International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 11, November - 2022, Pages: 49-55

Restrict Nearly Prime Submodules

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Abstract**:** *We note in this work Restrict Nearly prime submodules was defined as a new popularization of prime submodules. Several basic properties, characterizations and examples of Restrict Nearly Prime submodules were given. Furthermore, some of it is properties were established.*

Keywords: prime submodule, Restrict Nearly prime submodule.

1. INTRODUCTION

A prime submodule was first introduced by Dauns in 1978 [4]. This concept was generalized to primary submodules by Lu in1989 [8], and to semiprime submodules by Athab in 1996 [9], and to quasi prime submodules by Abdul-Razak in 1999 [11]. Many others focused on generalizing prime submodules such as quasi primary submodules by Shahcbaddin in 2006 [10], and in 2019 Ali Sh. And Haibut introduced the concept (Approximaitly prime, Approximaitly semi-prime) submodules as generalizations of prime submodules see [13, 14]. We denoted to the submodule by submodule.

2. Basic notions

Definition 2.1 [4]

A proper submodule $\mathcal{A}$ of an $\mathfrak{R}$_module Ω is named a prime submodule if $\mathrm{r}\,\omega \in \mathcal{A}$ for $\mathrm{r} \in \mathfrak{R}$, $\omega \in \Omega$ implies that either $\omega \in \mathcal{A}$ or $\mathrm{r} \in [\mathcal{A}:_{\mathfrak{R}} \Omega]$.

Definition 2.2 [5]

The resudule of a submodule $\mathcal{A}$ of an $\mathfrak{R}$_module Ω shortened by $[\mathcal{A}:_{\mathfrak{R}} \Omega]$ is an ideal of $\mathfrak{R}$ defined by $[\mathcal{A}:_{\mathfrak{R}} \Omega] = \{\mathrm{r} \in \mathfrak{R}: \mathrm{r}\,\Omega \subseteq \mathcal{A}\}$.

Definition 2.3 [6]

The socle of an $\mathfrak{R}$_module Ω shortened by $soc(\Omega)$ is defined as the intersection of all essential submodule of Ω.

Definition 2.4 [3]

A non-zero submodule $\mathcal{A}$ of an $\mathfrak{R}$_module Ω is named an essential in Ω if $\mathcal{A} \cap N \neq (0)$ for each non-zero submodule $\mathcal{A}$ of Ω.

Definition 2.5 [6]

The Jacobson of an $\mathfrak{R}$_module Ω shortened by $J(\Omega)$ is defined as the intersection of all max submodule of Ω.

Proposition 2.6 [10, p. 16]

Let Ω be an $\mathfrak{R}$_module, and $\mathcal{A}$ be a submodule of $\mathfrak{R}$, $\mathcal{J}$ is an ideal of $\mathfrak{R}$, then $[\mathcal{A}:_{\Omega} \mathfrak{R}] = \mathcal{A}$ and $[\mathcal{J}:\mathfrak{R}] = \mathcal{J}$.

Proposition 2.7 [1, Theo. (5.1)]

Let $\mathcal{J}$ be a proper ideal of a ring $\mathfrak{R}$. Then $\mathcal{J}$ is max ideal iff $\mathcal{J} + \langle \mathrm{r} \rangle = \mathfrak{R}$ for any $\mathrm{r} \notin I$.

Proposition 2.8 [7, Coro. (2.14)(i)]

If Ω be a faithful multiplication $\mathfrak{R}$_module, then $soc(\mathfrak{R})\,\Omega = soc(\Omega)$.

Definition 2.9[11, Remark, P14]

If Ω be a faithful multiplication $\mathfrak{R}$_module, then $J(\mathfrak{R})\,\Omega = J(\Omega)$.

Defenition 2.10 [9]

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 11, November - 2022, Pages: 49-55

A submodules $\mathcal{A}$, β of an R-module $\mathcal{M}$ is called comaximal if $+\beta = \mathcal{M}$

3. Charctrizations of RNP submodules Multiplication Modules.

Definition 3.1

A proper submodule $\mathcal{A}$ of an $\mathfrak{R}$_module Ω is named a (Restrict nearly Prime) (for short RNP) submodule of M, if whenever r$\omega \in \mathcal{A}$, for $r \in \mathfrak{R}$, $\omega \in \Omega$ implies that either $\omega \in \mathcal{A} + (J(\Omega) \cap \mathrm{Soc}(\Omega))$ Or $r \in \left[\left(\mathcal{A} + \left(J(\Omega) \cap Soc(\Omega)\right)\right) :_{\mathfrak{R}} \Omega\right]$.

And an ideal $\mathcal{J}$ of a ring $\mathfrak{R}$ is named an RNP ideal of $\mathfrak{R}$ if $\mathcal{J}$ is RNP $\mathfrak{R}$_submodule of an $\mathfrak{R}$-module $\mathfrak{R}$.

Example and Remarks 3.2

1.Consider the Z-module Z_{60}, we see that the only essential submodules of Z_{60} are Z_{60} itself and the submodules $<\bar{2}>$, so $\mathrm{Soc}(Z_{60}) = Z_{60} \cap <\bar{2}> = <\bar{2}>$. And the only maximal submodules of Z_{60} are $<\bar{2}>$, $<\bar{3}>$ and $<\bar{5}>$, so $J(Z_{60}) = <\bar{2}> \cap <\bar{3}> \cap <\bar{5}> = <\overline{30}>$. Hence $\mathrm{Soc}(Z_{60}) \cap J(Z_{60}) = <\bar{2}> \cap <\overline{30}> = <\overline{30}>$.

2. The submodules $<\bar{2}>$, $<\bar{3}>$, $<\bar{4}>$ and $<\bar{5}>$ of the Z-module Z_{60} are RNP submodules of Z_{60}. Since $r\varpi \in <\bar{4}>$, for $\varpi \in Z_{60}$, $r \in Z$, implies that either $\varpi \in <\bar{4}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) = <\bar{4}> + <\overline{30}> = <\bar{2}>$ or $r \in [<\bar{4}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) : Z_{60}] = [<\bar{2}>: Z_{60}] = 2Z$, for example $2.\bar{2} \in <\bar{4}>$, for $\bar{2} \in Z_{60}$, $2 \in Z$, implies that $\bar{2} \in <\bar{4}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) = <\bar{2}>$ or $2 \in [<\bar{4}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) : (Z_{60})] = [<\bar{2}>: (Z_{60})] = 2Z$. Similarly for the other elements of $<\bar{4}>$.

3. The submodules $<\bar{6}>$, $<\overline{10}>$, $<\overline{12}>$, $<\overline{15}>$, $<\overline{20}>$ and $<\overline{30}>$ of the Z-module Z_{60} are not RNP submodules of Z_{60}. Since $r\varpi \in <\bar{6}>$, for $\varpi \in Z_{60}$, $r \in Z$, implies that either $\varpi \in <\bar{6}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) = <\bar{6}> + <\overline{30}> = <\bar{6}>$ or $r \in [<\bar{6}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) : Z_{60}] = [<\bar{6}>: Z_{60}] = 6Z$, for example $3.\bar{2} \in <\bar{6}>$, for $\bar{2} \in Z_{60}$, $3 \in Z$, but $\bar{2} \notin <\bar{6}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) = <\bar{6}>$ and $3 \notin [<\bar{6}> + (\mathrm{Soc}(Z_{60}) \cap J(Z_{60})) : (Z_{60})] = [<\bar{6}>: (Z_{60})] = 6Z$. Similarly for the other elements of $<\bar{6}>$.

4. The submodule 4Z of Z_module Z is not RNP submodule of Z_module Z since $2.2 \in 4Z$, for $2 \in Z$, but $2 \notin 4Z + (\mathrm{Soc}(Z) \cap J(Z)) = 4Z + (0) = 4Z$.

5. The resudule of RNP submodule $\mathcal{A}$ of an R_module $\boldsymbol{\Omega}$ need not to be an RNP ideal of R. For example consider the Z_module Z_{60}, and the submodule $\mathcal{A} = <\bar{4}>$ is RNP submodule by (2) but $[\mathcal{A} :_Z Z_{60}] = [<\bar{4}> :_Z Z_{60}] = 4Z$ is not RNP ideal of Z by (4).

6. The intersection of two nonzero RNP submodules is not necessarily RNP submodules For example let $\mathfrak{R}$=Z, $\boldsymbol{\Omega} = Z_{60}$, $\mathcal{A}_1 = <\bar{2}>$, $\mathcal{A}_2 = <\bar{3}>$ we have $<\bar{2}>$ and $<\bar{3}>$ are RNP submodules of Z_{60}, but $\mathcal{A}_1 \cap \mathcal{A}_2 = <\bar{2}> \cap <\bar{3}> = <\bar{6}>$ This submodule is not RNP submodule since $3 \cdot \bar{2} \in <\bar{6}>$ for $3 \in Z$, $\bar{2} \in Z_{60}$, but $\bar{2} \notin <\bar{6}> + (J(Z_{60}) \cap Soc(Z_{60}) = <\bar{6}> + <\overline{30}> = <\bar{6}>$ and $3 \notin [<\bar{6}> + (J(Z_{60}) \cap Soc(Z_{60}) : Z_{60}] = [<\bar{6}> + <\overline{30}> : Z_{60}] = [<\bar{6}> : Z_{60}] = 6Z$.

7. If $\mathcal{A}_1$, $\mathcal{A}_2$ are submodules of $\boldsymbol{\Omega}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. If $\mathcal{A}_2$ is an RNP submodule of $\boldsymbol{\Omega}$, but $\mathcal{A}_1$ need not to be RNP submodule of $\boldsymbol{\Omega}$. As in the following:

The submodules $\mathcal{A}_1 = <\bar{6}>$, $\mathcal{A}_2 = <\bar{2}>$ of the Z-module Z_{60}, for $<\bar{6}> \subseteq <\bar{2}>$,we show that $<\bar{2}>$ is an RNP submodule of Z_{60} by (2). But $<\bar{6}>$ is not RNP submodule of Z_{60} by (3).

8.Every prime submodule $\mathcal{A}$ of an $\mathfrak{R}$-module $\boldsymbol{\Omega}$ is RNP submodule but not conversely.

Proof

Assume that $\mathcal{A}$ be a prime submodule of $\boldsymbol{\Omega}$, and $r\,\omega \in \mathcal{A}$, for $r \in \mathfrak{R}$, $\omega \in \boldsymbol{\Omega}$. Since $\mathcal{A}$ is a prime submodule, then either $\omega \in \mathcal{A} \subseteq \mathcal{A} + (J(\Omega) \cap \mathrm{Soc}(\boldsymbol{\Omega}))$ implies that $\omega \in \mathcal{A} + (J(\Omega) \cap \mathrm{Soc}(\boldsymbol{\Omega}))$ or $r \in [\mathcal{A} :_{\mathfrak{R}} \boldsymbol{\Omega}]$, that is $r\boldsymbol{\Omega} \subseteq \mathcal{A} \subseteq \mathcal{A} + (J(\Omega) \cap \mathrm{Soc}(\boldsymbol{\Omega}))$ implies that $r \in \left[\left(\mathcal{A} + \left(J(\Omega) \cap Soc(\Omega)\right)\right) : \Omega\right]$. Hence is an RNP submodule of $\boldsymbol{\Omega}$. ■

For the converse consider the following example the submodule of the Z-module Z_{60} $<\bar{4}>$ is not prime since $2\cdot\bar{2}\in<\bar{4}>$ for $2\in Z$, $\bar{2}\in Z_{60}$, but $\bar{2}\notin<\bar{4}>$ and $2\notin[<\bar{4}>:_Z Z_{60}]=4Z$. On the other hand $<\bar{4}>$ is RNP submodule since $Soc(Z_{60})\cap J(Z_{60}) = <\bar{2}>\cap<\overline{30}> = <\overline{30}>$, and $\mathcal{A}+(Soc(Z_{60})\cap J(Z_{60})) = <\bar{4}>+<\overline{30}> = <\bar{2}>$ it follows that for $r\bar{\varpi}\in<\bar{4}>$ for $r\in Z$, $\bar{\varpi}\in Z_{60}$ implies that $\bar{\varpi}\in<\bar{4}>+(J(Z_{60})\cap Soc(Z_{60})) = <\bar{2}>$ and, $r\in[(<\bar{4}>+(J(Z_{60})\cap Soc(Z_{60})):Z_{60}]=2Z$. For example whenever $2\cdot\bar{2}\in<\bar{4}>$ for $2\in Z$, $\bar{2}\in Z_{60}$, implies that $\bar{2}\in<\bar{4}>+<\overline{30}>=<\bar{2}>$, or $2\in[<\bar{4}>+<\overline{30}>:_Z Z_{60}]=[<\bar{2}>:Z_{60}]=2Z$. Similarly for other elements in $<\bar{4}>$.

The following properties are characterizations of RNP submodules.

Proposition 3.3

Suppose that Ω be an $\mathfrak{R}$_module, and $\mathcal{A}$ be a submodule of Ω. Then $\mathcal{A}$ is an RNP submodule of Ω if and only if for every submodule y of Ω and every ideal I of $\mathfrak{R}$ with $Iy\subseteq\mathcal{A}$, implies that either $y\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $I\subseteq[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$.

Proof

($\Longrightarrow$) Assume that $Iy\subseteq\mathcal{A}$, for y is a submodule of Ω and I is an ideal of $\mathfrak{R}$, with $y\nsubseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$, then there exists $x\in y$ such that $x\notin(soc(\Omega)\cap J(\Omega))$. Since $Iy\subseteq\mathcal{A}$ then for any $a\in I$, $ax\in\mathcal{A}$. But $\mathcal{A}$ is an RNP submodule of Ω and $x\notin\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ then $a\in[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$ that is $I\subseteq[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$.

($\Longleftarrow$) Suppose that $a\,\omega\in\mathcal{A}$, for $a\in\mathfrak{R}$, $\omega\in\Omega$, then $(a)(\omega)\subseteq\mathcal{A}$, by hypothesis either $(\omega)\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $(a)\subseteq[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$. That is either $\omega\in\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $a\in[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$. Hence $\mathcal{A}$ is an RNP submodule of Ω. ■

The following corollaries are a direct consequence of above proposition.

Corollary 3.4

Let Ω be an $\mathfrak{R}$_module and $\mathcal{A}$ be a submodule of Ω. Then $\mathcal{A}$ is an RNP submodule of Ω if and only if for every submodule y of Ω and every $r\in\mathfrak{R}$ with $ry\subseteq\mathcal{A}$, implies that either $y\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $r\in[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$.

Corollary 3.5

Let Ω be an $\mathfrak{R}$_module, and $\mathcal{A}$ be a proper submodule of M. Then $\mathcal{A}$ is an RNP submodule of Ω if and only if whenever $J\,\omega\subseteq\mathcal{A}$ for J is an ideal of $\mathfrak{R}$, $\omega\in\Omega$, implies that either $\omega\in\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $J\subseteq[\mathcal{A}+(soc(\Omega)\cap J(\Omega)):_{\mathfrak{R}}\Omega]$.

The following proposition indicates that the resudule of an RNP submodule is an RNP ideal of $\mathfrak{R}$ under certain condition.

Proposition 3.6

Let $\mathcal{A}$ be an RNP submodule of an $\mathfrak{R}$_module Ω, with $soc(\Omega)\subseteq\mathcal{A}$ and $J(\Omega))\subseteq\mathcal{A}$. Then $[\mathcal{A}:_{\mathfrak{R}}\Omega]$ is an RNP ideal of $\mathfrak{R}$.

Proof

Assume that $\mathcal{A}$ is an RNP submodule of Ω and $r\mathcal{J}\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]$ for $r\in\mathfrak{R}$, $\mathcal{J}$ is an ideal of $\mathfrak{R}$ implies that $r\mathcal{J}\Omega\subseteq\mathcal{A}$, where $\mathcal{J}\Omega$ is a submodule of Ω. Since $\mathcal{A}$ is an RNP submodule of Ω then by corollary (3.4) either $\mathcal{J}\Omega\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $r\Omega\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$. But $(soc(\Omega)\cap J(\Omega))\subseteq\mathcal{A}$ implies that $\mathcal{A}+(soc(\Omega)\cap J(\Omega))=\mathcal{A}$, it follows that either $\mathcal{J}\Omega\subseteq\mathcal{A}$ or $r\Omega\subseteq\mathcal{A}$. That is either $\mathcal{J}\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]+(soc(\mathfrak{R})\cap J(\mathfrak{R}))$ or $r\in[\mathcal{A}:_{\mathfrak{R}}\Omega]\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]+\big(soc(\mathfrak{R})\cap J(\mathfrak{R})\big)=\big[[\mathcal{A}:_{\mathfrak{R}}\Omega]+(soc(\mathfrak{R})\cap J(\mathfrak{R})):_{\mathfrak{R}}\mathfrak{R}\big]$ by proposition(2.7), hence by corollary (3.5) $[\mathcal{A}:_{\mathfrak{R}}\Omega]$ is an RNP ideal of $\mathfrak{R}$. ■

Proposition 3.7

Let Ω be an $\mathfrak{R}$_module, and $\mathcal{A},y$ are submodules of Ω, with $\mathcal{A}$ is a proper submodule of y, and $\mathcal{A}$ is an RNP submodule of Ω and $(soc(\Omega)\cap J(\Omega))\subseteq(soc(y)\cap J(y))$, then $\mathcal{A}$ is an RNP submodule of y.

Proof

Suppose that $r\,\omega\in\mathcal{A}$, for $r\in\mathfrak{R}$, $\omega\in y$. Since y is a submodule of Ω implies that $\omega\in\Omega$. Now, for $\mathcal{A}$ is an RNP submodule of Ω, then either $\omega\in\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $r\Omega\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$. But $(soc(\Omega)\cap J(\Omega))\subseteq(soc(y)\cap J(y))$, it follows that either $\omega\in\mathcal{A}+(soc(y)\cap J(y))$ or $r\Omega\subseteq\mathcal{A}+(soc(y)\cap J(y))$. That is $\mathcal{A}$ is an RNP submodule of y. ■

Proposition 3.8

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
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Let Ω be an $\mathfrak{R}$_module, and $\mathcal{A}$ be a submodule of Ω with $[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$ is a max ideal of $\mathfrak{R}$. Then $\mathcal{A}$ is an RNP submodule of Ω.

Proof

Suppose that $\mathrm{r}\,\mathrm{w}\in \mathcal{A}$, for $\mathrm{r}\in\mathfrak{R}$, $\mathrm{w}\in\Omega$, with $\mathrm{r}\notin[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$. Since $[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$ is a maximal ideal of $\mathfrak{R}$, then by proposition (2.6) $\mathfrak{R}=\langle \mathrm{r}\rangle+[\mathcal{A}+\big(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)\big):_{\mathfrak{R}}\Omega]$, where $\langle \mathrm{r}\rangle$ is an ideal of $\mathfrak{R}$ generated by r, it follows that $\exists\ a\in\mathfrak{R}$ and $b\in[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$ such that $1=a\mathrm{r}+b$, hence $\mathrm{w}=a\mathrm{r}\,\mathrm{w}+b\,\mathrm{w}\in\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$. Hence $\mathcal{A}$ is an RNP submodule of Ω. ■

Proposition 3.9

Let Ω be an $\mathfrak{R}$_module, and J be a maximal ideal of $\mathfrak{R}$, with $J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$ is a proper submodule of Ω. Then $J\Omega$ is an RNP submodule of Ω.

Proof

Since $J\Omega\subseteq J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$, implies that $J\subseteq[J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$, that is there exists $a\in[J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$ and $a\notin J$, but J is a maximal ideal then by proposition (2.7) $J+\langle a\rangle=\mathfrak{R}$, where $\langle a\rangle$ is an ideal of $\mathfrak{R}$ generated by a, thus $\exists\ r\in\mathfrak{R}$ and $j\in J$ such that $1=j+ar$, it follows that $\mathrm{w}=j\,\mathrm{w}+ar\,\mathrm{w}$ for each $\mathrm{w}\in\Omega$. Hence $\mathrm{w}\in J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$, for each $\mathrm{w}\in\Omega$, that is $\Omega\subseteq J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$, hence $\Omega=J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$ which is a contradiction. Then $a\in J$ and hence $[J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]\subseteq J$, it follows that $[J\Omega+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]=J$ is a maximal ideal of $\mathfrak{R}$, hence by proposition (3.8) $J\Omega$ is an RNP submodule of Ω. ■

Proposition 3.10

Let Ω be a faithful multiplication $\mathfrak{R}$_module and $\mathcal{A}$ a proper submodule of Ω. Then $\mathcal{A}$ is an RNP submodule of Ω if and only if $[\mathcal{A}:_{\mathfrak{R}}\Omega]$ is an RNP ideal of $\mathfrak{R}$.

Proof

($\Longrightarrow$) Let $r\,\mathrm{w}\in[\mathcal{A}:_{\mathfrak{R}}\Omega]$ for $r,\mathrm{w}\in\mathfrak{R}$, implies that $r(\mathrm{w}\Omega)\subseteq\mathcal{A}$. But $\mathcal{A}$ is an RNP submodule of Ω, then by corollary (3.5) either $\mathrm{w}\Omega\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$ or $r\Omega\subseteq\mathcal{A}+(soc(\Omega)\cap J(\Omega))$. Since Ω is multiplication, then $\mathcal{A}=[\mathcal{A}:_{\mathfrak{R}}\Omega]\Omega$, and since Ω is faithful multiplication, then by proposition(2.8) and(2.9) $\big(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R})\big)\Omega=(\mathrm{soc}(\mathrm{R})\Omega\cap \mathrm{J}(\mathrm{R})\Omega)=(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$. Thus either $\mathrm{w}\Omega\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]\Omega+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))\Omega$ or $r\Omega\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]\Omega+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))\Omega$, it follows that either $\mathrm{w}\in[\mathcal{A}:_{\mathfrak{R}}\Omega]+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))$ or $r\in[\mathcal{A}:_{\mathfrak{R}}\Omega]+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))=\big[[\mathcal{A}:_{\mathfrak{R}}\Omega]+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R})):_{\mathfrak{R}}\mathfrak{R}\big]$. Hence $[\mathcal{A}:_{\mathfrak{R}}\Omega]$ is RNP ideal of $\mathfrak{R}$.

($\Longleftarrow$) Let $rK\subseteq\mathcal{A}$ for $r\in\mathfrak{R}$ and K is a submodule of Ω. Since Ω is a multiplication, then $K=\mathcal{J}\Omega$ for some ideal $\mathcal{J}$ of $\mathfrak{R}$, that is $r\mathcal{J}\Omega\subseteq\mathcal{A}$, implies that $r\mathcal{J}\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]$, but $[\mathcal{A}:_{\mathfrak{R}}\Omega]$ is an RN-prime ideal of $\mathfrak{R}$, then by corollary (3.5) either $\mathcal{J}\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]+(\mathrm{soc}(\mathrm{R})\cap \mathrm{J}(\mathrm{R}))$ or $r\in\big[[\mathcal{A}:_{\mathfrak{R}}\Omega]+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R})):_{\mathfrak{R}}\mathfrak{R}\big]=[\mathcal{A}:_{\mathfrak{R}}\Omega]+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))$. Hence either $\mathcal{J}\Omega\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]\Omega+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))\Omega$ or $r\Omega\subseteq[\mathcal{A}:_{\mathfrak{R}}\Omega]\Omega+(\mathrm{soc}(\mathfrak{R})\cap \mathrm{J}(\mathfrak{R}))\Omega$. Hence by proposition (2.8) and (2.9) either $\mathcal{J}\Omega\subseteq\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$ or $r\Omega\subseteq\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$. That is either $K\subseteq\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$ or $r\in[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$. Thus by corollary (3.4) $\mathcal{A}$ is an RNP submodule of Ω. ■

Proposition 3.11

Let Ω be an $\mathfrak{R}$_module, and $\mathcal{A}$ be a proper submodule of Ω, with $[L:_{\mathfrak{R}}\Omega]\nsubseteq[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$, and $\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega))$ is a proper submodule of L for each submodule L of Ω such that $[\mathcal{A}+(\mathrm{soc}(\Omega)\cap \mathrm{J}(\Omega)):_{\mathfrak{R}}\Omega]$ is a prime ideal of $\mathfrak{R}$. Then $\mathcal{A}$ is an RNP submodule of Ω.

Proof

Suppose that $r\,w \in \mathcal{A}$, for $r \in \mathfrak{R}$, $w \in \Omega$, with $w \notin \mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$. Then $\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)) \subsetneq \mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)) + \langle w \rangle = L$ and so $[L{:}_{\mathfrak{R}}\,\Omega] \nsubseteq [\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)){:}_{\mathfrak{R}}\,\Omega]$, then there exists $a \in [L{:}_{\mathfrak{R}}\,\Omega]$ and $a \notin [\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)){:}_{\mathfrak{R}}\,\Omega]$. That is $a\Omega \subseteq L$ and $a\Omega \nsubseteq \mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$. Thus $a\Omega \subseteq L$, implies that $ra\Omega \subseteq r(\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)) + \langle w \rangle) \subseteq \mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$. It follows that $ra \in [\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)){:}_{\mathfrak{R}}\,\Omega]$. But $[\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)){:}_{\mathfrak{R}}\,\Omega]$ is a prime ideal of $\mathfrak{R}$, and $a \notin [\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)){:}_{\mathfrak{R}}\,\Omega]$ then $r \in [\mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)){:}_{\mathfrak{R}}\,\Omega]$. Hence $\mathcal{A}$ is an RNP submodule of Ω. ■

Proposition 3.12

Let Ω be an $\mathfrak{R}$-module, and $\mathcal{A}, \beta$ are a submodules of Ω, with $\mathcal{A}, \beta$ are comaximal and $\beta \subseteq (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$, if $\mathcal{A} + \beta$ is a RNP submodule of Ω then $\mathcal{A}$ is a RNP submodule of $\mathcal{A} + \beta$.

Proof

Suppose that $\mathcal{S}\,w \in \mathcal{A} + \beta$, for $\mathcal{S} \in \mathfrak{R}$, $w \in \Omega$. Now, $\mathcal{A} + \beta$ is a RNP submodule of Ω, then either $w \in (\mathcal{A} + \beta) + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)) = \mathcal{A} + (\beta + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)))$ or $\mathcal{S}\mathcal{M} \subseteq (\mathcal{A} + \beta) + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)) = \mathcal{A} + (\beta + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega)))$. But we have $\beta \subseteq (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$, it follows that either $w \in \mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$ or $\mathcal{S}\Omega \subseteq \mathcal{A} + (\mathrm{soc}(\Omega) \cap \mathrm{J}(\Omega))$. Then by (2.10) $\mathcal{A} + \beta = \Omega$, implies that $w \in \mathcal{A} + (\mathrm{soc}(\mathcal{A} + \beta) \cap \mathrm{J}(\mathcal{A} + \beta))$ or $\mathcal{S}(\mathcal{A} + \beta) \subseteq \mathcal{A} + (\mathrm{soc}(\mathcal{A} + \beta) \cap \mathrm{J}(\mathcal{A} + \beta))$ therefore $\mathcal{S} \in [\mathcal{A} + (\mathrm{soc}(\mathcal{A} + \beta) \cap \mathrm{J}(\mathcal{A} + \beta)){:}_R\,(\mathcal{A} + \beta)]$. That is $\mathcal{A}$ is a RNP submodule of $(\mathcal{A} + \beta)$. ■

4 . Conclusion

In this paper an RNP submodules are introduced and studied as a new generalization of prime submodules.
The main results of this study are the following.

1) A proper submodule $\boldsymbol{\Omega}$of an $\mathfrak{R}$_module $\mathcal{A}$ is an RNP submodule if and only if for every submodule y of $\boldsymbol{\Omega}$ and every ideal I of $\mathfrak{R}$ with $Iy \subseteq \mathcal{A}$, implies that either $y \subseteq \mathcal{A} + (soc(\boldsymbol{\Omega}) \cap J(\boldsymbol{\Omega}))$ or $I \subseteq [\mathcal{A} + (soc(\boldsymbol{\Omega}) \cap J(\boldsymbol{\Omega})){:}_{\mathfrak{R}}\,\boldsymbol{\Omega}]$.

2) Every prime submodule of an $\mathfrak{R}$_module $\boldsymbol{\Omega}$ is RNP submodule but not conversely see proposition (Example 8).

3) We introduced and studied and state several basic properties of this notion for example see(3.2).

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