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On the Fuzzy Normed Linear Space

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Abstract: *This research is a review of the fuzzy - normed linear space. We have presented some important theories and examples on this subject and its relationship with some other concepts.*

1. Introduction

In 1965 the concept of the fuzzy - set was introduced by L.A. Zadeh . It is a set has bounds that are not precise . This concept has been successfully applied in the study of sequence spaces by Nanda, Das, Savas and Nuray , Tripathy and many others. There is little works in different fields on this subject using the fuzzy- norm such as the work of Felbin and some others .the concept of the fuzzy norm has been studied after this concept was introduced in 2003 by Bag and Samanta. Also in 1986 the concept of intuitionistic- fuzzy set was introduced by Atanassov, as for the concept of intuitionistic- fuzzy norm on a linear space and defined an intuitionistic- fuzzy normed linear space , it was introduce by Samanta and Jebril. Intuitionistic- fuzzy set theory deals with the uncertain or imprecise situation by adopting the degree of non-membership and the degree of membership to an object through which it belongs to a collection .

Keywords: Fuzzy-Norm, Fuzzy- Normed Linear Space**,** Intuitionistic - Fuzzy Normed Linear Space

2. Fuzzy-Normed Linear Space

Definition 2.1. [5] Suppose that $\mho$ is a linear space on the field $\mathfrak{F}$. A fuzzy- subset $\mathfrak{M}$ *on* $\mho \times \mathfrak{R}$ is said a fuzzy- norm on $\mho$ iff $\gimel, \beth \in \mho$ *and* $c \in \mathfrak{F}$
(i) $\mathfrak{M}(\aleph, \gimel) = 0$ for each $\gimel$ *belongs to* $\mathfrak{R}$, $\gimel \leq 0$,
(ii) $\mathfrak{M}(\aleph, \gimel) = 1$ *if and only if* $\aleph = \theta$ for each $\gimel$ *belongs to* $\mathfrak{R}$, $\gimel > 0$.
(iii) For each $\gimel$ *belongs to* $\mathfrak{R}$, $\gimel > 0$,

$$\mathfrak{M}(c\aleph, \gimel) = \mathfrak{M}\left(\aleph, \frac{\gimel}{|c|}\right) \quad if \quad c \neq 0$$

(iv) For each $\mathfrak{s}, \gimel \in \mathbb{R}$, $\aleph, \beth \in \mho$,

$$\mathfrak{M}(\aleph + \beth, \gimel + \mathfrak{s}) \geq min\{\mathfrak{M}(\aleph, \gimel), \mathfrak{M}(\beth, \mathfrak{s})\}$$

(v) $\mathfrak{M}(\aleph, \gimel)$ is increasing function of $\mathfrak{R}$ and

$$\lim_{\gimel \to \infty} \mathfrak{M}(\aleph, \gimel) = 1 .$$

$(\mho, \mathfrak{M})$ is said a $\mathfrak{FNLS}$, which refers to fuzzy - normed linear space.

Example 2.2. [3] Suppose that $\mho$ is a linear space on the field $\mathfrak{F}$ and $\mathfrak{M} : \mho \times \mathfrak{R} \to [0,1]$ denote by

$$\mathfrak{M}(\aleph, \gimel) = \begin{cases} \dfrac{\gimel - \|\aleph\|}{\gimel + \|\aleph\|}, & \forall \gimel > \|\aleph\| \\ 0, & \forall \gimel \leq \|\aleph\| \end{cases}$$

Then $(\mho, \mathfrak{M}, *)$ be a $\mathfrak{FNLS}$.

Example 2.3. [3] Suppose that $(\mho, \|\aleph\|)$ is the normed linear space on the field $\mathfrak{F}$ and $: \mho \times \mathfrak{R} \to [0,1]$ denote by

$$\mathfrak{M}(\aleph, \gimel) = \begin{cases} 0, & \forall \gimel \leq \|\aleph\| \\ 1, & \forall \gimel > \|\aleph\| \end{cases}$$

Then $\mathfrak{M}$ be a fuzzy- norm on $\mho$ and $(\mho, \mathfrak{M}, *)$ be a $\mathfrak{FNLS}$.

Theorem 2.4. [3] Suppose that $(\mho, \mathfrak{M}, *)$ be a $\mathfrak{FNLS}$. $\forall \sigma \in [0,1)$ we denote

$$|\aleph|_\sigma = \wedge \{ \gimel > 0 : \mathfrak{M}(\aleph, \gimel) \geq \sigma \}$$

Then

(1) $|\aleph|_\sigma \geq 0$ for all $\aleph \in \mho$, for each $\sigma \in [0,1)$ and $|\aleph|_0 = 0$ for each $\aleph \in \mho$.
(2) $\left(|\aleph|_\sigma = 0 \text{ for each } \sigma \in [0,1)\right)$ ifand only if $\aleph = \theta$.
(3) $|c\aleph|_\sigma = |c||\aleph|_\sigma$ for each $\sigma \in (0,1)$.
(4) $\forall \sigma, \vartheta \in [0,1)$, $|\aleph + \beth|_{\sigma * \vartheta} \leq |\aleph|_\sigma + |\beth|_\vartheta$.
(5) $If\ \sigma \geq \vartheta\ then\ |\aleph|_\sigma \geq |z|_\vartheta$.

Theorem 2.5. [3] Suppose that $\varrho^* = \{|.|_\sigma : \sigma \in [0,1)\}$ is a $*$-quasi norm family on a linear space $\mho$, where $|\aleph|_\sigma = 0$ if and only if $\aleph = \theta \ \forall \sigma \in (0,1)$, and

$$\mathfrak{M}'(\aleph, \gimel) = \begin{cases} \vee\{\sigma \in [0,1): |\aleph|_\sigma \le \gimel\} & \forall (\aleph,\gimel) \neq (\theta, 0) \\ 0 & \forall (\aleph,\gimel) = (\theta, 0) \end{cases}$$

Then $(\mho, \mathfrak{M}', *)$ *is a* $\mathfrak{FNLS}$.

Theorem 2.6. [3] If $\{\aleph_n\}$ be a sequence in $\mho$ and $(\mho, \mathfrak{M}, *)$ be a $\mathfrak{FNLS}$ such that $\mathfrak{M}(\aleph, \gimel) > 0 \ \forall \gimel > 0 \longrightarrow \aleph = \theta$, thus $\{\aleph_n\} \to \aleph$ with respect to $\mathfrak{M} \longleftrightarrow \{\aleph_n\} \to \aleph$ with respect to ϱ^*.

Theorem 2.7. [3] Suppose that $\{\aleph_n\}$ be a sequence in a $\mathfrak{FNLS}$ $(\aleph, \mathfrak{M}, *)$ such that $\mathfrak{M}(\aleph, \gimel) > 0 \ \forall \gimel > 0 \longrightarrow \aleph = \theta$, then $\{\aleph_n\}$ be a Cauchy sequence in $(\mho, \mathfrak{M}, *)$ iff it is a Cauchy sequence in $(\mho, \varrho^*)$.

Theorem 2.8. [1] Suppose that $(\mho, \|\cdot\|)$ be a $\mathfrak{FNLS}$, then $bv_p^{\mathcal{F}}(\mho)$, $1 \le p < \infty$ is $\mathfrak{FNLS}$ - sequence valued space, such that $bv_p^{\mathcal{F}}(\mho)$ is the class of $p-$ bounded variation sequences.

Theorem 2.9. [1] Suppose that $(\mho, \|\cdot\|)$ be a $\mathfrak{FNLS}$, then $bv_p^{\mathcal{F}}(\mho)$, $1 \le p < \infty$ is complete with the fuzzy-norm

$$\|\aleph\| = \|\aleph_1\| \oplus \left\{\sum_{k=1}^{\infty} \|\aleph_k - \aleph_{k+1}\|^p\right\}^{1/p},$$

where $bv_p^{\mathcal{F}}(\mho)$ is the class of $p-$ bounded variation sequences, $\aleph = (\aleph_k) \in bv_p^{\mathcal{F}}(\mho)$ and $\mho$ is complete.

Theorem 2.10. [1] Suppose that $(\mho, \|\cdot\|)$ be a $\mathfrak{FNLS}$, then $bv_p^{\mathcal{F}}(\mho)$, $1 \le p < \infty$ is neither solid nor monotonous.

Theorem 2.11. [1] Suppose that $(\mho, \|\cdot\|)$ be a $\mathfrak{FNLS}$, then $bv_p^{\mathcal{F}}(\mho)$, $p > 1$ is not symmetrical.

Theorem 2.12. [1] Suppose that $(\mho, \|\cdot\|)$ be a $\mathfrak{FNLS}$, then $bv_p^{\mathcal{F}}(\mho)$, $1 \le p < \infty$ is not free of convergence.

Theorem 2.13. [1] Suppose that $(\mho, \|\cdot\|)$ be a $\mathfrak{FNLS}$, then $bv_q^{\mathcal{F}}(\mho) \subset bv_p^{\mathcal{F}}(\mho)$.

Theorem 2.14. [4] Let $\{\aleph_n\}_n$ be a sequence in an intuitionistic - $\mathfrak{FNLS}$ $(\xi, \mathfrak{H})$, then $\{\aleph_n\}_n$ converges to $\aleph \in \xi$ iff $\lim_{n\to\infty} \mathcal{M}(\aleph_n - \aleph, \gimel) = 1$ and $\lim_{n\to\infty} \mathcal{M}(\aleph_n - \aleph, \gimel) = 0$.

Theorem 2.15. [4] Suppose that $\{\aleph_n\}_n$ is convergent in an intuitionistic - $\mathfrak{FNLS}$ $(\xi, \mathfrak{H})$, then it's limit is unique.

Theorem 2.16. [4] Suppose that $\lim_{n\to\infty} \aleph_n = \aleph$ and $\lim_{n\to\infty} \beth_n = \beth$ *then* $\lim_{n\to\infty} \aleph_n + \beth_n = \aleph + \beth$ in an intuitionistic - $\mathfrak{FNLS}$ $(\xi, \mathfrak{H})$.

Theorem 2.17. [4] Suppose that $\lim_{n\to\infty} \aleph_n = \aleph$ and $\neq 0 \in \mathfrak{F}$, then $\lim_{n\to\infty} c\aleph_n = c\aleph$ in an intuitionistic - $\mathfrak{FNLS}$ $(\xi, \mathfrak{H})$.

Theorem 2.18. [4] Suppose that $(\xi, \mathfrak{H})$ be an intuitionistic - $\mathfrak{FNLS}$ and assume that

(1) $\begin{matrix} b \circ b = b \\ b * b = b \end{matrix}$ for all $b \in [1,0]$

(2) $\mathfrak{M}(\aleph, \gimel) > 0$ for all $\gimel > 0 \rightarrow \gimel = \underline{0}$

(3) $\mathfrak{N}(\aleph, \gimel) > 0$ for all $\gimel > 0 \rightarrow \gimel = \underline{0}$.

Denote $\|\aleph\|_\sigma^1 = \wedge\{\gimel : \mathfrak{M}(\aleph, \gimel) \ge \sigma\}$ and $\|\aleph\|_\sigma^2 = \vee\{\gimel : \mathfrak{N}(\aleph, \gimel) \le \sigma\}, \sigma \in (0, 1)$, then $\{\|\aleph\|_\sigma^1, \sigma \in (0,1)\}$ and $\{\|\aleph\|_\sigma^2, \sigma \in (0,1)\}$ ascend to the family of norms on ξ. these norms are said $\sigma - norm\ on\ \xi$ corresponding to the $\mathfrak{H}$ on ξ.

Theorem 2.19. [4] Suppose that $(\xi, \mathfrak{H})$ be an intuitionistic - $\mathfrak{FNLS}$ and $\{\aleph_1, \aleph_2, \dots, \aleph_n\}$ be a finite collection of linearly independent vectors of ξ. Then $\forall \sigma \in (0,1)$ there is a constant $C_\sigma > 0$, where $\forall \sigma_1, \sigma_2, \dots, \sigma_n$,

$$\|\aleph_1\sigma_1 + \aleph_2\sigma_2 + \cdots + \aleph_n\,\sigma_n\|_\sigma^1 \geq C_\sigma \sum_{i=1}^{n} |\,\sigma_i|.$$

Example 2.20. [4] Suppose that $(\xi\,, \|\aleph\|\,)$ be a normed linear space , $\beta * \delta = min\,\{\,\beta\,, \delta\,\}$ and $\beta \circ \delta = max\,\{\,\beta\,, \delta\,\}$, $\forall\, \beta\,, \delta$ belongs to $[0, 1\,]$. $\forall\, \lambda > 0$, there is $\mathfrak{M}\,(\aleph, \lambda) = \frac{\lambda}{\lambda + d\|\aleph\|}$ and $\mathfrak{N}\,(\aleph, \lambda) = \frac{d\|\aleph\|}{\lambda + d\|\aleph\|}$, such that $d > 0$. This leads to $\mathfrak{H} = \{\,((\aleph, \lambda)\,, \mathfrak{N}\,(\aleph, \lambda)\,, \mathfrak{M}\,(\aleph, \lambda)\,)$:
$(\aleph, \lambda)$ belongs to $\xi \times \mathfrak{R}^+\,\}$ is an intuitionistic - fuzzy normed over ξ , also the sequence $\{\,\aleph_n\,\}_n$ is Cauchy in $(\xi\,, \|\aleph\|\,)$, iff it is a Cauchy in $(\,\xi\,, \mathfrak{H}\,)$ and the sequence $\{\,\aleph_n\,\}_n$ is convergent in $(\xi\,, \|\aleph\|\,)$ iff it is convergent in $(\,\xi\,, \mathfrak{H}\,)$.

Theorem 3.1. [2] suppose that $(\mathfrak{R}^n, \|\cdot\|\,)$ be a normed linear space on $\mathfrak{F}$, $\beta * \delta = min\,\{\,\beta\,, \delta\,\}$ and $\beta \circ \delta = max\,\{\,\beta\,, \delta\,\}$, $\forall\, \beta\,, \delta$ belongs to $[0, 1\,]$. Also $\forall\, \lambda > 0$, there is $\mathfrak{Q}\,(\aleph, \lambda) = \lambda/(\lambda + \|\aleph\|\,)$ *and* $\mathfrak{G}\,(\aleph, \lambda) = \|\aleph\|/(\lambda + \|\aleph\|\,)$, then
$\mathfrak{H} = \left\{\left((\aleph, \lambda)\,, \mathfrak{Q}\,(\aleph, \lambda)\,, \mathfrak{G}\,(\aleph, \lambda)\,\right), (\aleph, \lambda) \in \mathfrak{R}^n \times \mathfrak{R}^+\right\}$ is an intuitionistic – fuzzy normed over $\mathfrak{R}^n$, thus $(\mathfrak{R}^n, \mathfrak{H})$ *is an* intuitionistic - $\mathfrak{FNLS}$.

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