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Compactness on $\left(T_{\tilde{p}VS}\right)$-Spaces

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Abstract: In the present work, we introduce some result about the compactness properties in new type of convergence topological spaces $T_{\tilde{p}VS}$.

Introduction

Many researchers studied the topological properties, including the property of compactness, and after expanding the topological spaces to the convergence spaces, it was necessary to study those topological properties.

1- Preliminaries

Definition 1.1. [1, 3]

A filter F on a vector space K is a collection of a non-empty subsets of Ξ, such that: 1- Don't contain $\emptyset$;2- $A \in F$ and $A \subset B \Rightarrow B \in F$;

3- $A, B \in F \Rightarrow A \cap B \in F$.

Definition 1.2. [1, 3]

A filter-basis in Ξ is a collection β of non-empty subset of Ξ which satisfies:

1- Don't contain $\emptyset$; 2- For all $A_1, A_2 \in \beta$ there exists $A_3 \in \beta$ such that $A_1 \cap A_2 \supset A_3$.

Definition 1.3.[3]

Let $\mathfrak{F}(\Xi)$ denote the system of all filters on Ξ. A pseudo-topology (or a limit structure) τ on Ξ is a map $m \mapsto \tau(m)$. If F converges to m in τ, we write $F \in \tau(m)$ (or $F \downarrow_m$).

Definition 1.4. [4]

$T_{\tilde{p}VS}$ space is a locally convex pseudo metrizable pseudo topological vector space.

Definition 1.5.[2, 3]

$\mathfrak{N}_\tau(m)$ is neighborhood filter of $m \in \Xi$, it is defined as:

$\mathfrak{N}_\tau(m) = \cap\{F{:}F \in \tau(m)\}$.We call a set $N \in \mathfrak{N}_\tau(m)$ a neighborhood of m.

Definition 1.6.[3]

Let (Ξ_1,τ_1) and (Ξ_2,τ_2) be a two $T_{\tilde{p}VS}$. The all filters $\mathfrak{F}(\Xi)$ on Ξ is partially ordered by the (included relation) $F_1 \leq F_2$ iff $F_1 \supseteq F_2$. Also if $\tau_1 \geq \tau_2$ that mean $\tau_1(m) \subseteq \tau_2(m) \forall m \in \Xi_1$.

Definition 1.7.[2,6]

Ultrafilter F on a set Ξ is a maximal filter on Ξ. And also if a filter F satisfies the condition (for any $A \subset \Xi$ either $A \in \mathfrak{f}$ or $(\Xi \setminus A) \in F$,

then F is ultrafilter on Ξ).The filter $[x]$, for any $x \in \Xi$ is an ultrafilter.

Example 1.8.:

Let Ξ= {a, b, c, d},

F= {{a, d}, {a, b, d}, {a, c, d}, Ξ} is ultrafilter on Ξ.

K= {Ξ, {a}, {a, b}, {a, c}, {a, b, c}} is ultrafilter on Ξ.

Proposition 1.9 [6]:

Any filter in is Ξ contained in an ultrafilter.

Definition 1.10.[5]:

A $T_{\tilde{p}VS}$ Ξ is said to be separated iff there exists a filter F in Ξ, $F \in \tau(m)$ and
$F \in \tau(n)$, then $m = n$.

Definition 1.11. [2, 6]:
Let (Ξ, τ) be a $T_{\tilde{p}VS}$ and $A \subseteq \Xi$, then the closure of A =(CL $A) = \{m \in \Xi$ / $\exists$ $F \in \tau(m)$ and $A \in F\}$. If $A =$ CL A we call A is τclosed set (for easily closed set).
Definition 1.12. [5]

Let (Ξ_1,τ_1) and (Ξ_2,τ_2) be two $T_{\tilde{p}VS}$and f: $\Xi_1 \to \Xi_2$ a map. Then we say that a mapping f: $(\Xi_1,\tau_1)\to(\Xi_2,\tau_2)$ is continuous at a point $m \in \Xi_1$ if for all filter $F \in \tau_1$ (m) the filter $f(F) \in \tau_2(f(m))$. The mapping f is called continuous on Ξ_1 if it is continuous at each point of Ξ_1.
Definition 1.13. [5]:
Let $(\Xi,\ \tau)$ be a $T_{\tilde{p}VS}$ and $A \subseteq \Xi$. The subspace pseudo structure τ_A on A is the initial pseudo structure with respect to the inclusion mapping $in: A \to \Xi$
Let $F \in$ Ƒ (A) and $m \in A$. We say that $F \in \tau_A$ (m) if and only if
$[F]_\Xi \in \tau$ (m).

2-Theorems

We introduced the notion of compactness on $T_{\tilde{p}VS}$, and some results about this notion.
Definition 2.1.[2]
Let (Ξ,τ) be a $T_{\tilde{p}VS}$ over R , and F any filter on Ξ, then define

$F = \{m \in \Xi: F \in \tau(m)\}$.

Definition 2.2.[2]
Let (Ξ,τ) be a $T_{\tilde{p}VS}$. The point $m \in \Xi$ is called adherent to the filter F if $\exists$ a filter J, $J \leq F$, $J \in \tau(m)$.
$A_\tau(F) = \{m \in \Xi: m\ is\ adherent\ to\ F\}$, $A_\tau(F)$ is called the adherence of Ξ.

Definition 2.3.
Let (Ξ,τ) be a $T_{\tilde{p}VS}$ space, it is called compact space if every ultrafilter on Ξ converges in Ξ.
Definition 2.4.
A covering system S of a $T_{\tilde{p}VS}$ Ξ is a non-empty subset of Ξ such that if each covering filter on Ξ contain some elements of S.
Theorem 2.5:
Let (Ξ,τ_1) and (Ξ,τ_2) be two $T_{\tilde{p}VS}$ spaces, such that $\tau_1 \geq \tau_2$ then for all filter F on Ξ we have $A_{\tau_1}(F) \subseteq A_{\tau_2}(F)$
Proof:
Let $m \in A_{\tau_1}(F)$, then there exists a filter J such that $F \leq J$ and
$J \in \tau_1(m)$. Since $\tau_1(m) \subseteq \tau_2(m)$, then we get that $J \in \tau_2(m)$. Hence,
$m \in A_{\tau_2}(F)$.
Theorem 2.6:
Let (Ξ,τ) be a $T_{\tilde{p}VS}$ space. The following statements are equivalent:

1- The space (Ξ,τ) is compact.
2- Any filter F on Ξ has member of $A_\tau(F)$.
3- There are finitely many elements of which the union is Ξ, in every S covering system.

Proof: $(1 \Longrightarrow 2)$ Since for every filter F on Ξ there is an ultrafilter J on Ξ such that $J \leq F$, then J is an adherent to F.

$(2 \Longrightarrow 3)$ Let S be a covering system, such that there is no finite subcover, hence$\{\Xi - A: A \in S\}$generates a filter as F on Ξ. By hypothesis, F has an adherence point as m.Therefore, there exists a filter J such that $J \leq F$ and

$J \in \tau(m)$ for some $m \in \Xi$. By definition of covering system there exists $A \in S$ such that $A \in J.A \cap (\Xi - A) = \emptyset$, but it is contradiction, then, there are finitely many elements of which their union is Ξ in every covering system.

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$(3 \Longrightarrow 1)$ Let p be some ultrafilter J on Ξ such that does not converge in Ξ, then J can not be finer than any convergent filter F. For any $A \subseteq \Xi$ either A or $\Xi - A \in J$. Then, we find in any convergent filter F a member $A_F \in F$ for which $\Xi - A_F$ belongs to p the system $\{A_F / F$ is converge in $\Xi\}$ is a covering system of Ξ. Then there exists finitely many elements of this system that covers Ξ, then J would have to contain $\emptyset$. Hence, every ultrfilter on Ξ is a convergent filter. Therefore, (Ξ, τ) is compact.

Theorem 2.7:

Let (Ξ,τ_1) and (Ξ,τ_2) be two $T_{\tilde{p}VS}$ spaces, and $\tau_1 \geq \tau_2$. Then, if (Ξ,τ_2) is compact, then (Ξ,τ_1) is compact.

Proof:

Since, $\tau_1 \geq \tau_2$, $A_{\tau_2}(F) \subseteq A_{\tau_1}(F)$ for all $F \in \mathfrak{F}(\Xi)$ and (Ξ,τ_2) is compact, then $A_{\tau_2}(F) \neq \emptyset$. Hence, $A_{\tau_2}(F) \neq \emptyset$, by theorem 2.6.we obtain that (Ξ,τ_1) is compact.

Theorem 2.8:

Let (Ξ, τ) be a separated and compact $T_{\tilde{p}VS}$ space, a filter F converges in Ξ iff $A_\tau(F)$ is a singleton set.

Proof:

Since (Ξ, τ) is separated and let $F \in \tau(m)$ then, $m \in A_\tau(F)$. Assume that $m \neq n$ and $m \in A_\tau(F)$ then there exists $J \in \tau(n)$ such that $J \leq F$.

Hence, $J \in \tau(m) \cap \tau(n)$ this is contradiction with hypothesis. Then,

$A_\tau(F) = \{m\}$.

A Conversely, let $A_\tau(F) = \{m\}$ and $F \notin \tau(m)$. Define $\tau_1$on Ξ as follows:

$J \in \tau_1(m)$ iff $J \leq q \cap F$ where $q \in \tau(m)$ and $J \in \tau_1(n)$ iff $J \in \tau(m)$ where $m \neq n$. It is clear that τ_1 is a limit structure.

To proof that τ_1 is separated, let $m \neq n \neq k$ then

$\tau_1(n) \cap \tau_1(k) = \tau(n) \cap \tau(k) = \emptyset$ as τ is separated.

Let $J \in \tau_1(m) \cap \tau_1(n)$ where, $m \neq n$. Then, there exists a filter q such that $J \leq q \cap F$ where $q \in \tau(m)$ and $J \in \tau(n)$. We can take, J to be an ultrafilter, so there exists $F \in F$ such that $\Xi - F \in J$ can not be finer than J is separated, so there exists $g \in J$ such that $\Xi - g \in J$.

Since $J \leq q \cap F$ and

$\{A \cup B : A \in F, B \in q\}$ is a filter base generating $q \cap F$ we have

$(F \cup g) \in J$. But $(\Xi - F) \cap (\Xi - g) \in J$.

Then, $(F \cup g) \cap [(\Xi - F) \cap (\Xi - g)] = \emptyset \in J$ which is contradiction. Then, (Ξ, τ_1) is separated space and since, $\tau_1 \geq \tau$ we get a contradiction as (Ξ, τ) is separated space.

Hence, $F \in \tau(m)$.

Theorem 2.9.
Let (Ξ, τ) be a $T_{\tilde{p}VS}$ space is compact separated space and (Ξ, τ_1) be a $T_{\tilde{p}VS}$ separated space. If $\tau_1 \geq \tau$ then $\tau_1 = \tau$.
Proof:
Since (Ξ, τ) is separated and $\tau_1 \geq \tau$, $A_\tau(F) \subseteq A_{\tau_1}(F)$ for all filter F.
Let $F \in \tau_1(m) \Longrightarrow A_{\tau_1}(F) = \{m\}$ by theorem…
Since τ is compact and $A_\tau(F) \subseteq A_{\tau_1}(F) = \{m\}$ we have, $A_\tau(F) = \{m\}$.
Hence, $F \in \tau(m)$. Thus for all $m \in \Xi$ we have $\tau_1(m) \subseteq \tau(m)$ which mean $\tau \leq \tau_1$. Therefore, $\tau_1 = \tau$.

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Corollary 2.10.:

Let (Ξ, τ) be compact topological space and τ_1 be a separated limit structure on Ξ such that $\tau_1 \geq \tau$, then $\tau_1 = \tau$.

Proof:

From theorem 2.9.

Definition 2.11.:

A subset of a pseudo space is compact if it is compact with respect to the subspace limit structure.

Theorem 2.12:

Let (Ξ, τ) be a $T_{\tilde{p}VS}$ space, and $A \subseteq \Xi$ be a subspace. Then, the following is true:

1- If Ξ is compact and A is closed $\Longrightarrow A$ is compact.
2- If Ξ is separated and A is compact $\Longrightarrow A$ is closed.

Proof:

1) Since a filter $F \downarrow_a$, $a \in A$ if and only if $[F]_\Xi \downarrow_a$(definition 1.13)
Let F be an ultrafilter in A therefore, $[F]_\Xi$ is an ultrafilter in Ξ
And it converges in Ξ and Ξ is compact .
Assume that $A_\tau([F]_\Xi) \cap A = \emptyset$ so $A_\tau([F]_\Xi) \subseteq (\Xi - A)$.
Therefore, $(\Xi - A) \in [F]_\Xi$ as $(\Xi - A)$ is open
and also $[F]_\Xi = A_\tau([F]_\Xi) \neq \emptyset$. But $\in [F]_\Xi$,
so $(\Xi - A) \cap A = \emptyset \in [F]_\Xi$, which is contradiction.
This implies that there exists $m \in A$ such that $[F]_\Xi \downarrow_m$.
Hence, $F \downarrow_m$. Therefore, A is compact.
2) Let $m \in Cl(A)$ then, there exists $F \in \tau(m)$ such that $A \in F$.
$F_A = \{A \cap F / F \in F\}$ is a filter in$A$.
Let J be the ultrafilter in A containing F_A. Then, $J \in \tau_A(n)$ for some $n \in A$. But, $[J]_\Xi$ is an ultrafilter converges to n and $[J]_\Xi \leq F$.
Therefore, $[J]_\Xi \downarrow_m$. Hence, $m - n$, and also Ξ is separated space.
Thus, $m \in A$. Hence, $Cl(A) = A$, this mean A is closed set in Ξ.

Corollary 2.13:

A subspace (A, τ_A) of a compact separated (Ξ, τ) space is compact if and only if A is closed.

Proof:

Follows by theorem 2.12.

Theorem 2.14:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$ such that the mapping $f: (\Xi_1 , \tau_1) \to (\Xi_2 , \tau_2)$ is continuous surjective mapping from a compact (Ξ_1, τ_1) onto (Ξ_2, τ_2). Then, (Ξ_2, τ_2) is compact.

Proof:

Let F be an ultrafilter on Ξ_2, then $\{f^{-1}(F)/F \in F\}$ is a basis of a filter F on Ξ_1. Choose a finer ultrafilter $\leq F$.
Then H converges and $f\ (H) \leq F$. Since F is an ultrafilter we get $F = f(H)$. Since f is continuous map, then F converges in Ξ_2 .
Therefore, Ξ_2 is compact.

Corollary 2.15:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$. If $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is a continuous mapping, then the image of a compact subset of (Ξ_1, τ_1) is compact in (Ξ_2, τ_2)

Proof:

Let $A \subseteq \Xi_1$be a compact set .The restriction mapping $f_A: A \to f(A) \subseteq \Xi_2$is continuous map. Hence, by above theorem 2.14, $f(A)$ is compact.

Theorem 2.16:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$, and the mapping $f: (\Xi_1 , \tau_1) \to (\Xi_2 , \tau_2)$ is continuous mapping from a compact (Ξ_1, τ_1) onto separated space (Ξ_2, τ_2). Then, if A is a closed set in Ξ_1, then, $f(A)$ is a closed set in Ξ_2.

Proof:

A closed subset A of $\Xi_1$is compact by theorem 2.12 (1), $f(A)$ is compact by corollary 2.15 therefore, $f(A)$ is closed by theorem 2.12(2) .

Theorem 2.17:

Let (Ξ_1, τ_1) and (Ξ_2, τ_2) be two $T_{\tilde{p}VS}$, and the mapping
$f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ is continuous mapping from a compact
(Ξ_1, τ_1) onto separated space (Ξ_2, τ_2). If $B \subseteq \Xi_2$ is compact, then $f^{-1}(B)$is compact.

Proof:

Let $B \subseteq \Xi_2$be compact then B is closed set in If Ξ_2 by theorem 2.12 (2).
$f^{-1}(B)$ is closed because $f$is continuous. $f^{-1}(B)$ is compact by theorem 2.12(1).

Theorem 2.18:

If τ_1 and τ_2 are limit structures on the set Ξ, with τ_2 is separated, τ_1 is compact and $\tau_2 \leq \tau_1$ .Then, $\tau_2(m) \cap UF(\Xi) = \tau_1(m) \cap UF(\Xi)$.where $UF(\Xi)$ is the set of ultrafilters on Ξ.

Proof:

$\tau_1(m) \subseteq \tau_2(m)$ as $\tau_2 \leq \tau_1$. Hence, $\tau_1(m) \cap UF(\Xi) \subseteq \tau_2(m) \cap UF(\Xi)$.
Let $F \in \tau_2(m) \cap UF(\Xi)$ so that F is an ultrafilter converges to m.
Since τ_2 is separated then, $F = A_{\tau_2}(F) = \{m\}$ by theorem 2.8. τ_1 is compact therefore, $A_{\tau_1}(F) \neq \emptyset$, and since
$A_{\tau_1}(F) \subseteq A_{\tau_2}(F) = \{m\}$ by theorem 2.5, we get:
$F = A_{\tau_1}(F) = \{m\}$. Hence, $F \in \tau_1(m)$.
Therefore, $\tau_2(m) \cap UF(\Xi) \subseteq \tau_1(m) \cap UF(\Xi)$.
Hence, $\tau_2(m) \cap UF(\Xi) = \tau_1(m) \cap UF(\Xi)$.

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