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On Local Irregularity Vertex Coloring of Tree Graph Family

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Abstract*: The graph* $\boldsymbol{G}$ *is a pair of finite sets* $(\boldsymbol{V},\boldsymbol{E})$ *where* $\boldsymbol{V}$ *is the set of vertex and* $\boldsymbol{E}$ *is the set of edge. A graph* $\boldsymbol{G}$ *is called local irregularity vertex coloring if there is a function* $\boldsymbol{l}: \boldsymbol{V}(\boldsymbol{G}) \rightarrow \{\boldsymbol{1},\boldsymbol{2},\dots,\boldsymbol{k}\}$ *is label function and weight function* $\boldsymbol{w}: \boldsymbol{V}(\boldsymbol{G}) \rightarrow \boldsymbol{N}$ *is desined as* $\boldsymbol{w}(\boldsymbol{u}) = \sum_{\boldsymbol{v}\in \boldsymbol{N}(\boldsymbol{u})} \boldsymbol{l}(\boldsymbol{v})$. *The function* $\boldsymbol{w}$ *is called local irregularity vertex coloring if: (i)* $\boldsymbol{opt}(\boldsymbol{l}) = \boldsymbol{min}\{\boldsymbol{maks}(\boldsymbol{l_i}); \boldsymbol{l_i}$ *is label function}, (ii) for every* $\boldsymbol{uv} \in \boldsymbol{E}(\boldsymbol{G}), \boldsymbol{w}(\boldsymbol{u}) \neq \boldsymbol{w}(\boldsymbol{v})$. *The chromatic number of local irregularity vertex coloring denoted by* $\boldsymbol{\chi_{lis}}(\boldsymbol{G})$ *is defined as* $\boldsymbol{\chi_{lis}}(\boldsymbol{G}) = \boldsymbol{min}\{|\boldsymbol{w}(\boldsymbol{V}(\boldsymbol{G}))|; \boldsymbol{w}$ *local irregularity vertex coloring}. In this paper, we will learn about local irregularity vertex coloring of tree graph family. All graph in this paper are member of tree graph family, namely double star graph, caterpillar graph, double broom graph, and banana tree graph.*

Keywords: local irregularity; vertex coloring; tree graph family

1. INTRODUCTION

Graph G is a pair of finite sets (V, E) where V is the set of vertex and E is the set of edges. In this paper, we will learn about local irregularity vertex coloring. Local irregular vertex coloring is a combination of the concept of anti-magic local vertex coloring that applies labeling to the vertex coloring and distance irregularity labeling by minimizing vertex labels and minimizing the number of vertex colors in the graph. The definition of local irregularity vertex coloring that was first introduced by Kristiana et.al [3] is as follows:

Definition 1. Let $l: V(G) \rightarrow \{1,2,\dots,k\}$ is label function and weight function $w: V(G) \rightarrow N$ is desined as $w(u) = \sum_{v\in N(u)} l(v)$. The function w is called local irregularity vertex coloring if:

(i) $opt(l) = min\{maks(l_i); l_i \textit{ is label function}\}$,
(ii) *for every* $uv \in E(G), w(u) \neq w(v)$.

Definition 2. The chromatic number of local irregularity vertex coloring denoted by $\chi_{lis}(G)$ is defined as $\chi_{lis}(G) = min\{|w(V(G))|; w$ local irregularity vertex coloring}.

An illustration of local irregularity vertex coloring and the chromatic number of local irregularity vertex coloring can be seen in Figure 1.

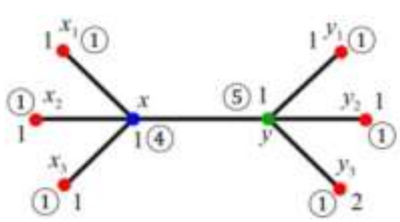

Fig. 1. *The chromatic number of local irregularity vertex coloring,* $\chi_{lis}(S_{n,m}) = 3$.

Lemma. For graph G, $\chi_{lis}(G) \geq (G)$. [3]

This observation uses the degree of vertex $d(v_{i,j})$ so that it can be used to determine the label of the vertex on the graph.

Observation 1. A connected graph G, if every two adjacent vertices have different degrees then $opt(l) = 1$. [4]

Observation 2. A connected graph G, if two adjacent vertices have the same degrees then $opt(l) \geq 2$. [4]

The results of previous research regarding local irregularity vertex coloring, namely Arumugam et al [1] defined local anti-magic labeling that applies a label to the vertex coloring. Futhermore, Slamin [2] defined the labeling of the irregularity of the distance graph G with vertices v labeled positive numbers to vertices v so that the weights calculated at the vertices are different, the weight is defined as the number of labels from all vertices adjacent to x. Kristiana et al [3] defined local irregular vertex coloring on graphs and obtained chromatic numbers from several specials graphs, namely path graph, circle graph, complete graph, complete graph, friendship graph, wheel graph, and graph complete bipartite. Furthermore, Kristiana et al [4] investigated local irregularity vertex coloring associated with wheel graphs, namely network graphs, helmet graphs, closed helmet graphs, gear graphs, fan graphs, sun graphs, and double helmet graphs. Then, Azahra et al [5] examined the chromatic number of local irregularity vertex coloring in the family of grid graphs, namely the grid graph, ladder graph, triangular ladder graph, and H graph. Furthermore, Kristiana et al [6] examined local irregularity vertex coloring in triangular book graphs, rectangular book graphs, pan graphs, subdivisions of pan graphs, and grid graphs.

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2. RESULT

In this paper, we discuss some new results of the chromatic number local irregular of tree graph family.

Theorem 2.1. The chromatic number of local irregularity vertex coloring on a double star graph $(\boldsymbol{S_{n,m}})$ with $\boldsymbol{n \geq 3}$, $\boldsymbol{m \geq 3}$ is $\boldsymbol{\chi_{lis}(S_{n,m}) = 3}$.

Proof. A double star graph $(\boldsymbol{S_{n,m}})$ has a vertex set $\boldsymbol{V(S_{n,m}) = \{x, y\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq m\}}$ and the edge set $\boldsymbol{E(S_{n,m}) = \{xy\} \cup \{xx_i; 1 \leq i \leq n\} \cup \{yy_j; 1 \leq j \leq m\}}$. The proof of the theorem will be divided into two cases.

Case 1: for $n \neq m$

Each adjacent vertex on a double star graph $(S_{n,m})$ has a different degree. Based on Observation 1 then $opt(l) = 1$, so lower bound of a double star graph is $\chi_{lis}(S_{n,m}) \geq 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(S_{n,m}) \to \{1\}$ then $l(v_{i,j}) = 1$. Therefore, a vertex-weight function of a double star graph $(S_{n,m})$ is as follows:

$$w(x_i) = w(y_j) = 1; 1 \leq i \leq n, 1 \leq j \leq m$$
$$w(x) = n + 1$$
$$w(y) = m + 1$$

For every $uv \in E(S_{n,m}), w(x) \neq w(y), w(x) \neq w(x_i)$ with $1 \leq i \leq n, w(y) \neq w(y_j)$ with $1 \leq j \leq m$, so upper bound of a double star graph is $\chi_{lis}(S_{n,m}) \leq 3$.

Based on the lower bound and upper bounds, we get $3 \leq \chi_{lis}(S_{n,m}) \leq 3$, so the chromatic number of local irregularity vertex coloring on a double star graph for $n \neq m$ is $\chi_{lis}(S_{n,m}) = 3$.

Case 2: for $n = m$

There are adjacent vertex on a double star graph $(S_{n,m})$ that has the same degree. Based on Observation 2 then $opt(l) \geq 2$. Next, we will prove the lower bound of a double star graph, based on the Lemma we get $\chi_{lis}(S_{n,m}) \geq 2$. Assume $\chi_{lis}(S_{n,m}) = 2$, if $l(x) = l(y) = l(x_i) = l(y_j) = 1; i = j$ then $w(x) = w(y)$. It contradicts the definition of local irregularity vertex coloring because $uv \in E(S_{n,m})$, so $w(x) \neq w(y)$. Based on this probability, the lower bound of a double star graph is $\chi_{lis}(S_{n,m}) \geq 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(S_{n,m}) \to \{1,2\}$ is as follows:

$$l(x) = l(y) = l(x_i) = 1; 1 \leq i \leq n$$
$$l(y_j) = \begin{cases} 1, & 1 \leq j \leq m-1 \\ 2, & m \end{cases}$$

Based on the label function, a vertex-weight function of a double star graph $(S_{n,m})$ is as follows:

$$w(x_i) = w(y_j) = 1; 1 \leq i \leq n, 1 \leq j \leq m$$
$$w(x) = n + 1$$
$$w(y) = m + 2$$

For every $uv \in E(S_{n,m}), w(x) \neq w(y), w(x) \neq w(x_i)$ with $1 \leq i \leq n, w(y) \neq w(y_j)$ with $1 \leq j \leq m$, so upper bound of a double star graph is $\chi_{lis}(S_{n,m}) \leq 3$.

Based on the lower bound and upper bounds, we get $3 \leq \chi_{lis}(S_{n,m}) \leq 3$, so the chromatic number of local irregularity vertex coloring on a double star graph for $n = m$ is $\chi_{lis}(S_{n,m}) = 3$.

Theorem 2.2. The chromatic number of local irregularity vertex coloring on a caterpillar graph $(\boldsymbol{C_{n,m}})$ with $\boldsymbol{n \geq 3}$, $\boldsymbol{m \geq 3}$ is $\boldsymbol{\chi_{lis}(C_{n,m}) = 3}$.

Proof. A caterpillar graph $(\boldsymbol{C_{n,m}})$ has a vertex set $\boldsymbol{V(C_{n,m}) = \{x_i; 1 \leq i \leq n\} \cup \{y_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}}$ and the edge set $\boldsymbol{E(C_{n,m}) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i y_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}}$. The proof of the theorem will be divided into two cases.

Case 1: for $n = 3, m \geq 3$

Each adjacent vertex on a caterpillar graph $(C_{n,m})$ has a different degree. Based on Observation 1 then $opt(l) = 1$, so lower bound of a caterpillar graph is $\chi_{lis}(C_{n,m}) \geq 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(C_{n,m}) \to \{1\}$ then $l(v_{i,j}) = 1$. Therefore, a vertex-weight function of a caterpillar graph $(C_{n,m})$ is as follows:

$$w(y_{i,j}) = 1; 1 \leq i \leq 3, 1 \leq j \leq m$$
$$w(x_i) = \begin{cases} m+1, & i = 1,3 \\ m+2, & i = 2 \end{cases}$$

For every $uv \in E(C_{n,m}), w(x_i) \neq w(y_{i,j})$ with $1 \leq i \leq 3, 1 \leq j \leq m, w(x_i) \neq w(x_2)$ with $i = 1,3$, so upper bound of a caterpillar graph is $\chi_{lis}(C_{n,m}) \leq 3$.

Based on the lower bound and upper bounds, we get $3 \leq \chi_{lis}(C_{n,m}) \leq 3$, so the chromatic number of local irregularity vertex coloring on a caterpillar graph for $n = 3, m \geq 3$ is $\chi_{lis}(C_{n,m}) = 3$.

Case 2: for $n \geq 4, m \geq 3$

There are adjacent vertex on a caterpillar graph $(C_{n,m})$ that has the same degree. Based on Observation 2 then $opt(l) \geq 2$. Next, we will prove the lower bound of a caterpillar graph, based on the Lemma we get $\chi_{lis}(C_{n,m}) \geq 2$. Assume $\chi_{lis}(C_{n,m}) = 2$, if $l(y_{1,1}) = l(y_{n,1}) = 2$, $l(x_i) = 1; 1 \leq i \leq n$, $l(y_{i,j}) = 1; 2 \leq i \leq n-1, 1 \leq j \leq m$ and $i =$

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$1, n, 2 \le j \le m$ then $w(x_i) = w(x_{i+1})$. It contradicts the definition of local irregularity vertex coloring because $uv \in E(C_{n,m})$, so $w(x_i) \ne w(x_{i+1})$. Based on this probability, the lower bound of a caterpillar graph is $\chi_{lis}(C_{n,m}) \ge 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. The proof of the upper bound will be divided into two subcases.

Subcase 1 when $n \ge 4, m \ge 3, n \equiv 0(mod2)$. Defined function $l: V(C_{n,m}) \to \{1,2\}$ is as follows:

$$l(x_i) = 1; 1 \le i \le n$$

$$l(y_{i,j}) = \begin{cases} 1, & i = 1, 1 \le j \le m-1 \\ & i = n, 1 \le j \le m-2 \\ & i \equiv 1(mod2), 1 \le i \le n, 1 \le j \le m \\ & i \equiv 0(mod2), 1 \le i \le n-1, 1 \le j \le m-1 \\ 2, & i = n, j = m-1 \\ & i = 1, i \equiv 0(mod2), 1 \le i \le n, j = m \end{cases}$$

Based on the label function, a vertex-weight function of a caterpillar graph $(C_{n,m})$ is as follows:

$$w(y_{i,j}) = 1; 1 \le i \le n, 1 \le j \le m$$
$$w(x_i) = \begin{cases} m+2, & i \text{ odd} \\ m+3, & i \text{ even} \end{cases}$$

Subcase 2 when $n \ge 4, m \ge 3, n \equiv 1(mod2)$. Defined function $l: V(C_{n,m}) \to \{1,2\}$ is as follows:

$$l(x_i) = 1; 1 \le i \le n$$

$$l(y_{i,j}) = \begin{cases} 1, & i = 1, n, 1 \le j \le m-1 \\ & i \equiv 0(mod2), 1 \le i \le n, 1 \le j \le m-1 \\ & i \equiv 1(mod2), 1 \le i \le n-1, 1 \le j \le m \\ 2, & i = 1, n, i \equiv 0(mod2), 1 \le i \le n, j = m \end{cases}$$

Based on the label function, a vertex-weight function of a caterpillar graph $(C_{n,m})$ is as follows:

$$w(y_{i,j}) = 1; 1 \le i \le n, 1 \le j \le m$$
$$w(x_i) = \begin{cases} m+2, & i \text{ odd} \\ m+3, & i \text{ even} \end{cases}$$

For every $uv \in E(C_{n,m}), w(x_i) \ne w(y_{i,j})$ with $1 \le i \le n, 1 \le j \le m, w(x_i) \ne w(x_{i+1})$ with $1 \le i \le n-1$, so upper bound of a caterpillar graph is $\chi_{lis}(C_{n,m}) \le 3$.

Based on the lower bound and upper bounds, we get $3 \le \chi_{lis}(C_{n,m}) \le 3$, so the chromatic number of local irregularity vertex coloring on a caterpillar graph for $n \ge 4, m \ge 3$ is $\chi_{lis}(C_{n,m}) = 3$.

Theorem 2.3. The chromatic number of local irregularity vertex coloring on a double broom graph $(\boldsymbol{B2_{n,m}})$ with $\boldsymbol{n \ge 3, m \ge 3}$ is

$$\boldsymbol{\chi_{lis}(B2_{n,m}) = \begin{cases} 3, & n = 3, m \ge 3 \\ & n \ge 5, m \ge 3, n \equiv 1(mod2) \\ 4, & n \ge 4, m \ge 3, n \equiv 0, 2(mod4) \end{cases}}$$

Proof. A double broom graph $(\boldsymbol{B2_{n,m}})$ has a vertex set $\boldsymbol{V(B2_{n,m}) = \{x_i; 1 \le i \le n\} \cup \{y_{i,j}; i = 1, n, 1 \le j \le m\}}$ and the edge set $\boldsymbol{E(B2_{n,m}) = \{x_i x_{i+1}; 1 \le i \le n-1\} \cup \{x_i y_{i,j}; i = 1, n, 1 \le j \le m\}}$. The proof of the theorem will be divided into three cases.

Case 1: for $n = 3, m \ge 3$

Each adjacent vertex on a double broom graph $(B2_{n,m})$ has a different degree. Based on Observation 1 then $opt(l) = 1$, so lower bound of a double broom graph is $\chi_{lis}(B2_{n,m}) \ge 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(B2_{n,m}) \to \{1\}$ then $l(v_{i,j}) = 1$. Therefore, a vertex-weight function of a double broom graph $(B2_{n,m})$ is as follows:

$$w(y_{i,j}) = 1; i = 1, n, 1 \le j \le m$$
$$w(x_i) = \begin{cases} 2, & i = 2 \\ m+1, & i = 1,3 \end{cases}$$

For every $uv \in E(B2_{n,m}), w(y_{i,j}) \ne w(x_i)$ with $i = 1, n, 1 \le j \le m, w(x_i) \ne w(x_2)$ with $i = 1,3$, so upper bound of a double broom graph is $\chi_{lis}(B2_{n,m}) \le 3$.

Based on the lower bound and upper bounds, we get $3 \le \chi_{lis}(B2_{n,m}) \le 3$, so the chromatic number of local irregularity vertex coloring on a double broom graph for $n = 3, m \ge 3$ is $\chi_{lis}(B2_{n,m}) = 3$.

Case 2: for $n \ge 5, m \ge 3, n \equiv 1(mod2)$

There are adjacent vertex on a double broom graph $(B2_{n,m})$ that has the same degree. Based on Observation 2 then $opt(l) \ge 2$. Next, we will prove the lower bound of a double broom graph, based on the Lemma we get $\chi_{lis}(B2_{n,m}) \ge 2$. Assume $\chi_{lis}(B2_{n,m}) = 2$, if $l(x_i) = 2; i$ odd, $l(y_{i,j}) = 1; i = 1, n, 1 \le j \le 3$, $l(x_i) = 1; i$ even then $w(x_i) = w(x_{i+1}); i = 1, n-1$. It contradicts the definition of local irregularity vertex coloring because $uv \in E(B2_{n,m})$, so $w(x_i) \ne w(x_{i+1}); i = 1, n-1$. Based on this probability, the lower bound of a double broom graph is $\chi_{lis}(B2_{n,m}) \ge 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(S_{n,m}) \to \{1,2\}$ when $n \ge 5, m \ge 3, n \equiv 1(mod2)$ is as follows:

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$$l(x_i)=\begin{cases}1, & i \text{ even}\\ 2, & i \text{ odd}\end{cases}$$
$$l(y_{i,j})=\begin{cases}1, & i=1,n,1\le j\le m-1\\ 2, & i=1,n,j=m\end{cases}$$

Based on the label function, a vertex-weight function of a double broom graph $(B2_{n,m})$ is as follows:

$$w(y_{i,j})=2; i=1,n,1\le j\le m$$
$$w(x_i)=\begin{cases}2, & i\equiv 1(mod2),1\le i\le n-1\\ 4, & i\equiv 0(mod2),1\le i\le n\\ m+2, & i=1,n\end{cases}$$

For every $uv\in E(B2_{n,m}), w(x_i)\neq w(y_{i,j})$ with $i=1,n,1\le j\le m, w(x_i)\neq w(x_{i+1})$ with $1\le i\le n-1$, so upper bound of a double broom graph is $\chi_{lis}(B2_{n,m})\le 3$.

Based on the lower bound and upper bounds, we get $3\le \chi_{lis}(B2_{n,m})\le 3$, so the chromatic number of local irregularity vertex coloring on a double broom graph for $n\ge 5, m\ge 3, n\equiv 1(mod2)$ is $\chi_{lis}(B2_{n,m})=3$.

Case 3: for $n\ge 4, m\ge 3, n\equiv 0,2(mod4)$

There are adjacent vertex on a double broom graph $(B2_{n,m})$ that has the same degree. Based on Observation 2 then $opt(l)\ge 2$. Next, we will prove the lower bound of a double broom graph, based on the Lemma we get $\chi_{lis}(B2_{n,m})\ge 2$. Assume $\chi_{lis}(B2_{n,m})=3$, if $l(x_i)=l(y_{i,j})=1; 1\le i\le n, 1\le j\le m$ then $w(x_i)=w(x_{i+1})$ $; 2\le i\le n-2$. It contradicts the definition of local irregularity vertex coloring because $uv\in E(B2_{n,m})$, so $w(x_i)\neq w(x_{i+1}); 2\le i\le n-2$. Based on this probability, a lower bound of a double broom graph is $\chi_{lis}(B2_{n,m})\ge 4$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. The proof of the upper bound will be divided into two subcases.

Subcase 1 when $n\ge 4, m\ge 3, n\equiv 0(mod4)$. Defined function $l: V(B2_{n,m})\to\{1,2\}$ is as follows:

$$l(x_i)=\begin{cases}1, & i=1, i\equiv 0(mod2),1\le i\le n\\ & i\equiv 1(mod4),5\le i\le n\\ 2, & i=3, i\equiv 3(mod4),3\le i\le n\end{cases}$$
$$l(y_{i,j})=\begin{cases}1, & i=1,1\le j\le m-1\\ & i=n,1\le j\le m\\ 2, & i=1,j=m\end{cases}$$

Based on the label function, a vertex-weight function of a double broom graph $(B2_{n,m})$ is as follows:

$$w(y_{i,j})=1; i=1,n,1\le j\le m$$
$$w(x_i)=\begin{cases}2, & i\equiv 1(mod2),1\le i\le n\\ 3, & i\equiv 0(mod2),1\le i\le n-1\\ m+2, & i=1,n\end{cases}$$

Subcase 2 when $n\ge 4, m\ge 3, n\equiv 2(mod4)$. Defined function $l: V(C_{n,m})\to\{1,2\}$ is as follows:

$$l(y_{i,j})=1; i=1,n,1\le j\le m$$
$$l(x_i)=\begin{cases}1, & i=1, i\equiv 0(mod2),1\le i\le n\\ & i\equiv 1(mod4),5\le i\le n\\ 2, & i=3, i\equiv 3(mod4),3\le i\le n\end{cases}$$

Based on the label function, a vertex-weight function of a double broom graph $(B2_{n,m})$ is as follows:

$$w(y_{i,j})=1; i=1,n,1\le j\le m$$
$$w(x_i)=\begin{cases}2, & i\equiv 1(mod2),1\le i\le n\\ 3, & i\equiv 0(mod2),1\le i\le n-1\\ m+2, & i=1,n\end{cases}$$

For every $uv\in E(B2_{n,m}), w(x_i)\neq w(y_{i,j})$ with $i=1,n,1\le j\le m, w(x_i)\neq w(x_{i+1})$ with $1\le i\le n-1$, so upper bound of a double broom graph is $\chi_{lis}(B2_{n,m})\le 4$.

Based on the lower bound and upper bounds, we get $4\le \chi_{lis}(B2_{n,m})\le 4$, so the chromatic number of local irregularity vertex coloring on a double broom graph for $n\ge 4, m\ge 3, n\equiv 0,2(mod4)$ is $\chi_{lis}(B2_{n,m})=4$.

Theorem 2.4. The chromatic number of local irregularity vertex coloring on a banana tree graph $(\boldsymbol{Bt_{n,m}})$ with $\boldsymbol{n\ge 2}$, $\boldsymbol{m\ge 3}$ is

$$\boldsymbol{\chi_{lis}(Bt_{n,m})}=\begin{cases}\mathbf{3}, & \boldsymbol{n\ge 3, m\ge 3, n=m}\\ & \boldsymbol{n=2, m\ge 3}\\ \mathbf{4}, & \boldsymbol{n\ge 3, m\ge 3, n\neq m}\end{cases}$$

Proof. A banana tree graph $(\boldsymbol{Bt_{n,m}})$ has a vertex set $\boldsymbol{V(Bt_{n,m})=\{x_i; 1\le i\le n\}\cup\{y\_(i,j); 1\le i\le n, 1\le j\le m\}\cup\{z\}}$ and the edge set $\boldsymbol{E(Bt_{n,m})=\{x_iy_{i,j}; 1\le i\le n\}\cup\{zy_{i,m}; 1\le i\le n\}}$. The proof of the theorem will be divided into three cases.

Case 1: for $n\ge 3, m\ge 3, n=m$

Each adjacent vertex on a banana tree graph $(Bt_{n,m})$ has a different degree. Based on Observation 1 then $opt(l)=1$, so lower bound of a banana tree graph is $\chi_{lis}(Bt_{n,m})\ge 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(Bt_{n,m})\to\{1\}$ then $l(v_{i,j})=1$. Therefore, a vertex-weight function of a banana tree graph $(Bt_{n,m})$ is as follows:

$$w(y_{i,j})=\begin{cases}1, & 1\le i\le n, 2\le j\le m\\ 2, & 1\le i\le n, j=1\end{cases}$$
$$w(z)=w(x_i)=n; 1\le i\le 3$$

For every $uv \in E(Bt_{n,m}), w(y_{i,j}) \neq w(x_i)$ with $1 \leq i \leq n, 1 \leq j \leq m, w(y_{i,1}) \neq w(z)$ with $1 \leq i \leq n$, so upper bound of a banana tree graph is $\chi_{lis}(Bt_{n,m}) \leq 3$.

Based on the lower bound and upper bounds, we get $3 \leq \chi_{lis}(Bt_{n,m}) \leq 3$, so the chromatic number of local irregularity vertex coloring on a banana tree graph for $n \neq m$ is $\chi_{lis}(Bt_{n,m}) = 3$.

Case 2: for $n = 2, m \geq 3$

There are adjacent vertex on a banana tree graph $(Bt_{n,m})$ that has the same degree. Based on Observation 2 then $opt(l) \geq 2$. Next, we will prove the lower bound of a banana tree graph, based on the Lemma we get $\chi_{lis}(Bt_{n,m}) \geq 2$. Assume $\chi_{lis}(Bt_{n,m}) = 2$, if $l(x_i) = 2; i = 1,2, l(z) = l(y_{i,j}) = 1; 1 \leq i \leq n, 1 \leq j \leq 3$ then $w(x_i) = w(y_{i,1}); i = 1,2$. It contradicts the definition of local irregularity vertex coloring because $uv \in E(Bt_{n,m})$, so $w(x_i) \neq w(y_{i,1}); i = 1,2$. Based on this probability, the lower bound of a banana tree graph is $\chi_{lis}(Bt_{n,m}) \geq 3$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(Bt_{n,m}) \to \{1,2\}$ is as follows:

$$l(z) = 1$$
$$l(x_i) = 1; 1 \leq i \leq n$$
$$l(y_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, 1 \leq j \leq m-1 \\ 2, & 1 \leq i \leq n, j = m \end{cases}$$

Based on the label function, a vertex-weight function of a banana tree graph $(Bt_{n,m})$ is as follows:

$$w(z) = 2$$
$$w(y_{i,j}) = \begin{cases} 2, & 1 \leq i \leq n, 2 \leq j \leq m \\ 3, & 1 \leq i \leq n, j = 1 \end{cases}$$
$$w(x_i) = m + 1; 1 \leq i \leq n$$

For every $uv \in E(Bt_{n,m}), w(x_i) \neq w(y_{i,j})$, with $1 \leq i \leq n, 1 \leq j \leq m, w(y_{i,1}) \neq w(z)$ with $1 \leq i \leq n$, so upper bound of a banana tree graph is $\chi_{lis}(Bt_{n,m}) \leq 3$.

Based on the lower bound and upper bounds, we get $3 \leq \chi_{lis}(Bt_{n,m}) \leq 3$, so the chromatic number of local irregularity vertex coloring on a banana tree graph for $n = 2, m \geq 3$ is $\chi_{lis}(Bt_{n,m}) = 3$.

Case 3: for $n \geq 3, m \geq 3, n \neq m$

Each adjacent vertex on a banana tree graph $(Bt_{n,m})$ has a different degree. Based on Observation 1 then $opt(l) = 1$, so lower bound of a banana tree graph is $\chi_{lis}(Bt_{n,m}) \geq 4$.

Next, we will prove the upper bound of the chromatic number of local irregularity vertex coloring. Defined function $l: V(Bt_{n,m}) \to \{1\}$ then $l(v_{i,j}) = 1$. Therefore, a vertex-weight function of a banana tree graph $(Bt_{n,m})$ is as follows:

$$w(y_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, 2 \leq j \leq m \\ 2, & 1 \leq i \leq n, j = 1 \end{cases}$$
$$w(z) = n$$
$$w(x_i) = m; 1 \leq i \leq 3$$

For every $uv \in E(Bt_{n,m}), w(y_{i,j}) \neq w(x_i)$ with $1 \leq i \leq n, 1 \leq j \leq m, w(y_{i,1}) \neq w(z)$ with $1 \leq i \leq n$, so upper bound of a banana tree graph is $\chi_{lis}(Bt_{n,m}) \leq 4$.

Based on the lower bound and upper bounds, we get $4 \leq \chi_{lis}(Bt_{n,m}) \leq 4$, so the chromatic number of local irregularity vertex coloring on a banana tree graph for $n = m$ is $\chi_{lis}(Bt_{n,m}) = 4$.

3. CONCLUSION

Based on the results described in the previous chapter, four new theorems were obtained regarding local irregularity vertex coloring of tree graphs family. The resulting theorem is, $\chi_{lis}(S_{n,m}) = 3$ for $n \geq 3, m \geq 3$, $\chi_{lis}(C_{n,m}) = 3$ for $n \geq 3, m \geq 3$, $\chi_{lis}(B2_{n,m}) = 3$ for $n = 3, m \geq 3$ and $n \geq 5, m \geq 3, n \equiv 1(mod2)$, $\chi_{lis}(B2_{n,m}) = 4$ for $n \geq 4, m \geq 3, n \equiv 0,2(mod4)$, $\chi_{lis}(Bt_{n,m}) = 3$ for $n \geq 3, m \geq 3, n = m$ and $n = 2, m \geq 3$, and $\chi_{lis}(Bt_{n,m}) = 4$ for $n \geq 3, m \geq 3, n \neq m$.

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