International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 5 Issue 8, August - 2021, Pages: 1-4

On Three − Monotone Approximation

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Abstract: This paper is review about three monotone approximation. We recall some general properties with important propositions.

1. Introduction

A mapping $\mathfrak{H}$ which is defined on $\mathcal{I} := [\mathfrak{d},\mathfrak{v}]$, is real-valued and $\mathfrak{C}$ is belong to $\mathbb{N}$. Denote by

$$\mathfrak{H}\ [\mathfrak{z}_0,\ldots,\mathfrak{z}_{\mathfrak{C}}] := \sum_{i=0}^{\mathfrak{C}} \frac{\mathfrak{H}\ (\mathfrak{z}_\iota)}{\prod_{j=0,j\neq i}^{\mathfrak{C}}(\mathfrak{z}_\iota - x_j)},$$

the $\mathfrak{C}th$ size divided of $\mathfrak{H}$ at the points $\mathfrak{z}_0,\ldots,\mathfrak{z}_{\mathfrak{C}}$. The mapping $\mathfrak{H}$ is called $\mathfrak{C}$ −monotone in $[\mathfrak{d},\mathfrak{v}]$, if $\mathfrak{H}\ [\mathfrak{z}_0,\ldots,\mathfrak{z}_{\mathfrak{C}}] \geq 0$ for all $\mathfrak{C}+1$ distinct points $\mathfrak{z}_0,\ldots,\mathfrak{z}_{\mathfrak{C}} \in [\mathfrak{d},\mathfrak{v}]$. The set of all $\mathfrak{C}$ −monotone mapping in $[\mathfrak{d},\mathfrak{v}]$ is denote by $\Delta^{\mathfrak{C}}_{[\mathfrak{d},\mathfrak{v}]}$, hence in particular, the sets of nondecreasing and convex mappings in $[\mathfrak{d},\mathfrak{v}]$ are $\Delta^1_{[\mathfrak{d},\mathfrak{v}]}$ and $\Delta^2_{[\mathfrak{d},\mathfrak{v}]}$ respectively. The set of all bounded mapping which is having a convex derivative on$(\mathfrak{d},\mathfrak{v})$ is $\Delta^3_{[\mathfrak{d},\mathfrak{v}]}$. Note that if $\mathfrak{H} \in \Delta^{\mathfrak{C}}_{[\mathfrak{d},\mathfrak{v}]}, \mathfrak{C} \geq 2$, then $\mathfrak{H}$ is continuous on $(\mathfrak{d},\mathfrak{v})$ and $\mathfrak{H}\ (\mathfrak{d}+), \mathfrak{H}\ (\mathfrak{v}-)$ exist and are finite.

For $\mathfrak{H} \in \mathcal{C}_{[\mathfrak{d},\mathfrak{v}]}$, and an interval $\mathcal{I} \subset [\mathfrak{d},\mathfrak{v}]$, the symbol $\|\mathfrak{H}\|_I$ is denoted to the usual supnorm of $\mathfrak{H}$ on $\mathcal{I}$, and for $\hbar > 0$ the $\mathfrak{v}th$ modulus of smoothness of $\mathfrak{H}$ is denoted by $\omega_\kappa(\mathfrak{H}, h; \mathcal{I})$, with the step h on $\mathcal{I}$. For the interval $[\mathfrak{d},\mathfrak{v}]$ itself we write$\|\mathfrak{H}\| := \|\mathfrak{H}\|_{[\mathfrak{d},\mathfrak{v}]}$ and $\omega_k(\mathfrak{H}, \hbar) := \omega_k(\mathfrak{H}, \hbar;\ [\mathfrak{d},\mathfrak{v}])$.

2. Review

Proposition 2.1: [4]

Let $\mathfrak{K} \in \Delta^3_{[\mathfrak{d},\mathfrak{v}]}$ and $\mathfrak{H}\ (\mathfrak{z}) := \mathfrak{K}'(\mathfrak{z}), \mathfrak{z} \in (\mathfrak{d},\mathfrak{v})$. Given an integer $\kappa \geq 2$, a partition $\mathfrak{d} =: \mathfrak{z}_0 < \mathfrak{z}_1 < \cdots < \mathfrak{z}_{\mathfrak{C}} := \mathfrak{v}$, and a piecewise polynomial $\delta \in \Delta^2_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa - 1$, with k-nots $\mathfrak{z}_\iota, i = 1,\ldots,\mathfrak{C}-1$, such that

$$\delta(\mathfrak{z}_\iota) = \mathfrak{H}\ (\mathfrak{z}_\iota), \iota = 1,\ldots,\mathfrak{C}-1,$$

there exists a piecewise polynomial $\xi \in \Delta^3_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa$ with k-nots $\mathfrak{z}_\iota, \iota = 1,\ldots,\mathfrak{C}-1$, for which

$$\|\mathfrak{K} - \xi\| \leq \epsilon\, max_{1\leq \iota \leq \mathfrak{C}}\ \|\mathfrak{H} - \xi\|_{L_{1[\mathfrak{z}_{\iota-1},\mathfrak{z}_\iota]}},$$

whereϵ is an absolute constant, and $\|\cdot\|_{L_{1[\mathfrak{z}_{\iota-1},\mathfrak{z}_\iota]}}$ denotes the L_1-norm on $[\mathfrak{z}_{\iota-1},\mathfrak{z}_\iota]$. In fact $\epsilon \leq 25$.

Proposition 2.2 : [4]

Suppose $\mathfrak{H} \in \Delta^2_{[\mathfrak{d},\mathfrak{v}]}, \kappa \geq 2$, and $\mathfrak{z}_{-1} := \mathfrak{d} =: \mathfrak{z}_0 < \mathfrak{z}_1 < \ldots < \mathfrak{z}_{\mathfrak{C}} := \mathfrak{v} =: \mathfrak{z}_{\mathfrak{C}+1}$. Then for each piecewise polynomial $\delta \in \Delta^2_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa - 1$ with k-nots $\mathfrak{z}_\iota, \iota = 1,\ldots,\mathfrak{C}-1$, there is a piecewise polynomial $\delta_1 \in \Delta^2_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa - 1$, with the same k-nots such that

- $\mathfrak{H}\ (\mathfrak{z}_\iota) = \delta_1(\mathfrak{z}_{\iota_i}), \iota = 0,\ldots,\mathfrak{C}$,
- $\|\mathfrak{H} - \delta_1\|_{[\mathfrak{z}_{\iota-1},\mathfrak{z}_\iota]} \leq \epsilon(\mathscr{r})\,\|\mathfrak{H} - \delta\|_{[\mathfrak{z}_{\iota-2},\mathfrak{z}_\iota+1]}, \iota = 1,\ldots,\mathfrak{C}$,

where $\epsilon(\mathscr{r})$ is depending only on $\mathscr{r}$,($\epsilon(\mathscr{r})$ is constant), the scale of the partition $\mathfrak{z}_0, \ldots, \mathfrak{z}_{\mathfrak{C}}$, i.e.,

$$\mathscr{r} := max_{1\leq \iota \leq \mathfrak{C}-1} \left\{\frac{\mathfrak{z}_{\iota+1}-\mathfrak{z}_\iota}{\mathfrak{z}_\iota-\mathfrak{z}_{\iota-1}}; \frac{\mathfrak{z}_\iota-\mathfrak{z}_{\iota-1}}{\mathfrak{z}_{\iota+1}-\mathfrak{z}_\iota}\right\}.$$

Proposition 2.3 : [4]

Let $\mathfrak{K} \in \Delta^3_{[\mathfrak{d},\mathfrak{v}]}$ *and* $\mathfrak{H}(\mathfrak{z}) := \mathfrak{K}'(\mathfrak{z}), \mathfrak{z} \in (\mathfrak{d},\mathfrak{v})$. Given an integer $\kappa \geq 2$, a partition $\mathfrak{z}_{-1} := a =: \mathfrak{z}_0 < \mathfrak{z}_1 < \ldots < \mathfrak{z}_{\mathfrak{C}} := \mathfrak{v} =: \mathfrak{z}_{\mathfrak{C}+1}$, and a piecewise polynomial $\delta \in \Delta^2_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa - 1$, with k-nots $\mathfrak{z}_\iota$, $\iota = 1, \ldots, \mathfrak{C} - 1$, there is a piecewise polynomial $\xi \in \Delta^3_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa$ with k-nots $\mathfrak{z}_\iota$, $\iota = 1, \ldots, \mathfrak{C} - 1$, for which

$$\|\mathfrak{K} - \xi\| \leq \epsilon(r)\, max_{1\leq i\leq \mathfrak{C}}\ (\mathfrak{z}_\iota - \mathfrak{z}_{\iota-1})\, \|\mathfrak{H} - \delta\|_{[\mathfrak{z}_{\iota-2},\mathfrak{z}_{\iota+1}]},$$

where m is the scale, and $\epsilon(r) \leq \epsilon r$ for some absolute constant ϵ.

Proposition 2.4 : [4]

Let $\kappa \geq 1$ and $\mathfrak{D} \geq 0$, be integers such that either $\mathfrak{D} \geq 2$ or $2 \leq \kappa + \mathfrak{D} \leq 3$. Then for each $\mathfrak{H} \in \mathcal{C}^{(\mathfrak{D})}_{[-1,1]} \cap \Delta^2_{[-1,1]}$ there exist piecewise polynomials $\delta_1, \delta_2 \in \Delta^2_{[-1,1]}$ of degree $\leq \kappa + \mathfrak{D} - 1$ such that δ_1 has $\mathfrak{C}$ equidistant k-nots, satisfying

$$\|\mathfrak{H} - \delta_1\|_{[-1,1]} \leq \frac{\epsilon(\kappa,\mathfrak{D})}{\mathfrak{D}^{\mathfrak{C}}}\ \omega_\kappa(\mathfrak{H}^{(\mathfrak{D})}, 1/\mathfrak{C};\ [-1,1]),$$

and δ_2 has k-nots on the Chebyshev partition, satisfying

$$\|\mathfrak{H} - \delta_2\|_{[-1,1]} \leq \frac{\epsilon(\kappa,\mathfrak{D})}{\mathfrak{D}^{\mathfrak{C}}}\ \omega^\varphi_\kappa(\mathfrak{H}^{(\mathfrak{D})}, 1/\mathfrak{C};\ [-1,1]).$$

Proposition 2.5 : [4]

Let $\kappa \geq 1$ and $\mathfrak{D} \geq 0$, be integers such that either $\mathfrak{D} \geq 3$ or $3 \leq \kappa + \mathfrak{D} \leq 4, (\kappa,\mathfrak{D}) \neq (4,0)$. Then for each $\mathfrak{K} \in \mathcal{C}^{(\mathfrak{D})}_{[-1,1]} \cap \Delta^3_{[-1,1]}$ there exist piecewise polynomials $\xi_1, \xi_2 \in \Delta^3_{[-1,1]}$ of degree $\leq \kappa + \mathfrak{D} - 1$, such that ξ_1 has $\mathfrak{C}$ equidistant k-nots, satisfying

$$\|\mathfrak{K} - \xi_1\|_{[-1,1]} \leq \frac{\epsilon(\kappa,\mathfrak{D})}{\mathfrak{D}^{\mathfrak{C}}}\ \omega_\kappa(\mathfrak{K}^{(\mathfrak{D})}, 1/\mathfrak{C};\ [-1,1]),$$

And ξ_2 has k-nots on the Chebyshev partition, satisfying

$$\|\mathfrak{K} - \xi_2\|_{[-1,1]} \leq \frac{\epsilon(\kappa,\mathfrak{D})}{\mathfrak{D}^{\mathfrak{C}}}\ \omega^\varphi_\kappa(\mathfrak{K}^{(\mathfrak{D})}, 1/\mathfrak{C};\ [-1,1]).$$

Proposition 2.6 : [4]

Suppose $\xi \in \Delta^3_{[\mathfrak{d},\mathfrak{v}]}$ is a piecewise polynomial of degree $\leq \kappa$, $\kappa \geq 3$, with k-nots on the partition $\mathfrak{z}_{-1} := \mathfrak{d} =: \mathfrak{z}_0 < \mathfrak{z}_1 < \ldots < \mathfrak{z}_{\mathfrak{C}} := \mathfrak{v} =: \mathfrak{z}_{\mathfrak{C}+1}$. Then there is a piecewise polynomial ξ_1 of degree $\leq \kappa$ with the same k-nots, such that

$$\xi_1 \in \Delta^3_{[\mathfrak{d},\mathfrak{v}]} \cap \mathcal{C}^{(2)}_{[\mathfrak{d},\mathfrak{v}]},$$

and

$$\|\xi - \xi_1\| \leq \epsilon(\kappa, r, \varsigma)\, max_{1\leq j\leq \mathfrak{C}-1}\ \omega_{k+1}(\xi, \mathfrak{z}_{t+1} - \mathfrak{z}_{t-1});\ [\mathfrak{z}_{t+1}, \mathfrak{z}_{t-1}]),$$

where $\epsilon(\kappa, r, \varsigma)$depends only on κ, r, ς, where $r$is given and

$$\varsigma = max_{0\leq t\leq \mathfrak{C}}\ \frac{(t-\iota)(\mathfrak{z}_{t+1} - \mathfrak{z}_{t-1})}{\mathfrak{z}_t - \mathfrak{z}_\iota}.$$

Properties 2.7 : [4]

Let $\kappa \geq 1$ and $\mathfrak{D} \geq 0$, be integers such that either $\mathfrak{D} \geq 3$ or $\kappa + \mathfrak{D} = 4, (\kappa,\mathfrak{D}) \neq (4,0)$. Then for each $\mathfrak{K} \in \mathcal{C}^{(\mathfrak{D})}_{[-1,1]} \cap \Delta^3_{[-1,1]}$ there exist piecewise polynomials $\xi_1, \xi_2 \in \Delta^3_{[-1,1]} \cap \mathcal{C}^{(2)}_{[\mathfrak{d},\mathfrak{v}]}$ of degree $\leq \kappa + \mathfrak{D} - 1$, such that ξ_1 has $\mathfrak{C}$ equidistant k-nots, satisfying

$$\|\mathfrak{K} - \xi_1\|_{[-1,1]} \leq \frac{\epsilon(\kappa,\mathfrak{D})}{\mathfrak{D}^{\mathfrak{C}}}\ \omega_\kappa(\mathfrak{K}^{(\mathfrak{D})}, 1/\mathfrak{C};\ [-1,1]),$$

and ξ_2 has k-nots on the Chebyshev partition, satisfying

$$\|\mathfrak{K} - \xi_2\|_{[-1,1]} \leq \frac{\epsilon(\kappa,\mathfrak{D})}{\mathfrak{D}^{\mathfrak{C}}}\ \omega^\varphi_\kappa(\mathfrak{K}^{(\mathfrak{D})}, 1/\mathfrak{C};\ [-1,1]).$$

Proposition 2. 8 : [4]

Let $\mathfrak{z}_{\mathfrak{C}} < \cdots < \mathfrak{z}_1 < \mathfrak{z}_0$ be given and let $\mathfrak{K} \in \Delta^3[\mathfrak{z}_{\mathfrak{C}}, \mathfrak{z}_0]$ be a mapping with a derivative $\mathfrak{H} := \mathfrak{K}' \in \Delta^2(\mathfrak{z}_{\mathfrak{C}}, \mathfrak{z}_0)$. Suppose, that $s \in \Delta^2(\mathfrak{z}_{\mathfrak{C}}, \mathfrak{z}_0)$ is a piecewise polynomial of sizeκ (degree $\kappa - 1$) with nodes $\mathfrak{z}_{\mathfrak{C}}, \ldots, \mathfrak{z}_0$, satisfying

- $\delta(\mathfrak{z}_\iota) = \mathfrak{H}(\mathfrak{z}_\iota), \iota = 0, \ldots, \mathfrak{C}$,

- $\delta'(\mathfrak{z}_\iota+) \geq \mathfrak{H}'(\mathfrak{z}_\iota+), \iota = 1, \dots, \mathfrak{C}$,
- $\mathfrak{H}'(\mathfrak{z}_\iota-) \geq \delta'(\mathfrak{z}_\iota-), \iota = 0, \dots, \mathfrak{C} - 1$.

Then, there are at most $\mathfrak{C}$additional nodes $\theta_{\mathfrak{C}}, \dots, \theta_1$, such that

$$\mathfrak{z}_{\mathfrak{C}} < \theta_{\mathfrak{C}} < \mathfrak{z}_{\mathfrak{C}-1} < \theta_{\mathfrak{C}-1} < \mathfrak{z}_{\mathfrak{C}-2} < \cdots < \theta_1 < \mathfrak{z}_0,$$

and a piecewise polynomial $\xi \in \Delta 3[\mathfrak{z}_{\mathfrak{C}}, \mathfrak{z}_0]$ of size $\kappa + 1$ with the nodes $\mathfrak{z}_{\mathfrak{C}}, \theta_{\mathfrak{C}}, \mathfrak{z}_{\mathfrak{C}-1}, \dots, \theta_1, \mathfrak{z}_0$, satisfying

$$\| \mathfrak{K} - \xi \|_{\epsilon[\mathfrak{z}_\iota \mathfrak{z}_{\iota-1}]} \leq 2 \left\| \int_{\mathfrak{z}_\iota}^{(\cdot)} \left(\mathfrak{H}(\mathfrak{z}) - \delta(\mathfrak{z}) \right) d\mathfrak{z} \right\|_{\epsilon[\mathfrak{z}_\iota \mathfrak{z}_{\iota-1}]}, \iota = 1, \dots, \mathfrak{C}$$

and such that $\mathfrak{K}(\mathfrak{z}_\iota) = \xi(\mathfrak{z}_\iota), \iota = 0, \dots, \mathfrak{C}$.

Proposition 2. 9 : [1]

For each mapping $\mathfrak{K} \in \Delta^3$ and every $\mathfrak{C} \geq 1$, there is a quadratic spline $\xi \in \Delta^3$ on the Chebyshev partition $-1 = \mathfrak{z}_{\mathfrak{C}} < \cdots < \mathfrak{z}_1 < \mathfrak{z}_0 = 1$, satisfying

$$|\mathfrak{K}(\mathfrak{z}) - \xi(\mathfrak{z})| \leq \epsilon\, \omega_3(\mathfrak{K}, \rho_{\mathfrak{C}}(\mathfrak{z})), \mathfrak{z} \in [-1, 1],$$

where ϵ is a constant in an absolute value.

Proposition 2.10 : [1]

For each mapping $\mathfrak{K} \in \Delta^3$ and every $\mathfrak{C} \geq 2$, there exists a polynomial $P_{\mathfrak{C}} \in \Delta^3$of degree $\leq \mathfrak{C}$, satisfying

$$|\mathfrak{K}(\mathfrak{z}) - P_{\mathfrak{C}}(\mathfrak{z})| \leq \epsilon \omega_3(\mathfrak{K}, \rho_{\mathfrak{C}}(\mathfrak{z})), \mathfrak{z} \in [-1, 1],$$

where ϵ is a constant in an absolute value.

Proposition 2.11 : [2]

For every $\eta \geq 1$, there exists a constant $\epsilon_1(\eta) > 0$ so that the following statement is valid. Let $\mathfrak{H} \in \Delta^3 \cap L_p, 0 < p \leq \infty$, and let $\mathfrak{z}_{\mathfrak{C}}$ be a partition of $[-1, 1]$ such that $\eta(\mathfrak{z}_{\mathfrak{C}}) \leq \eta$. Then there exist a partition λ_r of $[-1, 1], r \leq 20\mathfrak{C}$, and a cubic ppf $\delta \in \xi_3(\lambda_r) \cap \Delta 3$ such that, for each $0 \leq \kappa \leq r - 1$, there exists $1 \leq t \leq \mathfrak{C} - 1$ such that

$$[\lambda_\kappa, \lambda_{\kappa+1}] \subseteq [\mathfrak{z}_{t-1}, \mathfrak{z}_{t+1}]$$

and

$$\lambda_{\kappa+1} - \lambda_\kappa \geq \epsilon_1(\eta)(\mathfrak{z}_{t+1} - \mathfrak{z}_{t-1}).$$

Also, for each $0 \leq t \leq \mathfrak{C} - 1$,

$$\| \mathfrak{H} - \delta \|_{L_{p[\mathfrak{z}_t, \mathfrak{z}_{t+1}]}} \leq \epsilon(\eta, p)\omega_4(f, [\mathfrak{z}_{t-1}, \mathfrak{z}_{t+2}])_p.$$

Proposition 2.12: [2]

Let $\kappa \geq 1$ and $r \geq 3$. For any $\mathfrak{H} \in \Delta^3 \cap C$ and every partition $\mathfrak{z}_{\mathfrak{C}}$ of $[-1, 1]$ such that $\kappa(\mathfrak{z}_{\mathfrak{C}}) \leq \kappa$, there exists a spline $\delta \in \xi_{\mathfrak{D}}(\mathfrak{z}_{\mathfrak{C}}) \cap \Delta^3$ of minimal defect such that

$$\| \mathfrak{H} - \delta \|_{L_\infty} \leq \epsilon(\mathfrak{D}, \kappa)\, max_{1 \leq t \leq \mathfrak{C}-1}\ \omega_4(\mathfrak{H}, [\mathfrak{z}_{t-1}, \mathfrak{z}_{t+1}])_\infty.$$

Proposition 2.13 : [2]

Let $\mathfrak{D} \geq 3$ and $\mathfrak{C} \in N$. For any $\mathfrak{H} \in \Delta^3 \cap C$, there exists a spline $\delta \in \xi_r(u_{\mathfrak{C}}) \cap \Delta^3$ of minimal defect such that

$$\|\mathfrak{H} - \delta\|_{L_\infty} \leq \epsilon(r)\omega_4(\mathfrak{H}, \mathfrak{C}^{-1}, [-1, 1])_\infty.$$

Proposition 2.14 : [2]

Let $\mathfrak{D} \geq 3$ and $\mathfrak{C} \in N$. For any $\mathfrak{H} \in \Delta^3 \cap C$, there exists a spline $\delta \in \xi_{\mathfrak{D}}(t_{\mathfrak{C}}) \cap \Delta^3$of minimal defect such that

$$\|\mathfrak{H} - \delta\|_{L_\infty} \leq \epsilon(\mathfrak{D})\omega_4^{\varphi}(\mathfrak{H}, \mathfrak{C}^{-1})_\infty,$$

where $\omega_4^{\varphi}(\mathfrak{H}, \mathfrak{C}^{-1})_\infty$ is the Ditzian–Totik modulus of smoothness of size 4.

Proposition 2.15 : [3]

Let $\kappa, \mathfrak{D} \in N, \kappa \geq 2, \mathfrak{D} \geq \kappa - 1, 0 < p \leq \infty, \mathfrak{H} \in \Delta_*^\kappa(\mathfrak{d}, \mathfrak{v}) \cap L_p[\mathfrak{d}, \mathfrak{v}]$, and let $s$be such that either $s \in \Pi_{\mathfrak{D}} \cap \Delta^\kappa(\mathfrak{d}, \mathfrak{v})$ or $(-s) \in (\Pi_{\mathfrak{D}} \setminus \Pi_\kappa) \cap \Delta^\kappa(\mathfrak{d}, \mathfrak{v})$. Then there exists s such that

$$\delta \in \xi_{c(\kappa),\mathfrak{D}}[\mathfrak{d}, \mathfrak{v}] \cap \Delta^\kappa[\mathfrak{H}](\mathfrak{d}, \mathfrak{v})$$

and

$$\| \mathfrak{H} - \delta \|_{L_{p[\mathfrak{d},\mathfrak{v}]}} \leq \epsilon(p, \mathfrak{D}, \kappa) \| \mathfrak{H} - s \|_{L_{p[\mathfrak{d},\mathfrak{v}]}}.$$

Proposition 2.16 : [2]

Let $\kappa, \mathfrak{D} \in N, \kappa \geq 2, \mathfrak{D} \geq \kappa - 1, 0 < p \leq \infty, \mathfrak{H} \in \Delta_*^\kappa(\mathfrak{d}, \mathfrak{v}) \cap L_p, \mathfrak{z}_{\mathfrak{C}}$ be a partition of $[-1, 1]$, and let σ be any ppf from $\xi_{\mathfrak{D}}(\mathfrak{z}_{\mathfrak{C}})$. Then there exist a constant $\epsilon_2 = \epsilon_2(\kappa, \mathfrak{D}) \in N$ and a ppf $\in \xi_{\epsilon_2 \mathfrak{C}, \mathfrak{D}} \cap \Delta^\kappa$, such that

- δ has $\leq \epsilon_2$ pieces in each interval $[\mathfrak{z}_t, \mathfrak{z}_{t+1}], 0 \leq t \leq \mathfrak{C} - 1$, and
- $\| \mathfrak{H} - \delta \|_{L_{p[\mathfrak{z}_t, \mathfrak{z}_{t+1}]}} \leq \epsilon(\kappa, r, p) \| \mathfrak{H} - \sigma \|_{L_{p[\mathfrak{z}_t, \mathfrak{z}_{t+1}]}}, 0 \leq t \leq \mathfrak{C} - 1.$

Proposition 2.17 : [2]

For any $\kappa \in N, A > 0, 0 < p \leq \infty, \mathfrak{D} \in N, \mathfrak{C} \in N$ and $0 < \epsilon < 2$ there exists a mapping $\mathfrak{H} \in \mathcal{C} \cap \Delta^\kappa$ such that

$$\| \mathfrak{H} - s_{\mathfrak{D}} \|_{L_{p[1-\epsilon,1]}} > A\omega_{\kappa+2}(\mathfrak{H}, [-1, 1])_p$$

for any $s_{\mathfrak{D}} \in \Pi_{\mathfrak{D}}$ satisfying $s_{\mathfrak{D}}^{(\kappa)}(1) \geq 0$.

Proposition 2.18 : [2]

For *any* $\kappa \geq 4, \mathfrak{D} \in N, 0 < p \leq \infty$ and $A > 0$, there is $\mathfrak{C} \in N$ such that, for any partition $\sigma_{\mathfrak{C}}$ of $[-1, 1]$ (into $\mathfrak{C}$ subintervals), there exists a mapping $\mathfrak{H} \in \Delta^\kappa \cap C^{\kappa-2}$ such that

$$\| \mathfrak{H} - \delta \|_{L_p} > A\omega_3(\mathfrak{H}, \mathfrak{C}^{-1}, [-1, 1])_p,$$

for any $\delta \in \xi_{\mathfrak{D}}(\sigma_{\mathfrak{C}}) \cap \Delta^\kappa$.

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