\mathcal{S}, \mathbb{P})$ and $(\mathrm{G}, \mathbb{P})$ over X, denoted by $(\mathcal{S}, \mathbb{P}) \backslash (\mathrm{G}, \mathbb{P})$, is defined as $\mathrm{H}(\mathcal{P}) = \mathcal{S}(\mathcal{P}) \backslash \mathrm{G}(\mathcal{P})$ for all $\mathcal{P} \in \mathbb{P}$.

Definition 2.7 [22] The soft sets $(\mathcal{S},\mathbb{A})$ and $(\mathrm{G},\mathbb{B})$ on a common universe set X, we can say that $(\mathcal{S},\mathbb{A})$ is a sub-set (soft sub-set) of $(\mathrm{G},\mathbb{B})$ if $\mathbb{A}\subseteq\mathbb{B}$ and $\mathcal{S}(\mathcal{P})\subseteq \mathrm{G}(\mathcal{P})$, for all $\mathcal{P}\in\mathbb{A}$.
then $(\mathcal{S},\mathbb{A})$ is called the soft super set of the soft set $(\mathrm{G},\mathbb{B})$, and if $(\mathrm{G},\mathbb{B})$ is a soft sub-set of $(\mathcal{S},\mathbb{A})$.
Definition 2.8 [22] The Union of the soft sets $(\mathcal{S},\mathbb{P})$ and $(\mathrm{G},\mathbb{P})$ on the universe set X is the soft set $(\mathrm{H},\mathbb{P})$, such that $\mathrm{H}(\mathcal{P})=\mathcal{S}(\mathcal{P})\cup \mathrm{G}(\mathcal{P})$ for every $\mathcal{P}\in\mathbb{P}$. And we can write $(\mathcal{S},\mathbb{P})\cup(\mathrm{G},\mathbb{P})=(\mathrm{H},\mathbb{P})$.
Definition 2.9 [22] The intersection of the soft sets $(\mathcal{S},\mathbb{P})$ and $(\mathrm{G},\mathbb{P})$ over the universe set X is $(\mathrm{H},\mathbb{P})$ soft set, such that $\mathrm{H}(\mathcal{P})=\mathcal{S}(\mathcal{P})\cap \mathrm{G}(\mathcal{P})$ for all $\mathcal{P}\in\mathbb{P}$. And we can write $(\mathcal{S},\mathbb{P})\cap(\mathrm{G},\mathbb{P})=(\mathrm{H},\mathbb{P})$.
Definition 2.10 [19] Let the family τ be a family of soft sets on the universe X, τ is called a soft topology on the X, if:
(1) the soft sets $\emptyset$ and $\mathcal{X}$ belong to τ.
(2) the soft union of the soft sets in τ belongs to τ.
(3) the soft intersection of any two soft sets in τ also belongs to τ.
Then, $(\mathcal{X},\tau,\mathbb{P})$ can said to be a soft topological space on the X, the member of the soft topology τ is called soft open set.
Definition 2.11 [19] If $(\mathcal{X},\tau,\mathbb{P})$ be a soft topological space on X. The soft open set $(\mathcal{S},\mathbb{P})$ on $\mathcal{X}$ is called a soft closed set in $\mathcal{X}$, if the complement $(\mathcal{S},\mathbb{P})^{c}$ belongs to soft topology τ.
Definition 2.12 [18] The soft set $(\mathcal{S},\mathbb{P})$ in $(\mathcal{X},\tau,\mathbb{P})$, then:
(1) The soft closure of $(\mathrm{F},\mathbb{P})$, is defined as:
$\mathrm{s.cl}\,(\mathcal{S},\mathbb{P})=\bigcap\{(\mathrm{G},\mathbb{P}):(\mathcal{S},\mathbb{P})\subseteq(\mathrm{G},\mathbb{P}),\text{and}(\mathrm{G},\mathbb{P})\ \text{soft closed}\}$
(2) The soft interior of $(\mathcal{S},\mathbb{P})$, is defined as:
$\mathrm{s.int}\,(\mathcal{S},\mathbb{P})=\bigcup\{(\mathrm{H},\mathbb{P}):(\mathrm{H},\mathbb{P})\subseteq(\mathcal{S},\mathbb{P}),\text{and }(\mathrm{H},\mathbb{P})\ \text{soft open}\}$
Definition 2.13 [18] If $(\mathcal{X},\tau,\mathbb{P})$is a soft topological space on X and $\mathrm{Y}\neq\emptyset$ and be a sub-set of the universe set X. Then; $\tau_{\mathrm{Y}}=\{(\mathcal{S}_{\mathrm{Y}},\mathbb{P})=\mathrm{Y}\cap(\mathcal{S},\mathbb{P})\,|\,(\mathcal{S},\mathbb{P})\in\text{top. space }\tau\}$ is called the soft relative (sub-space) topology on Y and $(\mathcal{X},\tau_{\mathrm{Y}},\mathbb{P})$ is called a soft relative (sub-space) of soft topological Space $(\mathcal{X},\tau,\mathbb{P})$.
Definition 2.14 [10] Let $(\mathcal{X},\tau_1,\mathbb{P})$, $(\mathcal{X},\tau_2,\mathbb{P})$ and $(\mathcal{X},\tau_3,\mathbb{P})$ be the three soft topological spaces on $\mathcal{X}$. Then $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ is called a soft tri-topological space.
The three soft topological spaces $(\mathcal{X},\tau_1,\mathbb{P})$, $(\mathcal{X},\tau_2,\mathbb{P})$ and $(\mathcal{X},\tau_3,\mathbb{P})$ are independently satisfy the axioms of soft topological space.
Definition 2.15 [10] Let $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ be a soft tri-topological space, then a soft set $(\mathrm{F},\mathbb{P})$ is called soft δ^*-open set iff $(\mathrm{F},\mathbb{P})\subseteq \mathrm{S.}\tau_1-\mathrm{int}(\mathrm{S.}\tau_2-\mathrm{cl}(\mathrm{S.}\tau_3-\mathrm{int}(\mathrm{F},\mathbb{P})))$. The family of all soft δ^*-open sets is denoted by $\mathrm{S.}\delta^*.\mathrm{O}(\mathcal{X})$. The complement of soft δ^*-open set is called a soft δ^*-closed set, and the family of all soft δ^*-closed sets is denoted by $\mathrm{S.}\delta^*.\mathrm{C}(\mathcal{X})$.

3. Soft Sub-space For a Soft Tri-topological Space (w.r.t. soft δ^*-open set)

In the following section, the definition and some foundations concepts about soft sub-space of a soft tri-topological space are given.
Definition 3.1. Let $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ be a soft tri-topological space over X, $\mathrm{Y}\neq\emptyset$ and $\mathrm{Y}\subseteq\mathrm{X}$. Then:
$\tau_{1\mathrm{Y}}=\{(\mathcal{M}_{\mathrm{Y}},\mathbb{P})=\mathcal{Y}\cap(\mathcal{M},\mathbb{P})|(\mathcal{M},\mathbb{P})\in\tau_1\}$, where $\mathcal{Y}$ is the soft set $(\mathrm{Y},\mathbb{P})$.
$\tau_{2\mathrm{Y}}=\{(\mathrm{G}_{\mathrm{Y}},\mathbb{P})=\mathcal{Y}\cap(\mathrm{G},\mathbb{P})|\,(\mathrm{G},\mathbb{P})\in\tau_2\}$
$\tau_{3\mathrm{Y}}=\{(\mathcal{H}_{\mathrm{Y}},\mathbb{P})=\mathcal{Y}\cap(\mathcal{H},\mathbb{P})|\,(\mathcal{H},\mathbb{P})\in\tau_3\}$
be a three soft relative topologies on Y for τ_1 , τ_2 and τ_3 resp. The space $(\mathrm{Y},\tau_{1\mathrm{Y}},\tau_{2\mathrm{Y}},\tau_{3\mathrm{Y}},\mathbb{P})$ is called a soft sub-space of a soft tri-topological space $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$.
And the soft relative space (soft sub-space) for $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ on Y (with respect to a soft δ^*-open set) is the collection $\mathrm{S.}\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$, given by: $\mathrm{S.}\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}=\{(\mathcal{S}_{\mathrm{Y}},\mathbb{P})=\mathcal{Y}\cap(\mathcal{S},\mathbb{P})\,|\,(\mathcal{S},\mathbb{P})\in \mathrm{S.}\delta^*.\mathrm{O}(\mathcal{X})\}$,where $\mathcal{Y}$ is a soft set $(\mathrm{Y},\mathbb{P})$ $\mathrm{F}_{\mathrm{Y}}(\mathcal{P})=\mathrm{Y}\cap\mathrm{F}(\mathcal{P})$, for all $\mathcal{P}\in\mathbb{P}$. And the members of $\mathrm{S.}\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$ are said to be soft $\delta^*{}_{\mathrm{Y}}-$open sets in Y.

Note 3.2. Throughout this thesis, $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ [or simply $\mathcal{X}$] denote the soft tri-topological space over X, [and simply $\mathbb{Y}$] denote to the soft sub-space of a soft tri-topological space $\mathcal{X}$ with respect to a soft δ^*-open set.

Example 3.3. This example shows that how can we find the soft sub-space of a soft tri-topological space $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ with respect to a soft δ^*-open set (i.e. the family of all soft $\delta^*{}_{\mathrm{Y}}-open$ sets $S.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$).
Let $\mathrm{X}=\{x_1,x_2,x_3,x_4\}$ be a universe set and $\mathbb{P}=\{\mathcal{P}_1,\mathcal{P}_2\}$ be the set of parameters, The soft set $(\mathrm{X},\mathbb{P})=\{(\mathcal{P}_1,\{x_1,x_2,x_3,x_4\}),(\mathcal{P}_2,\{x_1,x_2,x_3,x_4\})\}$. Then by [4], its cardinality is $|\mathrm{P}(\mathcal{X})|=2^{\sum_{\mathcal{P}\in\mathbb{P}}|\mathcal{S}(\mathcal{P})|}$, s.t. $|\mathcal{S}(\mathcal{P})|$ is the cardinality of $\mathcal{S}(\mathcal{P})$. (that is mean $|\mathrm{P}(\mathcal{X})|=2^8=256$ soft set).
And let $(\mathcal{X},\tau_1,\mathbb{P})$, $(\mathcal{X},\tau_2,\mathbb{P})$ and $(\mathcal{X},\tau_3,\mathbb{P})$ be the three soft topological spaces on X, and the soft topologies defined as follows:
$\tau_1=\{\varphi,\mathcal{X},(\mathcal{M},\mathbb{P})\}$
$\tau_2=\{\varphi,\mathcal{X},(\mathrm{G}_1,\mathbb{P}),(\mathrm{G}_2,\mathbb{P}),(\mathrm{G}_3,\mathbb{P}),(\mathrm{G}_4,\mathbb{P})\}$
$\tau_3=\{\varphi,\mathcal{X},(\mathcal{H}_1,\mathbb{P}),(\mathcal{H}_2,\mathbb{P}),(\mathcal{H}_3,\mathbb{P}),(\mathcal{H}_4,\mathbb{P}),(\mathcal{H}_5,\mathbb{P}),(\mathcal{H}_6,\mathbb{P})\}$

Where $(\mathcal{M},\mathbb{P}),(\mathrm{G}_1,\mathbb{P}),(\mathrm{G}_2,\mathbb{P}),(\mathrm{G}_3,\mathbb{P}),(\mathrm{G}_4,\mathbb{P}),(\mathcal{H}_1,\mathbb{P}),(\mathcal{H}_2,\mathbb{P}),(\mathcal{H}_3,\mathbb{P}),(\mathcal{H}_4,\mathbb{P}),(\mathcal{H}_5,\mathbb{P}),(\mathcal{H}_6,\mathbb{P})$ are $soft-open$ sets over X, defined as follows:

$\mathcal{M}(\mathcal{P}_1)=\{x_3,x_4\}$, $\mathcal{S}(\mathcal{P}_2)=\emptyset$

$\mathrm{G}_1(\mathcal{P}_1)=\{x_1,x_2\}$, $\mathrm{G}_1(\mathcal{P}_2)=\emptyset$
$\mathrm{G}_2(\mathcal{P}_1)=\{x_1\}$, $\mathrm{G}_2(\mathcal{P}_2)=\mathrm{X}$
$\mathrm{G}_3(\mathcal{P}_1)=\{x_1\}$, $\mathrm{G}_3(\mathcal{P}_2)=\emptyset$
$\mathrm{G}_4(\mathcal{P}_1)=\{x_1,x_2\}$, $\mathrm{G}_4(\mathcal{P}_2)=\mathrm{X}$

$\mathcal{H}_1(\mathcal{P}_1)=\{x_3,x_4\}$, $\mathcal{H}_1(\mathcal{P}_2)=\emptyset$
$\mathcal{H}_2(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{H}_2(\mathcal{P}_2)=\{x_2,x_3,x_4\}$
$\mathcal{H}_3(\mathcal{P}_1)=\emptyset$, $\mathcal{H}_3(\mathcal{P}_2)=\{x_1,x_4\}$
$\mathcal{H}_4(\mathcal{P}_1)=\{x_4\}$, $\mathcal{H}_4(\mathcal{P}_2)=\emptyset$
$\mathcal{H}_5(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{H}_5(\mathcal{P}_2)=\emptyset$
$\mathcal{H}_6(\mathcal{P}_1)=\emptyset$, $\mathcal{H}_6(\mathcal{P}_2)=\{x_1,x_3,x_4\}$
$\mathcal{H}_7(\mathcal{P}_1)=\{x_3,x_4\}$, $\mathcal{H}_7(\mathcal{P}_2)=\{x_1,x_4\}$
$\mathcal{H}_8(\mathcal{P}_1)=\{x_4\}$, $\mathcal{H}_8(\mathcal{P}_2)=\{x_1,x_4\}$
$\mathcal{H}_9(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{H}_9(\mathcal{P}_2)=\{x_1,x_4\}$
$\mathcal{H}_{10}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{H}_{10}(\mathcal{P}_2)=\{x_1,x_3,x_4\}$
$\mathcal{H}_{11}(\mathcal{P}_1)=\{x_4\}$, $\mathcal{H}_{11}(\mathcal{P}_2)=\{x_1,x_3,x_4\}$
$\mathcal{H}_{12}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{H}_{12}(\mathcal{P}_2)=\{x_3,x_4\}$
$\mathcal{H}_{13}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{H}_{13}(\mathcal{P}_2)=\{x_4\}$
$\mathcal{H}_{14}(\mathcal{P}_1)=\{x_4\}$, $\mathcal{H}_{14}(\mathcal{P}_2)=\{x_3,x_4\}$
$\mathcal{H}_{15}(\mathcal{P}_1)=\{x_4\}$, $\mathcal{H}_{15}(\mathcal{P}_2)=\{x_4\}$

The complements of the soft $open$ sets of soft topology τ_2 Are;
${\mathrm{G}_1}^{\mathrm{c}}(\mathcal{P}_1)=\{x_3,x_4\}$, ${\mathrm{G}_1}^{\mathrm{c}}(\mathcal{P}_2)=\mathrm{X}$
${\mathrm{G}_2}^{\mathrm{c}}(\mathcal{P}_1)=\{x_2,x_3,x_4\}$, ${\mathrm{G}_2}^{\mathrm{c}}(\mathcal{P}_2)=\emptyset$
${\mathrm{G}_3}^{\mathrm{c}}(\mathcal{P}_1)=\{x_2,x_3,x_4\}$, ${\mathrm{G}_3}^{\mathrm{c}}(\mathcal{P}_2)=\mathrm{X}$
${\mathrm{G}_4}^{\mathrm{c}}(\mathcal{P}_1)=\{x_3,x_4\}$, ${\mathrm{G}_4}^{\mathrm{c}}(\mathcal{P}_2)=\emptyset$

If we take the soft set $(\mathcal{A},\mathbb{P})$ which defined as:
$\mathcal{A}(\mathcal{P}_1)=\{x_3,x_4\}$, $\mathcal{A}(\mathcal{P}_2)=\emptyset$,
then $S.\tau_1-int(S.\tau_2-cl(S.\tau_3-int(\mathrm{F},\mathbb{P})))=$
$S.\tau_1-int(S.\tau_2-cl((\mathcal{H}_1,\mathbb{P}))= S.\tau_1-int((\mathrm{G}_4,\mathbb{P})^{\mathrm{c}})=(\mathcal{M},\mathbb{P})$
thus $(\mathcal{A},\mathbb{P})\subseteq(\mathcal{M},\mathbb{P})$, therefore $(\mathcal{A},\mathbb{P})$ is a soft δ^*-open set.
If we take the soft set $(\mathcal{B},\mathbb{P})$ which defined as:
$\mathcal{B}(\mathcal{P}_1)=\emptyset$, $\mathcal{B}(\mathcal{P}_2)=\{x_1,x_4\}$,
then $S.\tau_1-int(S.\tau_2-cl(S.\tau_3-int(\mathcal{B},\mathbb{P})))=$
$S.\tau_1-int(S.\tau_2-cl((\mathcal{H}_3,\mathbb{P}))= S.\tau_1-int((\mathrm{G}_1,\mathbb{P})^{\mathrm{c}})=(\mathcal{M},\mathbb{P})$
But $(\mathcal{B},\mathbb{P})\nsubseteq(\mathcal{M},\mathbb{P})$. Therefore $(\mathcal{B},\mathbb{P})$ is not a soft δ^*-open set.
And so on to other $soft$ −sets (i.e. examine all 256 $soft$ −sets) ….
Hence all soft δ^*-open sets of the soft tri-topological space $(\mathcal{X},\tau_1,\tau_2,\tau_3,\mathbb{P})$ are,
$S.\delta^*.\mathrm{O}(\mathcal{X})=\{\mathcal{X},\varphi,(\mathcal{S}_i,\mathbb{P})\}$, $(with\ i\ =\ 1,\dots,16\ )$
$\mathcal{S}_1(\mathcal{P}_1)=\{x_3,x_4\}$, $\mathcal{S}_1(\mathcal{P}_2)=\emptyset$
$\mathcal{S}_2(\mathcal{P}_1)=\{x_4\}$, $\mathcal{S}_2(\mathcal{P}_2)=\emptyset$
$\mathcal{S}_3(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_3(\mathcal{P}_2)=\{x_2,x_3,x_4\}$
$\mathcal{S}_4(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_4(\mathcal{P}_2)=\emptyset$
$\mathcal{S}_5(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_5(\mathcal{P}_2)=\{x_1,x_4\}$
$\mathcal{S}_6(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_6(\mathcal{P}_2)=\{x_1,x_3,x_4\}$
$\mathcal{S}_7(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_7(\mathcal{P}_2)=\{x_1,x_2,x_4\}$
$\mathcal{S}_8(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_8(\mathcal{P}_2)=\{x_1,x_2,x_3\}$
$\mathcal{S}_9(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_9(\mathcal{P}_2)=\{x_3,x_4\}$
$\mathcal{S}_{10}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_{10}(\mathcal{P}_2)=\{x_1,x_3\}$
$\mathcal{S}_{11}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_{11}(\mathcal{P}_2)=\{x_2,x_3\}$
$\mathcal{S}_{12}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_{12}(\mathcal{P}_2)=\{x_1,x_2\}$
$\mathcal{S}_{13}(\mathcal{P}_1)=\mathrm{X}$, $\mathcal{S}_{13}(\mathcal{P}_2)=\{x_4\}$

$\mathcal{S}_{14}(\mathcal{P}_1) = \mathrm{X}$, $\mathcal{S}_{14}(\mathcal{P}_2) = \{x_3\}$
$\mathcal{S}_{15}(\mathcal{P}_1) = \mathrm{X}$, $\mathcal{S}_{15}(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_{16}(\mathcal{P}_1) = \mathrm{X}$, $\mathcal{S}_{16}(\mathcal{P}_2) = \{x_1\}$

Now, *let* $\mathrm{Y} = \{x_1, x_3, x_4\} \subseteq \mathrm{X}$. By definition **3.1.1**, the soft sub-sets $(\mathcal{S}_{iY}, \mathbb{P})$ $(with\ i = 1, \dots, 16)$ of the soft $\delta^* - open\ set$ $(\mathcal{S}_i, \mathbb{P})$ over Y, results to be defined by:
$\mathcal{S}_{1\mathrm{Y}}(\mathcal{P}_1) = \{x_3, x_4\}$, $\mathcal{S}_{1\mathrm{Y}}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{2\mathrm{Y}}(\mathcal{P}_1) = \{x_4\}$, $\mathcal{S}_{2\mathrm{Y}}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{3\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{3\mathrm{Y}}(\mathcal{P}_2) = \{x_3, x_4\}$
$\mathcal{S}_{4\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{4\mathrm{Y}}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{5\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{5\mathrm{Y}}(\mathcal{P}_2) = \{x_1, x_4\}$
$\mathcal{S}_{6\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{6\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{7\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{7\mathrm{Y}}(\mathcal{P}_2) = \{x_1, x_4\}$
$\mathcal{S}_{8\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{8\mathrm{Y}}(\mathcal{P}_2) = \{x_1, x_3\}$
$\mathcal{S}_{9\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{9\mathrm{Y}}(\mathcal{P}_2) = \{x_3, x_4\}$
$\mathcal{S}_{10\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{10\mathrm{Y}}(\mathcal{P}_2) = \{x_1, x_3\}$
$\mathcal{S}_{11\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{11\mathrm{Y}}(\mathcal{P}_2) = \{x_3\}$
$\mathcal{S}_{12\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{12\mathrm{Y}}(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{13\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{13\mathrm{Y}}(\mathcal{P}_2) = \{x_4\}$
$\mathcal{S}_{14\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{14\mathrm{Y}}(\mathcal{P}_2) = \{x_3\}$
$\mathcal{S}_{15\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{15\mathrm{Y}}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{16\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{16\mathrm{Y}}(\mathcal{P}_2) = \{x_1\}$
And so, being $(\mathcal{S}_{3Y}, \mathbb{P}) = (\mathcal{S}_{9Y}, \mathbb{P})$, $(\mathcal{S}_{4Y}, \mathbb{P}) = (\mathcal{S}_{15Y}, \mathbb{P})$, $(\mathcal{S}_{5Y}, \mathbb{P}) = (\mathcal{S}_{7Y}, \mathbb{P})$, $(\mathcal{S}_{8Y}, \mathbb{P}) = (\mathcal{S}_{10Y}, \mathbb{P})$, $(\mathcal{S}_{11Y}, \mathbb{P}) = (\mathcal{S}_{14Y}, \mathbb{P})$, $(\mathcal{S}_{12Y}, \mathbb{P}) = (\mathcal{S}_{16Y}, \mathbb{P})$and $(\mathcal{S}_{6Y}, \mathbb{P}) = (\mathrm{Y}, \mathbb{P}) = \mathcal{Y}$, then the soft sub-space of a soft tri-topological space $\mathcal{X}$ *(w.r.t.* $soft - \delta^* - open$ *set)* is the family $\mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}} =$
$\{\mathcal{Y}, \emptyset, (\mathcal{S}_{1Y}, \mathbb{P}), (\mathcal{S}_{2Y}, \mathbb{P}), (\mathcal{S}_{3Y}, \mathbb{P}), (\mathcal{S}_{4Y}, \mathbb{P}), (\mathcal{S}_{5Y}, \mathbb{P}), (\mathcal{S}_{8Y}, \mathbb{P}), (\mathcal{S}_{11Y}, \mathbb{P}), (\mathcal{S}_{12Y}, \mathbb{P}), (\mathcal{S}_{13Y}, \mathbb{P})\}$

Example 3.4. For the sake of clarity, here are some examples of soft sub-space of a soft tri-topological space $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$, the family $S.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$ that forms a soft topology or not.
(i) Let X be an initial universe set, $\mathbb{P}$ be the set of parameters and $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ be a soft tri-topological space, If we consider the family of all soft $\delta^* - open$ sets $S.\delta^*.\mathrm{O}(\mathcal{X}) = \{(\mathrm{X}, \mathbb{P}), (\emptyset, \mathbb{P})\}$, and let $\mathrm{Y} \neq \emptyset$, $\mathrm{Y} \subseteq \mathrm{X}$. Then the family $S.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}} = \{(\mathrm{Y}, \mathbb{P}), (\emptyset, \mathbb{P})\}$ is a soft sub-space of a soft indiscrete tri-topological space $\mathcal{X}$ *(w.r.t.* soft $\delta^* - open$ *set)*, then the soft sub-space $S.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$ is **verify soft topology**.
(ii) Let X be an initial universe set, $\mathbb{P}$ be the set of parameters, and $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ be soft tri-topological space, If we consider the family of soft $\delta^* - open$ sets $S.\delta^*.\mathrm{O}(\mathcal{X}) = \{(\mathcal{S}, \mathbb{P}) | (\mathcal{S}, \mathbb{P}) \text{ is a soft set over X}\}$, and let $\mathrm{Y} \neq \emptyset$, $\mathrm{Y} \subseteq \mathrm{X}$. By definition **3.1**, then the family $\mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}} = \{(\mathcal{S}_Y, \mathbb{P}) | (\mathcal{S}_Y, \mathbb{P}) \text{ is a soft set over Y}\}$ is a soft sub-space of a soft discrete tri-topological space $\mathcal{X}$ *(w.r.t.* $soft - \delta^* - open$ *set)*, then the soft sub-space $\mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$ is **verify soft topology**.
(iii) Let $\mathcal{X} = \{x_1, x_2, x_3\}$ be a universe set and $\mathbb{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$ be the set of parameters, The soft set $(\mathrm{X}, \mathbb{P}) = \{(\mathcal{P}_1, \{x_1, x_2, x_3\}), (\mathcal{P}_2, \{x_1, x_2, x_3\})\}$. Then by [4], its cardinality is $|\mathrm{P}(\mathcal{X})| = 2^{\sum_{e \in \mathbb{P}} |\mathcal{S}(e)|}$, s.t. $|\mathcal{S}(\mathrm{e})|$ is the cardinality of $\mathcal{S}(\mathrm{e})$. (that is mean $|\mathcal{P}(\mathcal{X})| = 2^6 = 64$ $soft$ −set).
Let $(\mathcal{X}, \tau_1, \mathbb{P})$, $(\mathcal{X}, \tau_2, \mathbb{P})$ and $(\mathcal{X}, \tau_3, \mathbb{P})$ be the three soft topological spaces on $\mathcal{X}$, where their soft topologies defined as follows:
$\tau_1 = \{\varphi, \mathcal{X}, (\mathcal{M}, \mathbb{P})\}$
$\tau_2 = \{\varphi, \mathcal{X}, (\mathrm{G}_1, \mathbb{P}), (\mathrm{G}_2, \mathbb{P}), (\mathrm{G}_3, \mathbb{P})\}$
$\tau_3 = \{\varphi, \mathcal{X}, (\mathcal{H}_1, \mathbb{P}), (\mathcal{H}_2, \mathbb{P}), (\mathcal{H}_3, \mathbb{P})\}$.
Where $(\mathcal{M}, \mathbb{P})$, $(\mathrm{G}_1, \mathbb{P})$, $(\mathrm{G}_2, \mathbb{P})$, $(\mathrm{G}_3, \mathbb{P})$, $(\mathcal{H}_1, \mathbb{P})$, $(\mathcal{H}_2, \mathbb{P})$, $(\mathcal{H}_3, \mathbb{P})$ are soft *open* sets over X, defined as follows:
$\mathcal{M}(\mathcal{P}_1) = \{x_2\}$, $\mathcal{M}(\mathcal{P}_2) = \{x_1, x_2\}$

$\mathrm{G}_1(\mathcal{P}_1) = \{x_2\}$, $\mathrm{G}_1(\mathcal{P}_2) = \{x_2\}$
$\mathrm{G}_2(\mathcal{P}_1) = \{x_1, x_2\}$, $\mathrm{G}_2(\mathcal{P}_2) = \mathrm{X}$
$\mathrm{G}_3(\mathcal{P}_1) = \{x_1\}$, $\mathrm{G}_3(\mathcal{P}_2) = \{x_1, x_3\}$

$\mathcal{H}_1(\mathcal{P}_1) = \emptyset$, $\mathcal{H}_1(\mathcal{P}_2) = \{x_2\}$
$\mathcal{H}_2(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{H}_2(\mathcal{P}_2) = \{x_1\}$
$\mathcal{H}_3(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{H}_3(\mathcal{P}_2) = \{x_1, x_2\}$
Then the complement of soft *open* sets of the soft topology τ_2 Are ;
${\mathrm{G}_1}^{\mathrm{c}}(\mathcal{P}_1) = \{x_1, x_3\}$, ${\mathrm{G}_1}^{\mathrm{c}}(\mathcal{P}_2) = \{x_1, x_3\}$

$G_2{}^c(\mathcal{P}_1) = \{x_3\}$, $G_2{}^c(\mathcal{P}_2) = \emptyset$
$G_3{}^c(\mathcal{P}_1) = \{x_2, x_3\}$, $G_3{}^c(\mathcal{P}_2) = \{x_2\}$
Hence, the family of all soft $\delta^* - open$ sets of soft tri-topological space $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ is,
$S.\delta^*.O(\mathcal{X}) = \{\mathcal{X}, \varphi, (\mathcal{S}_{iY}, \mathbb{P})\}$. Where (with $i = 1, \ldots, 7$),
$\mathcal{S}_1(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{S}_1(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_2(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{S}_2(\mathcal{P}_2) = \{x_1, x_2\}$
$\mathcal{S}_3(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{S}_3(\mathcal{P}_2) = \{x_1, x_3\}$
$\mathcal{S}_4(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{S}_4(\mathcal{P}_2) = X$
$\mathcal{S}_5(\mathcal{P}_1) = X$, $\mathcal{S}_5(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_6(\mathcal{P}_1) = X$, $\mathcal{S}_6(\mathcal{P}_2) = \{x_1, x_2\}$
$\mathcal{S}_7(\mathcal{P}_1) = X$, $\mathcal{S}_7(\mathcal{P}_2) = \{x_1, x_3\}$
It is clear, the family $S.\delta^*.O(\mathcal{X})$ **verify soft topology**.
Now, let $Y = \{x_1, x_2\} \subseteq X$. By definition **3.1.1**, the soft sub sets $(\mathcal{S}_{iY}, \mathbb{P})$ (with $i = 1, \ldots, 7$) of the $soft - \delta^* - open$ set $(\mathcal{S}_i, \mathbb{P})$ over Y, results to be defined by:
$\mathcal{S}_{1Y}(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_{1Y}(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{2Y}(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_{2Y}(\mathcal{P}_2) = Y$
$\mathcal{S}_{3Y}(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_{3Y}(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{4Y}(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_{4Y}(\mathcal{P}_2) = Y$
$\mathcal{S}_{5Y}(\mathcal{P}_1) = Y$, $\mathcal{S}_{5Y}(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{6Y}(\mathcal{P}_1) = Y$, $\mathcal{S}_{6Y}(\mathcal{P}_2) = Y$
$\mathcal{S}_{7Y}(\mathcal{P}_1) = Y$, $\mathcal{S}_{7Y}(\mathcal{P}_2) = \{x_1\}$
And so, being $(\mathcal{S}_{1Y}, \mathbb{P}) = (\mathcal{S}_{3Y}, \mathbb{P})$, $(\mathcal{S}_{2Y}, \mathbb{P}) = (\mathcal{S}_{4Y}, \mathbb{P})$, $(\mathcal{S}_{5Y}, \mathbb{P}) = (\mathcal{S}_{7Y}, \mathbb{P})$ and $(\mathcal{S}_{6Y}, \mathbb{P}) = (Y, \mathbb{P}) = \mathcal{Y}$, then the soft sub-space of a soft tri-topological space $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ (w.r.t. soft $\delta^* - open$ *set),* the family $S.\delta^*.O(\mathcal{X})_Y = \{\mathcal{Y}, \varphi, (\mathcal{S}_{1Y}, \mathbb{P}), (\mathcal{S}_{2Y}, \mathbb{P}), (\mathcal{S}_{5Y}, \mathbb{P})\}$
Since, it is a very simple routine to verify that all soft unions and soft intersections of members in $S.\delta^*.O(\mathcal{X})_Y$ still belong to $S.\delta^*.O(\mathcal{X})_Y$, it follows $S.\delta^*.O(\mathcal{X})_Y$ is a soft topology on Y.
Also, the example **3.1.3** shows that the family $S.\delta^*.O(\mathcal{X})$ verify soft topology over X, and the soft sub-space (*i.e. the family* $S.\delta^*.O(\mathcal{X})_Y$) of a soft tri-topological space $\mathcal{X}$ (w.r.t. soft $\delta^* - open$ set), also **verify soft topology** over Y.
(iv) Let $\mathcal{X} = \{x_1, x_2\}$ be the universe set and $\mathbb{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$ be the set of parameters. By [4], its cardinality defined by: $|P(\mathcal{X})| = 2^{\sum_{\mathcal{P}\in\mathbb{P}}|\mathcal{S}(\mathcal{P})|}$, where $|\mathcal{S}(\mathcal{P})|$ is the cardinality of $\mathcal{S}(\mathcal{P})$. (i.e. $|(P\mathcal{X})| = 2^4 = 16$ soft set)
Then the all soft sets of X are:
$\mathcal{S}_1(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_1(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_2(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_2(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_3(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_3(\mathcal{P}_2) = X$
$\mathcal{S}_4(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_4(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_5(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_5(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_6(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_6(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_7(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_7(\mathcal{P}_2) = X$
$\mathcal{S}_8(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_8(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_9(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_9(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{10}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{10}(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_{11}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{11}(\mathcal{P}_2) = X$
$\mathcal{S}_{12}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{12}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{13}(\mathcal{P}_1) = X$, $\mathcal{S}_{13}(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{14}(\mathcal{P}_1) = X$, $\mathcal{S}_{14}(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_{15}(\mathcal{P}_1) = X$, $\mathcal{S}_{15}(\mathcal{P}_2) = X$
$\mathcal{S}_{16}(\mathcal{P}_1) = X$, $\mathcal{S}_{16}(\mathcal{P}_2) = \emptyset$
Let $(\mathcal{X}, \tau_1, \mathbb{P})$, $(\mathcal{X}, \tau_2, \mathbb{P})$ and $(\mathcal{X}, \tau_3, \mathbb{P})$ be a three soft topological spaces on X, where their soft topologies defined as:
$\tau_1 = \{\emptyset, \mathcal{X}, (\mathcal{M}, \mathbb{P})\}$
$\tau_2 = \{\emptyset, \mathcal{X}, (G_1, \mathbb{P}), (G_2, \mathbb{P}), (G_3, \mathbb{P}), (G_4, \mathbb{P})\}$ $\tau_3 = \{\emptyset, \mathcal{X}, (\mathcal{H}_1, \mathbb{P}), (\mathcal{H}_2, \mathbb{P}), (\mathcal{H}_3, \mathbb{P}), (\mathcal{H}_4, \mathbb{P}), \ldots, (\mathcal{H}_9, \mathbb{P})\}$
Where all $soft - open$ sets over X above, are defined as:
$\mathcal{M}(\mathcal{P}_1) = \{x_1\}$, $\mathcal{M}(\mathcal{P}_2) = \{x_2\}$

$G_1(\mathcal{P}_1) = \{x_1\}$, $G_1(\mathcal{P}_2) = \emptyset$
$G_2(\mathcal{P}_1) = \{x_2\}$, $G_2(\mathcal{P}_2) = \{x_1\}$
$G_3(\mathcal{P}_1) = \{x_2\}$, $G_3(\mathcal{P}_2) = X$

$G_4(\mathcal{P}_1) = X$, $G_4(\mathcal{P}_2) = \{x_1\}$

$\mathcal{H}_1(\mathcal{P}_1) = \{x_1\}$, $\mathcal{H}_1(\mathcal{P}_2) = \{x_2\}$
$\mathcal{H}_2(\mathcal{P}_1) = X$, $\mathcal{H}_2(\mathcal{P}_2) = \{x_2\}$
$\mathcal{H}_3(\mathcal{P}_1) = \emptyset$, $\mathcal{H}_3(\mathcal{P}_2) = X$
$\mathcal{H}_4(\mathcal{P}_1) = X$, $\mathcal{H}_4(\mathcal{P}_2) = \{x_1\}$
$\mathcal{H}_5(\mathcal{P}_1) = \emptyset$, $\mathcal{H}_5(\mathcal{P}_2) = \{x_2\}$
$\mathcal{H}_6(\mathcal{P}_1) = \{x_1\}$, $\mathcal{H}_6(\mathcal{P}_2) = \emptyset$
$\mathcal{H}_7(\mathcal{P}_1) = \emptyset$, $\mathcal{H}_7(\mathcal{P}_2) = \{x_1\}$
$\mathcal{H}_8(\mathcal{P}_1) = X$, $\mathcal{H}_8(\mathcal{P}_2) = \emptyset$
$\mathcal{H}_9(\mathcal{P}_1) = \{x_1\}$, $\mathcal{H}_9(\mathcal{P}_2) = \{x_1\}$

Then the complement of all soft $open$ sets of the soft topology τ_2 Are ;
$G_1{}^c(\mathcal{P}_1) = \{x_2\}$, $G_1{}^c(\mathcal{P}_2) = X$
$G_2{}^c(\mathcal{P}_1) = \{x_1\}$, $G_2{}^c(\mathcal{P}_2) = \{x_2\}$
$G_3{}^c(\mathcal{P}_1) = \{x_1\}$, $G_3{}^c(\mathcal{P}_2) = \emptyset$
$G_4{}^c(\mathcal{P}_1) = \emptyset$, $G_4{}^c(\mathcal{P}_2) = \{x_2\}$

The family of all soft $\delta^* - open$ sets of the soft tri-topological space $(\mathcal{X}, \tau_1, \tau_2, \tau_3, E)$ is,
$S.\delta^*.O(\mathcal{X}) = \{\mathcal{X}, \varphi, (\mathcal{S}_1, \mathbb{P}), (\mathcal{S}_2, \mathbb{P}), (\mathcal{S}_3, \mathbb{P}), (\mathcal{S}_{13}, \mathbb{P}), (\mathcal{S}_{14}, \mathbb{P}), (\mathcal{S}_{16}, \mathbb{P})\}$, remember
$\mathcal{S}_1(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_1(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_2(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_2(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_3(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_3(\mathcal{P}_2) = \mathcal{X}$
$\mathcal{S}_{13}(\mathcal{P}_1) = \mathcal{X}$, $\mathcal{S}_{13}(\mathcal{P}_2) = \{x_1\}$
$\mathcal{S}_{14}(\mathcal{P}_1) = \mathcal{X}$, $\mathcal{S}_{14}(\mathcal{P}_2) = \{x_2\}$
$\mathcal{S}_{16}(\mathcal{P}_1) = \mathcal{X}$, $\mathcal{S}_{16}(\mathcal{P}_2) = \emptyset$

Clearly, $S.\delta^*.O(\mathcal{X})$ not verify soft topology over $\mathcal{X}$, because $(\mathcal{S}_1, \mathbb{P}) \cap (\mathcal{S}_2, \mathbb{P})$ is not belong to $S.\delta^*.O(\mathcal{X})$.
Now, let $Y = \{x_2\} \subseteq X$. By definition **3.1.1**, the soft sub sets $(\mathcal{S}_{iY}, \mathbb{P})$ (with $i = 1,2,3,13,14,16$) of the soft $\delta^* - open\ set$ $(\mathcal{S}_i, \mathbb{P})$ over Y, results to be defined by:
$\mathcal{S}_{1Y}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{1Y}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{2Y}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{2Y}(\mathcal{P}_2) = Y$
$\mathcal{S}_{3Y}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{3Y}(\mathcal{P}_2) = Y$
$\mathcal{S}_{13Y}(\mathcal{P}_1) = Y$, $\mathcal{S}_{13Y}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{14Y}(\mathcal{P}_1) = Y$, $\mathcal{S}_{14Y}(\mathcal{P}_2) = Y$
$\mathcal{S}_{16Y}(\mathcal{P}_1) = Y$, $\mathcal{S}_{16Y}(\mathcal{P}_2) = \emptyset$

and so, being $(\mathcal{S}_{2Y}, \mathbb{P}) = (\mathcal{S}_{3Y}, \mathbb{P})$ and $(\mathcal{S}_{13Y}, \mathbb{P}) = (\mathcal{S}_{16Y}, \mathbb{P})$, then the soft sub-space of a soft tri-topological space $\mathcal{X}$ (w.r.t. soft $\delta^* - open$ set) is the family;
$S.\delta^*.O(\mathcal{X})_Y = \{\mathcal{Y}, \varphi, (\mathcal{S}_{2Y}, \mathbb{P}), (\mathcal{S}_{13Y}, \mathbb{P})\}$
Since, it is a very simple routine to verify that all soft unions and soft intersections of members in $S.\delta^*.O(\mathcal{X})_Y$ still belong to $S.\delta^*.O(\mathcal{X})_Y$, it follows $S.\delta^*.O(\mathcal{X})_Y$ is a soft topology on Y.
This example shows that $S.\delta^*.O(\mathcal{X})$ **not verify soft topology** over X, but the soft sub-space (i.e. the family $S.\delta^*.O(\mathcal{X})_Y$) of a soft tri-topological space $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ (w.r.t. soft $\delta^* - open$ set), **verify soft topology** over Y.
(v) Let $X = \{x_1, x_2, x_3, x_4\}$ be the universe set and $\mathbb{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$ be the set of parameters. By [4], then its cardinality defined as: $|P(\mathcal{X})| = 2^{\sum_{\mathcal{P}\in\mathbb{P}}|\mathcal{S}(\mathcal{P})|}$, where $|\mathcal{S}(\mathcal{P})|$ is the cardinality of $\mathcal{S}(\mathcal{P})$. (i.e. $|P(\mathcal{X})| = 2^8 = 256$ soft set)
And let $(\mathcal{X}, \tau_1, E)$, $(\mathcal{X}, \tau_2, E)$ and $(\mathcal{X}, \tau_3, E)$ be the three soft topological spaces on X, where their soft topologies defined as follows:
$\tau_1 = \{\varphi, \mathcal{X}, (\mathcal{M}_1, E), (\mathcal{M}_2, E), (\mathcal{M}_3, E), (\mathcal{M}_4, E)\}$,
$\tau_2 = \{\varphi, \mathcal{X}, (G_1, E), (G_2, E), \dots, (G_{17}, E)\}$, $\tau_3 = \{\varphi, \mathcal{X}, (\mathcal{H}_1, E), (\mathcal{H}_2, E), (\mathcal{H}_3, E), (\mathcal{H}_4, E), \dots, (\mathcal{H}_8, E)\}$
And all τ_1, τ_2 , τ_3 soft open sets over X above are defined as follows:
$\mathcal{M}_1(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{M}_1(\mathcal{P}_2) = \emptyset$
$\mathcal{M}_2(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{M}_2(\mathcal{P}_2) = \emptyset$
$\mathcal{M}_3(\mathcal{P}_1) = \{x_3\}$, $\mathcal{M}_3(\mathcal{P}_2) = \emptyset$
$\mathcal{M}_4(\mathcal{P}_1) = \{x_1, x_2, x_3\}$, $\mathcal{M}_4(\mathcal{P}_2) = \emptyset$
$G_1(\mathcal{P}_1) = \{x_1, x_4\}$, $G_1(\mathcal{P}_2) = X$
$G_2(\mathcal{P}_1) = \{x_2, x_4\}$, $G_2(\mathcal{P}_2) = X$
$G_3(\mathcal{P}_1) = \{x_4\}$, $G_3(\mathcal{P}_2) = X$
$G_4(\mathcal{P}_1) = \{x_1, x_2, x_4\}$, $G_4(\mathcal{P}_2) = X$
$G_5(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $G_5(\mathcal{P}_2) = X$
$G_6(\mathcal{P}_1) = \{x_1, x_4\}$, $G_6(\mathcal{P}_2) = \{x_4\}$

$G_7(\mathcal{P}_1) = \{x_4\}$, $G_7(\mathcal{P}_2) = \{x_4\}$
$G_8(\mathcal{P}_1) = \emptyset$, $G_8(\mathcal{P}_2) = \{x_4\}$
$G_9(\mathcal{P}_1) = \{x_1, x_2, x_4\}$, $G_9(\mathcal{P}_2) = \{x_4\}$
$G_{10}(\mathcal{P}_1) = \{x_1, x_4\}$, $G_{10}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{11}(\mathcal{P}_1) = \{x_4\}$, $G_{11}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{12}(\mathcal{P}_1) = \{x_3, x_4\}$, $G_{12}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{13}(\mathcal{P}_1) = X$, $G_{13}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{14}(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $G_{14}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{15}(\mathcal{P}_1) = \{x_3\}$, $G_{15}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{16}(\mathcal{P}_1) = \emptyset$, $G_{16}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$G_{17}(\mathcal{P}_1) = \{x_1, x_2, x_4\}$, $G_{17}(\mathcal{P}_2) = \{x_2, x_3, x_4\}$
$\mathcal{H}_1(\mathcal{P}_1) = \{x_{1,}, x_3, x_4\}$, $\mathcal{H}_1(\mathcal{P}_2) = \emptyset$
$\mathcal{H}_2(\mathcal{P}_1) = \{x_{1,}, x_3, x_4\}$, $\mathcal{H}_2(\mathcal{P}_2) = X$
$\mathcal{H}_3(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{H}_3(\mathcal{P}_2) = \emptyset$
$\mathcal{H}_4(\mathcal{P}_1) = X$, $\mathcal{H}_4(\mathcal{P}_2) = \emptyset$
$\mathcal{H}_5(\mathcal{P}_1) = \emptyset$, $\mathcal{H}_5(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{H}_6(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $\mathcal{H}_6(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{H}_7(\mathcal{P}_1) = X$, $\mathcal{H}_7(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{H}_8(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{H}_8(\mathcal{P}_2) = \{x_1, x_3, x_4\}$

The complement of the soft open sets of soft topology τ_2 Are ;

${G_1}^c(\mathcal{P}_1) = \{x_2, x_3\}$, ${G_1}^c(\mathcal{P}_2) = \emptyset$
${G_2}^c(\mathcal{P}_1) = \{x_1, x_3\}$, ${G_2}^c(\mathcal{P}_2) = \emptyset$
${G_3}^c(\mathcal{P}_1) = \{x_1, x_2, x_3\}$, ${G_3}^c(\mathcal{P}_2) = \emptyset$
${G_4}^c(\mathcal{P}_1) = \{x_3\}$, ${G_4}^c(\mathcal{P}_2) = \emptyset$
${G_5}^c(\mathcal{P}_1) = \{x_2\}$, ${G_5}^c(\mathcal{P}_2) = \emptyset$
${G_6}^c(\mathcal{P}_1) = \{x_2, x_3\}$, ${G_6}^c(\mathcal{P}_2) = \{x_1, x_2, x_3\}$
${G_7}^c(\mathcal{P}_1) = \{x_1, x_2, x_3\}$, ${G_7}^c(\mathcal{P}_2) = \{x_1, x_2, x_3\}$
${G_8}^c(\mathcal{P}_1) = X$, ${G_8}^c(\mathcal{P}_2) = \{x_1, x_2, x_3\}$
${G_9}^c(\mathcal{P}_1) = \{x_3\}$, ${G_9}^c(\mathcal{P}_2) = \{x_1, x_2, x_3\}$
${G_{10}}^c(\mathcal{P}_1) = \{x_2, x_3\}$, ${G_{10}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{11}}^c(\mathcal{P}_1) = \{x_1, x_2, x_3\}$, ${G_{11}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{12}}^c(\mathcal{P}_1) = \{x_1, x_2\}$, ${G_{12}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{13}}^c(\mathcal{P}_1) = \emptyset$, ${G_{13}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{14}}^c(\mathcal{P}_1) = \{x_2\}$, ${G_{14}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{15}}^c(\mathcal{P}_1) = \{x_1, x_2, x_4\}$, ${G_{15}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{16}}^c(\mathcal{P}_1) = X$, ${G_{16}}^c(\mathcal{P}_2) = \{x_1\}$
${G_{17}}^c(\mathcal{P}_1) = \{x_3\}$, ${G_{17}}^c(\mathcal{P}_2) = \{x_1\}$

Hence the family of all soft $\delta^* - open$ sets of the soft tri-topological space $(\mathcal{X}, \tau_1, \tau_2, \tau_3, E)$ is,
$S.\delta^*.O(\mathcal{X}) = \{\mathcal{X}, \varphi, (\mathcal{S}_i, \mathbb{P}): i = 1, 2, \ldots, 21\}$ where;

$\mathcal{S}_1(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $\mathcal{S}_1(\mathcal{P}_2) = X$
$\mathcal{S}_2(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{S}_2(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_3(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_3(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_4(\mathcal{P}_1) = X$, $\mathcal{S}_4(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_5(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_5(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_6(\mathcal{P}_1) = \{x_2\}$, $\mathcal{S}_6(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_7(\mathcal{P}_1) = \{x_3\}$, $\mathcal{S}_7(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_8(\mathcal{P}_1) = \{x_4\}$, $\mathcal{S}_8(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_9(\mathcal{P}_1) = \{x_1, x_2\}$, $\mathcal{S}_9(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{10}(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{S}_{10}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{11}(\mathcal{P}_1) = \{x_2, x_3\}$, $\mathcal{S}_{11}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{12}(\mathcal{P}_1) = \{x_1, x_4\}$, $\mathcal{S}_{12}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{13}(\mathcal{P}_1) = \{x_2, x_4\}$, $\mathcal{S}_{13}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{14}(\mathcal{P}_1) = \{x_3, x_4\}$, $\mathcal{S}_{14}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{15}(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $\mathcal{S}_{15}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{16}(\mathcal{P}_1) = \{x_1, x_2, x_4\}$, $\mathcal{S}_{16}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$

$\mathcal{S}_{17}(\mathcal{P}_1) = \{x_1, x_2, x_3\}$, $\mathcal{S}_{17}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{18}(\mathcal{P}_1) = \{x_2, x_3, x_4\}$, $\mathcal{S}_{18}(\mathcal{P}_2) = \{x_1, x_3, x_4\}$
$\mathcal{S}_{19}(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $\mathcal{S}_{19}(\mathcal{P}_2) = \{x_1, x_4\}$
$\mathcal{S}_{20}(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $\mathcal{S}_{20}(\mathcal{P}_2) = \{x_2, x_4\}$
$\mathcal{S}_{21}(\mathcal{P}_1) = \{x_1, x_3, x_4\}$, $\mathcal{S}_{21}(\mathcal{P}_2) = \{x_3, x_4\}$

Thus, $S.\delta^*.\mathrm{O}(\mathcal{X})$ **does not verify a soft topology**. Because, the soft intersection of the two soft $\delta^* - open$ sets is not a soft $\delta^* - open$ set, and more exactly that; $(\mathcal{S}_2, E) \cap (\mathcal{S}_5, E) = (\mathcal{A}, E)$, where $\mathcal{A}(\mathcal{P}_1) = \{x_1\}$, $\mathcal{A}(\mathcal{P}_2) = \emptyset$ and $(\mathcal{A}, E) \notin S.\delta^*.\mathrm{O}(\mathcal{X})$.
Now, let $\mathrm{Y} = \{x_1, x_3, x_4\} \subseteq \mathrm{X}$. By the definition **3.1.1**, the soft sub-sets $(\mathcal{S}_{i\mathrm{Y}}, \mathbb{P})$ (with $\mathrm{i} = 1, 2, \ldots, 21$) of the soft $\delta^* - open$ set $(\mathcal{S}_{\mathrm{i}}, \mathbb{P})$ over Y, results to be defined by:

$\mathcal{S}_{1\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{1\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{2\mathrm{Y}}(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{S}_{2\mathrm{Y}}(\mathcal{P}_2) = \emptyset$
$\mathcal{S}_{3\mathrm{Y}}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{3\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{4\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{4\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{5\mathrm{Y}}(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_{5\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{6\mathrm{Y}}(\mathcal{P}_1) = \emptyset$, $\mathcal{S}_{6\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{7\mathrm{Y}}(\mathcal{P}_1) = \{x_3\}$, $\mathcal{S}_{7\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{8\mathrm{Y}}(\mathcal{P}_1) = \{x_4\}$, $\mathcal{S}_{8\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{9\mathrm{Y}}(\mathcal{P}_1) = \{x_1\}$, $\mathcal{S}_{9\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{10\mathrm{Y}}(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{S}_{10\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{11\mathrm{Y}}(\mathcal{P}_1) = \{x_3\}$, $\mathcal{S}_{11\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{12\mathrm{Y}}(\mathcal{P}_1) = \{x_1, x_4\}$, $\mathcal{S}_{12\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{13\mathrm{Y}}(\mathcal{P}_1) = \{x_4\}$, $\mathcal{S}_{13\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{14\mathrm{Y}}(\mathcal{P}_1) = \{x_3, x_4\}$, $\mathcal{S}_{14\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{15\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{15\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{16\mathrm{Y}}(\mathcal{P}_1) = \{x_1, x_4\}$, $\mathcal{S}_{16\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{17\mathrm{Y}}(\mathcal{P}_1) = \{x_1, x_3\}$, $\mathcal{S}_{17\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{18\mathrm{Y}}(\mathcal{P}_1) = \{x_3, x_4\}$, $\mathcal{S}_{18\mathrm{Y}}(\mathcal{P}_2) = \mathrm{Y}$
$\mathcal{S}_{19\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{19\mathrm{Y}}(\mathcal{P}_2) = \{x_1, x_4\}$
$\mathcal{S}_{20\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{20\mathrm{Y}}(\mathcal{P}_2) = \{x_4\}$
$\mathcal{S}_{21\mathrm{Y}}(\mathcal{P}_1) = \mathrm{Y}$, $\mathcal{S}_{21\mathrm{Y}}(\mathcal{P}_2) = \{x_3, x_4\}$

And so, being $(\mathcal{S}_{2Y}, \mathbb{P}) = (\mathcal{S}_{10Y}, \mathbb{P}) = (\mathcal{S}_{17Y}, \mathbb{P})$, $(\mathcal{S}_{3Y}, \mathbb{P}) = (\mathcal{S}_{6Y}, \mathbb{P})$, $(\mathcal{S}_{5Y}, \mathbb{P}) = (\mathcal{S}_{9Y}, \mathbb{P})$, $(\mathcal{S}_{7Y}, \mathbb{P}) = (\mathcal{S}_{11Y}, \mathbb{P})$, $(\mathcal{S}_{8Y}, \mathbb{P}) = (\mathcal{S}_{13Y}, \mathbb{P})$, $(\mathcal{S}_{12Y}, \mathbb{P}) = (\mathcal{S}_{16Y}, \mathbb{P})$, $(\mathcal{S}_{14Y}, \mathbb{P}) = (\mathcal{S}_{18Y}, \mathbb{P})$ and $(\mathcal{S}_{1Y}, \mathbb{P}) = (\mathcal{S}_{4Y}, \mathbb{P}) = (\mathcal{S}_{15Y}, \mathbb{P}) = \mathcal{Y}$, then the soft sub-space of a soft tri-topological space $\mathcal{X}$ (w.r.t. soft $\delta^* - open$ set) *is* the family;
$S.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}} = \{\mathcal{Y}, \varphi, (\mathcal{S}_{2Y}, \mathbb{P}), (\mathcal{S}_{3Y}, \mathbb{P}), (\mathcal{S}_{5Y}, \mathbb{P}), (\mathcal{S}_{7Y}, \mathbb{P}), (\mathcal{S}_{8Y}, \mathbb{P}),$
$(\mathcal{S}_{12Y}, \mathbb{P}), (\mathcal{S}_{14Y}, \mathbb{P}), (\mathcal{S}_{19Y}, \mathbb{P}), (\mathcal{S}_{20Y}, \mathbb{P}), (\mathcal{S}_{21Y}, \mathbb{P})\}$.
Thus, $\mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$ is not verified a soft topology. Because the soft intersection of the two soft ${\delta^*}_Y - open$ sets $(\mathcal{S}_{2Y}, E)$ and $(\mathcal{S}_{5Y}, E)$ is not a soft ${\delta^*}_Y - open$ set, for more exactly that $(\mathcal{S}_{2Y}, E) \cap (\mathcal{S}_{5Y}, E) = (\mathcal{A}_Y, E)$, where $\mathcal{A}_Y(\mathcal{P}_1) = \{x_1\}$, $\mathcal{A}_Y(\mathcal{P}_2) = \emptyset$ and $(\mathcal{A}_Y, E) \notin S.\delta^*.\mathrm{O}(\mathcal{X})$. It follows $\mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$ is a soft topology on Y.
This example shows that $S.\delta^*.\mathrm{O}(\mathcal{X})$ is not verified soft topology over X, and the soft sub-space of a soft tri-topological space $\mathcal{X}$ (w.r.t. soft $\delta^* - open$ set), also **does not verify soft topology** over Y.

Remark 3.5. From all the previous examples we can conclude :
1-When $S.\delta^*.\mathrm{O}(\mathcal{X})$ verify soft topology over X, then the soft sub-space of a soft tri-topological space $\mathcal{X}$ (w.r.t. soft $\delta^* - open$ set), is always verify soft topology over Y.
2- When $S.\delta^*.\mathrm{O}(\mathcal{X})$ is not verify soft topology over X, then the soft sub-space of a soft tri-topological space $\mathcal{X}$ (w.r.t. soft $\delta^* - open$ set), is not always (may be verify or not verify) a soft topology over Y.

Proposition 3.6. Let $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ be a soft tri-topological space over $\mathcal{X}$ and Y be a non-empty sub-set of X. Then $(\mathrm{Y}, \tau_{i\mathcal{P}\mathrm{Y}})$ are sub-spaces of the topological spaces $(\mathrm{X}, \tau_{i\mathcal{P}})$ for each $\in \mathbb{P}$, where $i = 1,2,3$.
Proof.
Since τ_1, τ_2 and τ_3 are soft topologies over X, then every $(\mathrm{Y}, \tau_{i\mathrm{Y}}, \mathbb{P})$, $i = 1,2,3$, are soft topological spaces over Y [23] so $(\mathrm{Y}, \tau_{i\mathcal{P}\mathrm{Y}})$ are topological spaces for each $\mathcal{P} \in \mathbb{P}$, and $i = 1,2,3$ [45].
Now, by definition of soft sub-space, for any $\in \mathbb{P}$, and $i = 1,2,3$.

$$\begin{aligned}\tau_{i\mathcal{P}\mathrm{Y}} &= \{\mathcal{S}_{\mathrm{Y}}(\mathcal{P}) \mid (\mathcal{S}, \mathbb{P}) \in \tau_i\}\\ &= \{\mathrm{Y} \cap \mathcal{S}(\mathcal{P}) \mid (\mathcal{S}, \mathbb{P}) \in \tau_i\}\\ &= \{\mathrm{Y} \cap \mathcal{S}(\mathcal{P}) \mid \mathcal{S}(\mathcal{P}) \in \tau_{i\mathcal{P}}\}\end{aligned}$$

Thus $(\mathrm{Y}, \tau_{i\mathcal{P}\mathrm{Y}})$ are sub-spaces of the topological spaces $(\mathcal{X}, \tau_{i\mathcal{P}})$ for each $\in \mathbb{P}$, where $i = 1,2,3$.

Note 3.7. All the following Propositions and Theorems are considered with respect to a soft $\delta^* - open$ set. And are valid when the family $S.\delta^*.\mathrm{O}(\mathcal{X})$ verify soft topology over $\mathcal{X}$.

Proposition 3.8. [12] Let $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{E})$ be a soft tri-topological space over X. Then, for each $e \in \mathbb{E}$, , the family $\delta^*.\tau_{\mathrm{e}} = \{F(e) | (F, E) \in S.\delta^*.\mathrm{O}(\mathcal{X})\}$ represent a crisp topology on X.
Remark 3.9.[12] Proposition 3.8 above shows that every family of soft $\delta^* - open$ set $S.\delta^*.\mathrm{O}(\mathcal{X})$ gives a parameterized family $\{(\mathrm{X}, \delta^*.\tau_{\mathrm{e}})\}_{e\in\mathbb{E}}$ of topological spaces.

Proposition 3.10. Let $(\mathcal{X}, \tau_1, \tau_2, \tau_3, \mathbb{P})$ be a soft tri-topological space over $\mathcal{X}$ and Y be a non-empty sub-set of $\mathcal{X}$. Then $\{(\mathrm{Y}, \delta^*.\tau_{\mathrm{Y}\mathcal{P}})\}_{\mathcal{P}\in\mathbb{P}}$ are sub-spaces of the parameterized family $\{(\mathrm{X}, \delta^*.\tau_{\mathcal{P}})\}_{\mathcal{P}\in\mathbb{P}}$ of topological spaces.
Proof. Since the soft $\mathrm{sub} - \mathrm{space}$ Y of a soft tri-topological space $\mathcal{X}$ is a soft topological space over Y, so $\{(\mathrm{Y}, \delta^*.\tau_{\mathrm{Y}\mathcal{P}})\}_{\mathcal{P}\in\mathbb{P}}$ are topological spaces to each parameter $\mathcal{P} \in \mathbb{P}$ by remark **3.9** .
Now, by definition of soft sub-space, for any $\in \mathbb{P}$,

$$\begin{aligned}\delta^*.\tau_{\mathrm{Y}\mathcal{P}} &= \{\mathcal{S}_{\mathrm{Y}}(\mathcal{P}) | (\mathcal{S}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})\}\\ &= \{\mathrm{Y} \cap \mathcal{S}(\mathcal{P}) | (\mathcal{S}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})\}\\ &= \{\mathrm{Y} \cap \mathcal{S}(\mathcal{P}) | \mathcal{S}(\mathcal{P}) \in \delta^*.\tau_{\mathcal{P}}\}\end{aligned}$$

Thus $\{(\mathrm{Y}, \delta^*.\tau_{\mathrm{Y}\mathcal{P}})\}_{\mathcal{P}\in\mathbb{P}}$ are sub-spaces of the parameterized family of topological spaces $\{(\mathrm{X}, \delta^*.\tau_{\mathcal{P}})\}_{\mathcal{P}\in\mathbb{P}}$.

Example 3.11. From the example **3.3** , since $\mathbb{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$ we find:
$\delta^*.\tau_{\mathcal{P}_1} = \{X, \emptyset, \{x_4\}, \{x_3, x_4\}\}$
$\delta^*.\tau_{\mathcal{P}_2} = \{X, \emptyset, \{x_1\}, \{x_2\}\{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\},$
$\{x_1, x_4\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\}$
$\delta^*.\tau_{Y\mathcal{P}_1} = \{\mathrm{Y}, \emptyset, \{x_4\}, \{x_3, x_4\}\}$
$\delta^*.\tau_{Y\mathcal{P}_2} = \{\mathrm{Y}, \emptyset, \{x_1\}, \{x_3\}, \{x_4\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_3, x_4\}\}$
Clear that, $\delta^*.\tau_{Y\mathcal{P}_1}$ and $\delta^*.\tau_{Y\mathcal{P}_2}$ represent a topology on Y.

Proposition 3.12. Let $\mathbb{Y}$ be a soft sub-space of a soft tri-topological space $\mathcal{X}$, when $S.\delta^*.\mathrm{O}(\mathcal{X})$ verify a soft topology, and $(\mathcal{S}, \mathbb{P})$ be a soft $\delta^* - open$ set in Y. If $\mathcal{Y} \in S.\delta^*.\mathrm{O}(\mathcal{X})$ then $(\mathcal{S}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$.
Proof. Let $(\mathcal{S}, \mathbb{P})$ be a soft $\delta^* - open$ set in Y (i.e. $(\mathcal{S}, \mathbb{P}) \in \mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$), then there exists a soft $\delta^* - open$ set $(\mathrm{G}, \mathbb{P})$ in $\mathcal{X}$ such that $(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \cap (\mathrm{G}, \mathbb{P})$. Now, if $\mathcal{Y} \in S.\delta^*.\mathrm{O}(\mathcal{X})$ then $\mathcal{Y} \cap (\mathrm{G}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$ by the soft topological space definition, hence $(\mathcal{S}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$.

Theorem 3.13. Let $\mathbb{Y}$ be a soft sub-space of a soft tri-topological space $\mathcal{X}$, and $(\mathcal{S}, \mathbb{P})$ be a soft set over X, then:
(1) $(\mathcal{S}, \mathbb{P})$ is soft $\delta^* - open$ in Y (i.e. $(\mathcal{S}, \mathbb{P}) \in \mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$) if and only if $(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \cap (\mathrm{G}, \mathbb{P})$ for some $(\mathrm{G}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$.
(2) $(\mathcal{S}, \mathbb{P})$ is soft $\delta^* - closed$ in Y (i.e. $(\mathcal{S}, \mathbb{P}) \in \mathrm{S}.\delta^*.\mathrm{C}(\mathcal{X})_{\mathrm{Y}}$) if and only if $(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \cap (\mathrm{G}, \mathbb{P})$ for somesoft $\delta^* - closed$ set $(\mathrm{G}, \mathbb{P})$ in $\mathcal{X}$ (i.e. for some $(\mathrm{G}, \mathbb{P}) \in S.\delta^*.\mathrm{C}(\mathcal{X})$).
Proof. (1) Follows from the definition of a soft sub-space of a soft tri-topological space.
(2) If $(\mathcal{S}, \mathbb{P})$ is soft $\delta^* - closed$ in a soft sub-space $\mathbb{Y}$, then we have
$(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \setminus (\mathrm{G}, \mathbb{P})$, for some $(\mathrm{G}, \mathbb{P}) \in \mathrm{S}.\delta^*.\mathrm{O}(\mathcal{X})_{\mathrm{Y}}$
Now, $(\mathrm{G}, \mathbb{P}) = \mathcal{Y} \cap (\mathrm{H}, \mathbb{P})$, for some $(\mathrm{H}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$.
For any $\in \mathbb{P}$,

$$\begin{aligned}\mathcal{S}(\mathcal{P}) &= \mathrm{Y}(\mathcal{P}) \setminus G(\mathcal{P})\\ &= \mathrm{Y} \setminus G(\mathcal{P})\\ &= \mathrm{Y} \setminus (\mathrm{Y}(\mathcal{P}) \cap \mathrm{H}(\mathcal{P}))\\ &= \mathrm{Y} \setminus (\mathrm{Y} \cap \mathrm{H}(\mathcal{P}))\\ &= \mathrm{Y} \setminus \mathrm{H}(\mathcal{P})\\ &= \mathrm{Y} \cap (\mathrm{X} \setminus \mathrm{H}(\mathcal{P}))\\ &= \mathrm{Y} \cap (\mathrm{H}(\mathcal{P}))^C\\ &= \mathrm{Y}(\mathcal{P}) \cap (\mathrm{H}(\mathcal{P}))^C\end{aligned}$$

Thus $(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \cap (H, \mathbb{P})^C$ where $(H, \mathbb{P})^C$ is $soft - \delta^* - closed$ in $\mathcal{X}$ as $(\mathrm{H}, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$.
Conversely, assume that $(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \cap (G, \mathbb{P})$ for some $soft - \delta^* - closed$ set $(G, \mathbb{P})$ in $\mathcal{X}$. This means that $(G, \mathbb{P})^C \in S.\delta^*.\mathrm{O}(\mathcal{X})$
Now, if $(G, \mathbb{P}) = (\mathcal{X}, \mathbb{P}) \setminus (H, \mathbb{P})$ where $(H, \mathbb{P}) \in S.\delta^*.\mathrm{O}(\mathcal{X})$ then for any $\in \mathbb{P}$,
$\mathcal{S}(\mathcal{P}) = \mathrm{Y}(\mathcal{P}) \cap G(\mathcal{P})$

$= \mathrm{Y} \cap G(\mathcal{P})$
$= \mathrm{Y} \cap (\mathrm{X}(\mathcal{P}) \setminus H(\mathcal{P}))$
$= \mathrm{Y} \cap (\mathrm{X} \setminus H(\mathcal{P}))$
$= \mathrm{Y} \setminus H(\mathcal{P})$
$= \mathrm{Y} \setminus (\mathrm{Y} \cap H(\mathcal{P}))$
$= \mathrm{Y}(\mathcal{P}) \setminus (\mathrm{Y}(\mathcal{P}) \cap H(\mathcal{P}))$.

Thus $(\mathcal{S}, \mathbb{P}) = \mathcal{Y} \setminus (\mathcal{Y} \cap (H, \mathbb{P}))$. Since $(H, \mathbb{P}) \in S.\delta^*.O(\mathcal{X})$, so $(\mathcal{Y} \cap (H, \mathbb{P})) \in \mathrm{S}.\delta^*.O(\mathcal{X})_{\mathrm{Y}}$ and hence $(\mathcal{S}, \mathbb{P})$ is soft $\delta^* - closed$ in Y.

4. Conclusions

A soft set with one or two specific topological structures is not sufficient to develop this theory. In that case, it becomes necessary to introduce an additional structure on the soft set. To confirm this idea, soft tri-topological space by soft tri-topological theory was introduced. It makes it more flexible to develop the theory of soft topological spaces. Thus, in this paper, we make a new approach to the soft tri-topological space theory.

In this work, we introduce the concept of soft sub-space of a soft tri-topological space as an original study. Firstly, we introduce the new concept in soft tri-topological space and investigate the properties by using many different examples and some important theorems or propositions, that and we analyse whether a soft sub-space of soft tri-topological space is a soft topology or not. Secondly, also we define another new concept which is the collection of soft sub-spaces of parametrized topologies on the initial universe set, i.e. corresponding to each parameter of a soft tri-topological space, we get a sub-spaces for the parametrized family of crisp topologies.We wish that findings in this work will be useful to characterize the soft sub-space of a soft tri-topological space; some further works can be done on the properties of hereditary soft tri-topology to carry out a general framework for applications of soft tri-topological space.

5. References

1. A.A. Ivanov, Bitopological spaces, Journal of Mathematical Scince, 26, (1984)1622–1636.
2. A. F. Hassan, δ^*-open set in tri-topological spaces, M.Sc thesis. Kufa University. (2004).
3. A. F. Hassan, Neeran T. and Alyaa Y., δ^*-Base in tri-topological space, journal of Kerbala univ., 9(3), (2011) 344- 352.
4. A. F. Hassan, δ^*-connectedness in tri-topological space, journal of Thi-Qar university, 6(3), (2011) 20-28.
5. A. F. Hassan, Countability and Separability in Tri-topological Spaces(δ^*-Countability and δ^*-Separability)., Mathematical Theory and Modeling., 7(1), (2017) 32-37.
6. A. F. Hassan and Z. Imran, On Compactness in Tri-topological Spaces. Sch J. Phys Math Stat., 6(7), (2019) 130-135.
7. A. F. Hassan, Tri-topological views in product spaces, Journal of New Theory, 30, (2020) 69-78.
8. A. F. Hassan, Quotient Tri-topology, Quotient Maps and Quotient Spaces in Tri-topological Spaces, International Journal of Engineering and Information Systems (IJEAIS), 4(1), (2020) 12-19.
9. A. F. Hassan, Z. Imran and I. Yahya, On lindelofness in Tri-topological Spaces., Jordan. J. Math. Sci., in press.
10. A. F. Hassan, Soft tri-topological Spaces, International Journal of Computer Applications, 176(9), (2017) 26-30.
11. A. F. Hassan, An Introduction to Fuzzy Soft tri-topological Spaces., International Journal of Fuzzy Mathematical Archive, 17(1), (2019) 41-49.
12. A. F. Hassan and A. H. Abass. Characterization Of Soft Open Set In Soft Tri-topological Space, Solid State Technology, Volume: 63 Issue: 3. 2020.
13. Asmhan Flieh Hassan and Ali Hussein Abaas. Separation Axioms in Soft Tri-topological Spaces with Respect to Ordinary Points. International Journal of Computer Applications, 175(10):10-19, August 2020.
14. A. F. Hassan, A. H. Abass, Separation Axioms In Soft Tri-topological Spaces With Respect to Soft Points. IOP Conference Series: Materials Science and Engineering, Volume 928, 2nd International Scientific Conference of Al-Ayen University (ISCAU-2020) 15-16 July 2020, Thi-Qar, Iraq.
15. B. M. Ittanagi, Soft Bitopological Spaces. International Journal of Computer Applications, 107(7), (2014) 1-4.
16. D.A. Molodtsov, Soft set theory -first results, Computers and Mathematics with Applications, 37 (4), (1999) 19–31.
17. H.M. Abu-Donia, New types of generalized closed sets in bitopological spaces, Journal of the Egyptian Mathematical Society, 21(3), (2013) 318-323.
18. M Shabir, M. Naz, On soft topological spaces, Computer and mathematics with applications, 61(7), (2011) 1786–1799.
19. N. Cağman, S. Karataş, S. Enginoglu. Soft Topology. Computer and mathematics with applications, 62(1), (2011) 351-358.
20. N. Giorgio., Soft N-Topological Spaces.Math.GN. 2019 May; arXiv:1905.12331 [math.GN]
21. P. K. Maji, A. R. Roy, R. Biswas. An application of soft sets in decision making problem. Computers & Mathematics with Applications 44(8), (2002) 1077-1083.

22. P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Computers & Mathematics with Applications, 45(4), (2003) 555-562