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Doubt Fuzzy BZ-ideals of BZ-algebra

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Abstract— ***In this paper, we introduce the notion of doubt fuzzy BZ − idealsin BZ − algebra, several appropriate examples are provided and some properties are investigated. The image and the inverse image of doubt fuzzy BZ − ideals in BZ − algebra are defined and how the image and the inverse image of doubt fuzzy BZ − ideals in BZ − algebra become doubt fuzzy BZ − ideals are studied.***

Keywords—component; $BZ - ideals$, doubt fuzzy subalgebra, doubt fuzzy $BZ - ideal$, image and pre − image of doubt fuzzy $BZ - ideals$.

1. INTRODUCTION

$BCK - algebras$ form an important class of logical algebras introduced by $K. Iseki$ [13] and was extensively investigated by several researchers. The class of all $BCK - algebras$ is quasi variety. $J. Meng$ and $Y.$ B. Jun posed an interesting problem (solved in [16]) whether the class of all $BCK - algebras$ is a variety. In connection with this problem, Komori introduced in [15] a notion of BCC algebras. $W. A. Dudek$ (cf. [4]) redefined the notion of $BCC - algebras$ by using a dual form of the ordinary definition in the sense of $Y. Komori$ and studied ideals and congruences of $BCC - algebras$. In [20,21]), C. Prabpayak and U. Leerawat introduced a new algebraic structure, which is called $KU - algebra$. They gave the concept of homomorphisms of $KU - algebras$ and investigated some related properties. $L. A. Zadeh$ [23] introduced the notion of fuzzy subsets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, $O. G. Xi$ [22] applied this concept to $BCK - algebras$, and he introduced the notion of fuzzy sub − algebras (ideals) of the $BCK - algebras$ with respect to minimum, and since then Jun et al studied fuzzy ideals (cf. [14]), and moreover several fuzzy structures in $BCC - algebras$ are considered (cf. [17]). In [6], the anti − fuzzy $AB - ideals$ of $AB - algebras$ and in [11], the anti − fuzzy $AT - ideals$ of $AT - algebras$ were introduced. Several theorems are stated and proved. Omar in [18] and Patthanangkoor in [19] have introduced the notion of $BZ - algebras$, ideals, subalgebras and studied the relations among them and gave the concept of homomorphism of $BZ - algebras$ and investigated some related properties. Abed in [1 − 3], studied the structure of $BZ - algebras$ and its properties, studied the isomorphism of $BZ - algebras$ and introduced on the special ideals in $BZ - algebras$. Ghabue and Hameed in [5], studied the fuzzy algebraic structures was started with the introduction of the concept of fuzzy $BZ - ideals$ of $BZ - algebras$ and investigated several basic properties which are related to fuzzy $BZ - ideals$. They described how to deal with the homomorphism *of image and inverse image of fuzzy* $BZ -$ *ideals*. And *they have also proved* that *the Cartesian product of fuzzy* $BZ -$ *ideals is a fuzzy* $BZ -$ *ideal. In this paper, we introduce* the *notion of* doubt fuzzy $BZ -$ *ideals of* $BZ -$ *algebras and then we* study t*he homomorphism* image *and inverse* image *of doubt fuzzy* $BZ -$ideals.

2. Preliminaries

In this section we *introduced* an al*gebraic structure called a* $BZ -$ *algebra*

***Definition* 2.1** ([**18, 19**]). *Let* $(X;*,0)$ *be* an al*gebra wit*h *operation* $(+)$ and *constant* (0). X *is called* a $BZ -$ *algebra* i*f it satisfies* the following *identities*: *f*or any $x, y, z \in X$,

$(BZ-1)$ $((x * z) * (y * z)) * (x * y) = 0$;

$(BZ-2)$ $x * 0 = x$;

$(BZ-3)$ $x * y = 0$ *and* $y * x = 0$ *implies that* $x = y$.

Re.2.2. ([18,19]).

On $BZ -$ *algebra* $(X,*,0)$, we de*fined a binary* relation $\leq$ *on* X b*y putting* $x \leq y$ *if and* only *if* $x * y = 0$.

Prop. 2.3 ([1-3,18,19]). Let $(X;*,0)$ be a BZ-algebra, th*en* $(X, \leq)$ is a partially order*ed s*et. It is eas*y to sho*w that the *foll*owing pro*perties ar*e true for a BZ -al*ge*bra. *F*or any $x, y, z \in X$:

(P-1) $x * ((x * y) * y) = 0$;

(P-2) $x * x = 0$;

(P-3) $x * (y * z) = y * (x * z)$;

(P-4) $((x * y) * y) * y = x * y$;

(P-5) $(x * y) * 0 = (x * 0) * (y * 0)$;

(P-6) $(x * y) * ((z * x) * (z * y)) = 0$;

(P-7) $x \leq y$ *implies* $y * z \leq x * z$;

(P-8) $x \leq y$ *implies* $z * x \leq z * y$.

Def.2.4. ([19]). A s*ubse*t S of a BZ-alg*ebra* X is *ca*lled **subal*gebra* of *X* (SA.)** if $x * y \in S$ wh*enever* $x, y \in S$.

Def.2.5. ([1-3]). A no*n* $-$ *em*pty s*ubse*t I of a BZ-al*gebr*a $(X,*,0)$ is c*alled* ***BZ*-ideal of *X* (BZI)** *if it sa*ti*sfie*s *the following co*ndi*tion*s: for any $x, y, z \in X$

(I-1) $0 \in I$

(I-2) $(x * y)*z \in I$ and $y \in I$ imply $x*z \in I$.

Prop. 2.6 ([18,19]). E*very* BZI o*f* BZ-al*ge*bra $(X,*,0)$ is a SA. of X.

Prop. 2.7 ([18,19]). Let $\{I_i \mid i \in \Lambda\}$ be a fa*mily of id*eals of BZ-al*ge*bra $(X,*,0)$. The inte*rsection of a*ny set of BZIs of X *is* al*s*o an BZI of X.

Def*. 2. 8** ([**10**]). Let $(X,*,0)$and $(Y,*',0')$ *be nonempty sets*. The *mapping* $f : (X;*,0) \rightarrow (Y;*',0')$ is called ***a homomorphism *if it satisfies*:

$f(x*y) = f(x)*'f(y)$, *for* all x , y $\in X$. *The set* $\{x \in X / f(x) = 0'\}$ *is called* ***the kernel of f*** *denoted by* $\ker f$.

***Th*. 2. 9** ([**1** − **3**]). *Let* $f : (X;*,0) \rightarrow (Y;*',0')$ *be a* homomorphism *of a BZ − algebra X* into an *BZ − algebra* Y, then:

A.

B. jective *if* and *only* if $\ker f = \{0\}$.

C. mplies $f(x) \leq f(y)$.

***Th*. 2. 10**([**1** − **3**]). *Let* $f : (X;*,0) \rightarrow (Y;*',0')$be *a homomorphism of an BZ − algebra X into an BZ − algebra Y*, *then* :

(*F*1) *If S is a SA. of X*, then $f(S)$ *is a SA. of Y* .

(*F*2) *If I is BZI of X*, then $f(I)$ *is BZI in Y* .

(*F*3) *If D is a SA. of* Y, then $f^{-1}(D)$ *is a SA. of X* .

(*F*4) *If J is BZI in Y*, *then* $f^{-1}(J)$ *is BZIin X* .

(*F*5) *ker* f is *BZI of X*.

(*F*6) $Im(f)$ is a *SA. of Y*.

***Def*. 2. 11**([**23**]). *Let* $(X,*,0)$ *be a nonempty* set, *a fuzzy* subset μ *of* X *is* a *mapping* $\mu: X \rightarrow [0,1]$.

***Def*. 2. 12.** [**23**] *Let* μ *be a fuzzy subset of a set* X . For $t \in [0,1]$, *the set* $\mu_t = U(\mu,t) = \{x \in X \mid \mu(x) \geq t\}$, *is called* upper *level cut* (level subset) *of* μ and the *set* $L(\mu,t) = \{x \in X \mid \mu(x) \leq t\}$ *is called* lower level *cut of* μ.

Def.2.13 ([9]).

Let $f: (X;*,0) \rightarrow (Y;*',0')$ *be a mapping* nonempty *sets X and Y* respectively. *If* μ is a *fuzzy* subset of X, *then the fuzzy subset* β of Y *defined by*:

$$f(\mu)(y) = \begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases}$$

is said to be *the image of* μ under f.

Similarly if β *is a fuzzy subset of Y* , *then the fuzzy* subset $\mu = (\beta \circ f)$ of X (*i.e the fuzzy subset* defined by $\mu(x) = \beta(f(x))$ *for all* $x \in X$) *is called* the *pre − image of* β under f .

***Def*. 2. 14** ([**9**]).

A fuzzy subset μ *of a set* X ***has sup property*** *if for* any subset T *of* X , *there exist* $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t)/t \in T\}$.

Def*. 2. 15**([**1**, **18**]). Let $(X,*,0)$ *be an* $BZ-algebra$, a fuzzy subset μ *of* X *is called* a ***fuzzy SA. of X (FSA) *if for* all $x, y \in X$, $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$

***Prop*. 2. 16**([**1**, **19**]). *Let* μ *be* a fuzzy *subset* of $BZ-$ algebra $(X,*,0)$. If μ is a *fuzzy SA. of* X , then *for any* $t \in [0,1]$, μ_t is a *SA. of* X .

Def*. 2. 17.** [**5**]. Let $(X;*,0)$ *be an* $BZ-algebra$. *A fuzzy* subset μ *of* X *is* called ***a fuzzy BZ* − *ideal of X *if it satisfies the following conditions*: for all $x, y \in X$,

(1) $\mu(0) \geq \mu(x)$.

(2) $\mu(x*z) \geq \min\{\mu((x*y)*z), \mu(y)\}$.

***Prop*. 2. 18**[**5**]. *Every fuzzy* $BZ-$ *ideal of* $BZ-$ *algebra is fuzzy SA.*.

***Prop*. 2. 21** ([**5**]). *Let* $f:(X;*,0) \to (Y;*`,0`)$ *be a homomorphism between* $BZ-$ *algebras* X *and* Y *respectively*.

1 − *For every fuzzy SA.* β *of* Y, $f^{-1}(\beta)$ *is a fuzzy SA. of* X.

2 − *For every fuzzy SA.* μ *of* X , $f(\mu)$ *is a fuzzy SA. of* Y.

3 − *For every* fuzzy *BZI* β *of* Y, $f^{-1}(\beta)$ *is a fuzzy BZI of* X.

4 − *For every fuzzy BZI* μ *of* X *with* sup property, $f(\mu)$ *is a* fuzzy *BZI f* Y, *where* f *is onto*.

3. Doubt Fuzzy Subalgebras and Homomorphism of BZ-algebras

In this section, we *will introduce* a new *notion called a doubt fuzzy* sub*algebra*.

Def.3.1.

Let $(X;*,0)$ be *an* BZ-al*gebra* (BZA), a fuzzy *subset* (FS) μ of X is *called* **a doubt fuzzy subalgebra of X** (DFSA) if *f*or all x, y $\in X$, $\mu(x*y) \leq max\{\mu(x), \mu(y)\}$.

Ex. 3.2.

Let $X = \{0, 1, 2, 3\}$ *be a set with the following table*:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X;*,0)$ is an BZA. It is easy to show that I ={0, 1}is an SA. of X .

Define a FS$v: X \to [0,1]$ by, $v(x) = \begin{cases} 0.2 & x \in \{0,1\} \\ 0.4 & x \in \{2,3\} \end{cases}$

Then μ is a DASA. of X.

Routine *calculation* gives that μ is a DASA. *of* BZAs X.

Prop. 3.3.

Let μ be a DFS of an BZA $(X;*,0)$. If μ is a DASA. of X , *then it* satisf*ies*: *f*or any t∈[0, 1], $L(\mu,t) \neq \emptyset \Rightarrow L(\mu,t)$ is a SA. of X.

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Prf.:

Assume that μ is a DASA. of X, let t∈ [0,1] be ∋ $L(\mu,t) \neq \emptyset$, and let x, $y \in X$ be ∋ x, $y \in L(\mu,t)$, then $\mu(x) \leq t$ and $\mu(y) \leq t$, so $\mu(x*y) \leq max\{\mu(x),\mu(y)\} \leq t$, so that $(x*y) \in L(\mu,t)$. ⟹ $L(\mu,t)$ is a SA. of X. ■

Prop. 3.4.

Let μ be a DFS of an BZA $(X;*,0)$. If $L(\mu,t)$ is a SA. of X, ∀ t∈[0, 1], $L(\mu,t) \neq \emptyset$, ⟹ μ is a DASA. of X .

Prf.:

Suppose that μ is not DASA. of X, satisfies $L(\mu,t)$ is a SA. of X. Now, assume $\mu(x*y) > max\{\mu(x),\mu(y)\}$, taking $t_0 = \frac{1}{2}(\mu(x*y) + max\{\mu(x),\mu(y)\})$, we have $t_0 \in [0, 1]$ and $max\{\mu(x),\mu(y)\} < t_0 < \mu(x*y)$, it follows that $x,y \in L(\mu,t_0)$ and $x*y \notin L(\mu,t_0)$, this is a C! since $L(\mu,t_0)$ is a SA. of X , ⟹μ is a DASA. of X. ■

Coro. 3.5.

If a FS μ of BZA $(X;*,0)$ is a DASA., ⟹ for every $t \in Im(\mu)$, $L(\mu,t)$is a SA. of X.

Coro. 3.6.

Let I be a SA. of an BZA $(X;*,0)$, ⟹ ∀ fixed number t in an open interval (0,1), there exist a DASA. μ of X ∋ $L(\mu,t)$=I.

Prf.:

Define $\mu: X \to [0:1]$ by $\mu(x) = \begin{cases} 0 & if x \in I \\ t & if x \notin I \end{cases}$. Where t is a fixed number in (0,1).

Clearly, $\mu(0) \leq \mu(x)$ and we have level sets $L(\mu,0) = I$ or $L(\mu,t) = X$, which are SA.s of X, ⟹ from Prop. (3.4), μ is a DASA. of X.■

Prop. 3.7.

The union of any set of DASA.s of BZA $(X;*,0)$ is also DASA..

Prf.:

Let $\{\mu_i \mid i \in \Lambda\}$ be a family of DASA.s of BZA X, ⟹ ∀ $x,y \in X$, i ∈Λ,

$$(\cup_{i\in\Lambda}\mu_i)(x*y) = sup(\mu_i(x*y)) \leq sup\{max\{\mu\_i(x), \mu\_i(y)\}\}$$
$$\leq max\{sup(\mu_i(x)), sup(\mu_i(y))\}$$
$$= max\{(\cup_{i\in\Lambda}\mu_i)(x)), (\cup_{i\in\Lambda}\mu_i)(y)\}.$$ ■

Re.3.8.

The intersection of any set of DASA.s of BZA $(X;*,0)$ is not necessarily DASA. as the example.

Ex. 3.9.

Let X = {0, 1, 2, 3} in which $(*)$ is defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\Longrightarrow$ $(X;*,0)$ is an BZA. It is $easy\ to$ show that I ={0, 2} and J ={0, 3} are SA.s of X . $Define$ a FS$\mu, v: X \rightarrow [0,1]$ by

$\mu(x)=\begin{cases}0.2 & x \in \{0,2\} \\ 0.7 & x \in \{3,1\}\end{cases}$ and $v(x)=\begin{cases}0.3 & x \in \{0,3\} \\ 0.8 & x = 1 \\ 0.9 & x = 2\end{cases}$. $\Longrightarrow \mu$ and v are DASA.s of X.

But $\mu \cap v(x) = min\{\mu(x), v(x)\} = \begin{cases}0.2 & x \in \{0,2\} \\ 0.7 & x = 1 \\ 0.3 & x = 3\end{cases}$

is not DASA. of $\boldsymbol{X}$, since $\mathbf{x = 2, y = 3}$, $\boldsymbol{\mu \cap v\,(2 * 3) = 0.7 \nleq max\,\{\mu \cap v\,(2), \mu \cap v\,(3)\} = 0.3}$

Prop. 3.10.

The inters$ection\ of\ a$ny set of DASA.s of BZA $(X;*,0)$ is also DASA. of X where is chain (Artinian).

Prf.:

Let$\{\mu_i \mid i \in \Lambda\}$ be a family of DASA.s of BZA X, $\Longrightarrow \forall\ x, y \in X$, i $\in \Lambda$,

$(\cap_{i\in\Lambda}\ \mu_i)\,(x*y) = \inf_{i\in\Lambda}(\mu_i(x*y)) \leq \inf_{i\in\Lambda}\{max\,\{\mu_i\,(x), \mu_i(y)\}\}$

$\leq max\,\{\inf_{i\in\Lambda}(\mu_i\,(x))\ ,\inf_{i\in\Lambda}(\mu_i\,(y)\,\}$

$= \max\,\{(\cap_{i\in\Lambda}\ \mu_i)(x))\ , (\cap_{i\in\Lambda}\ \mu_i)\,(y)\}$. ■

Th.3.11.

A hom$o.pre - im$age of DASA. is also a DASA..

Prf.:

Let $f:(X;\ *,0) \rightarrow (Y;\ *`,0`)$ be a homo. of BZAs, β is a DASA. of Y and μ th$e$ pre-imag$e$ of β under f , let $x, y \in X$, $\Longrightarrow$

$\mu\,(x*y) = \beta(\,f\,(x*y)) = \beta(\,f\,(x)*'\,f\,(y))$

$\leq max\{\beta(\,f\,(x)),\beta(\,f\,(y))\}$

$= max\{\mu\,(x),\mu\,(y)\}$. ■

Def.3.12.

A D FSμ of BZA $(X;*,0)$ has in$f\ pro$perty if $\forall$ subset T of X, $\exists\ t_0 \in$ T $\ni \mu\,(t_0) = inf_{t\in T}\,\mu\,(t)$.

Th.3.13.

Let $f:(X;\ *,0) \rightarrow (Y;\ *`,0`)$ be an epimo. between BZAs X and Y resp. and f has $inf\ pro$perty. For every DASA. of X, f (μ) is a DASA. of Y.

Prf.:

By Def.(2.15), $\beta(y') = f\,(\mu)(y') = inf_{x\in f^{-1}(y')}\,\mu\,(x)$, $\forall\ y' \in Y$.

$We\ have\ to\ prove$ that $\beta\,(x'*y') \leq max\{\beta(x'),\beta(y')\}$, $\forall\ x', y' \in Y$.

Let $f:(X;\ *,0) \rightarrow (Y;\ *`,0`)$ be an epimo. of BZAs , μ is a DASA. of X with i$nf\ pro$perty and β the image of μ under f , s$i$nce μ is DASA. of X, $\forall$ x', y'$\in Y$, let $x_0 \in f^{-1}$ (x'),$y_0 \in f^{-1}$ (y') be $\ni$ $\mu\,(x_0) = inf_{t\in f^{-1}(x')}\,\mu\,(t)$, $\mu\,(y_0) = inf_{t\in f^{-1}(y')}\,\mu\,(t)$ and $\mu\,(x_0*y_0) = inf_{t\in f^{-1}(x'*y')}\,\mu\,(t) \Longrightarrow$

$\beta\,(x'*y') = inf_{t\in f^{-1}(x'*y')}\,\mu\,(t) = \mu\,(x_0*y_0)$

$\leq max\{\mu\,(x_0)\,,\mu\,(y_0)\}$

$= max\{inf_{t\in f^{-1}(x')}\mu(t), inf_{t\in f^{-1}(y')}\mu(t)\}$

$= max\{\beta(x'),\beta(y')\}$.

$\Rightarrow \beta$ is a DASA. of Y. ■

4. DABZ-ideals and Homomorphism of BZAs

In this section, we *will introduce a* new notion *called* a *DABZI* of *BZA* and *study several basic properties of it.*

Def.4.1.

Let $(X; *, 0)$ be an BZA, a FSμ of X is called **a DABZI of X** if it satisfies the following conditions, $\forall\, x, y \in X$,

(DFBZI$_1$) $\mu(0) \le \mu(x)$,

(DFBZI$_2$) $\mu(x*z) \le max\{\mu((x*y)*z), \mu(y)\}$.

Ex. 4.2.

Let $X = \{0, 1, 2, 3\}$ be a set *with the following* table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\Rightarrow (X;*,0)$ is an BZA. It is *easy to show* that $I_1 = \{0, 1\}$ is BZ-ideals of X.

Define a FS$v: X \to [0,1]$, $v(x) = \begin{cases} 0.2 & x \in \{0,1\} \\ 0.4 & x \in \{2,3\} \end{cases}$.

Routine *calculation* gives *that* v is a DABZI of BZAs X.

Lem. 4.3.

Let μ be a DABZI of BZA$(X;*,0)$ *and if* $x \le y$, $\Rightarrow \mu(0*y) \le \mu(0*x)$, $\forall\, x, y \in X$.

Prf.:

Assume that $x \le y$, $\Rightarrow x*y = 0$, and

$$\mu(0*y) \le max\{\mu(x*y), \mu(0*x)\} = max\{\mu(0), \mu(0*x)\} = \mu(0*x)$$

$\Rightarrow \mu(0*y) \le \mu(0*x)$. ■

Prop. 4.4.

A FSμ of an *BZA* $(X;*,0)$ is a DABZI of X, $\Rightarrow \forall\, t \in [0,1]$, $L(\mu,t)$ is an BZI of X, where $L(\mu,t) \ne \emptyset$.

Prf.:

Assume that μ is a DABZI of X, by (DFBZI$_1$), we have $\mu(0) \le \mu(x)$ $\forall\, x \in X \Rightarrow \mu(0) \le \mu(x) \le t$, for $x \in L(\mu,t)$ and so $0 \in L(\mu,t)$.

Let $x, y, z \in X \ni x*y \in L(\mu,t)$ and $y \in L(\mu,t)$, $\Rightarrow \mu((x*y)*z) \le t$ and $\mu(y) \le t$, since μ is a DABZI it follows that $\mu(x*z) \le max\{\mu((x*y)*z), \mu(y)\} \le t$ and that $x*z \in L(\mu,t)$.

$\Rightarrow L(\mu,t)$ is an BZI of X. ■

Prop. 4.5.

Let μ be a FS of an BZA $(X;*,0)$, if $L(\mu,t)$ is an BZI of X, where $L(\mu,t) \neq \emptyset$, for every t∈[0,1], ⟹ μ is a DABZI of X .

Prf.:

Now, we only need to show that (DFBZI $_1$) and (DFBZI $_2$) are true. If (DFBZI $_1$) is false, ⟹ $\exists\ x \in X \ni \mu\ (0)\ > \mu\ (x)$.

If we take $t\ = \frac{1}{2}\ (\mu\ (x)\ +\ \mu\ (0))$, ⟹ $\mu\ (0)\ >\ t$ and $0\ \leq \mu\ (x)\ <\ t \leq 1$ ⟹ $x \in L(\mu,t)$ and $L(\mu,t) \neq \emptyset$. As $L(\mu,t)$is an BZI of X, we have $0 \in L(\mu,t)$ and so $\mu\ (0)\ \leq\ t$. This is a C! .

Now, assume (DFBZI $_2$) is not true, ⟹ $\exists x,y,z \in X \ni$, μ(x*z) > $max\{\mu$((x*y)∗ z),μ(y)}.

Putting $t = \frac{1}{2}$(μ(x*z)+ max{μ ((x*y)*z), μ (y)}),⟹

$\mu(x{*}z) > t$ and $0 \leq max\{\mu$ ((x*y)∗ z)$, \mu\ (y)\}, < t \leq 1$, ⟹

μ((x*y)∗ z) $<\ t$ and μ(y) < t, ⟹(x ∗z)∈$L(\mu,t)$, since $L(\mu$,t)is an DABZI, it follows that (x*z)∈$L(\mu,t)$ and that μ (x*z) $\leq t$, this is also a C! . ⟹ μ is a DABZI of X. ■

Prop. 4.6.

The union of any set of DABZI of BZA is also DABZI.

Prf.:

Let { μ_i | i $\in \Lambda$} be a family of DABZIs of BZA $(X;*,0)$, by (DFBZI $_1$), we have

$\mu(0) \leq \mu(x)\ \forall\ x\ \in\ X$, ⟹$\cup_{i\in\Lambda}\ \mu_i\ (0) \leq \cup_{i\in\Lambda}\ \mu_i\ (x)$.

By (DFBZI $_2$), $\forall\ x,y,z \in X$, i ∈Λ,

($\cup_{i\in\Lambda}\ \mu_i$)(x*z)=sup[f0](μ_i (x*z))

$\leq$ sup{max {μ_i ((x*y)*z), μ_i (y)}}

$\leq$ max {$sup\ (\mu_i\ ((x*y)*z)\ , sup\ (\mu_i\ (y)\}$

$=$ max {($\cup_{i\in\Lambda}\ \mu_i$)((x*y)∗ z) , ($\cup_{i\in\Lambda}\ \mu_i$) ($y$)} .

⟹ $\cup_{i\in\Lambda}\ \mu_i$ is a DABZI of X. ■

Re.4.7.

The intersection of any set of DABZIs of BZA $(X;*,0)$ is not necessary DABZ-ideal as the following Ex..

Ex. 4.8.

Let X = {0, 1, 2, 3} in which (∗) is defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

⟹ $(X;*,0)$ is an BZA. It is easy to show that I ={0, 1} and J ={0, 2} are BZIs of X .

Define a FS$\mu, \nu: X \to [0,1]$ by $\mu(x) = \begin{cases} 0.3 & x \in \{0,2\} \\ 0.5 & x \in \{1,3\} \end{cases}$ and $\nu(x) = \begin{cases} 0.2 & x \in \{0,1\} \\ 0.4 & x \in \{2,3\} \end{cases}$.$\Rightarrow \mu$ and ν are a DABZIs of X. But $\mu \cap \nu\,(x) = \begin{cases} 0.2 & x \in \{0,1\} \\ 0.3 & x = 2 \\ 0.4 & x = 3 \end{cases}$ is not DABZI of X, since $x = 2, y = 2$ and $z = 3$,$\Rightarrow$

$$\mu \bigcap \nu\,(2 * 3) = 0.4 \not\leq max\{\mu \bigcap \nu\,((2 * 2) * 3), \mu \bigcap \nu\,(2)\} = 0.3$$

Prop. 4.9.

The intersection of any set of DABZIs of BZA $(X; *, 0)$ is also DABZI of X where is chain (Artinian).

Prf.:

Let $\{\mu_i \mid i \in \Lambda\}$ be a family of DABZIs of BZA $(X; *, 0)$, by (DFBZI $_1$), we have $\mu(0) \leq \mu(x), \forall x \in X$, therefore

$\cap_{i\in\Lambda} \mu_i\,(0) \leq \cap_{i\in\Lambda} \mu_i\,(x)$.

By (DFBZI $_2$), $\forall x, y, z \in X$, $i \in \Lambda$,

$$\begin{aligned}(\cap_{i\in\Lambda} \mu_i)(x * z) &= \inf_{i\in\Lambda}(\mu_i(x * z)) \\ &\leq \inf_{i\in\Lambda}\{max\{\mu_i((x * y) * z), \mu_i(y)\}\} \\ &\leq max\{\inf_{i\in\Lambda}(\mu_i((x * y) * z)), \inf_{i\in\Lambda}(\mu_i(y))\} \\ &= max\{(\cap_{i\in\Lambda} \mu_i)((x * y) * z), (\cap_{i\in\Lambda} \mu_i)(y)\}.\end{aligned}$$

$\Rightarrow \cap_{i\in\Lambda} \mu_i$ is a DABZI of X. ■

Prop. 4.10.

Every DABZI of BZA $(X; *, 0)$ is a DASA. of X.

Prf.:

Since μ is a DABZI of BZA X, $\Rightarrow$ by Prop. (4.4) , for every $t \in [0,1]$,$L(\mu,t)$ is BZI of X . By Def.(2.10), for every $t \in [0,1]$, $L(\mu,t)$ is SA. of X . $\Rightarrow \mu$ is a DASA. of X by Prop. (3.4). ■

Re.4.11.

The converse of Prop. (4.10) is not true as the following Ex.:

Ex. 4.12.

Let $X = \{0, 1, 2, 3\}$ in which $(*)$ is defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\Rightarrow$ $(X; *, 0)$ is an BZA. It is easy to show that I $=\{0, 3\}$ is SA. of X . Define a FS$\nu: X \to [0,1]$ by

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$$\nu(x) = \begin{cases} 0.3 & x \in \{0,3\} \\ 0.8 & x = 1 \\ 0.9 & x = 2 \end{cases}.$$

$\Longrightarrow \nu$ is a DASA. of X, but ν is a DABZI of X . But it is not a DABZI since $x = 1, y = 1 and\ z = 2$ $, \Longrightarrow$ $\nu(1 * 2) = 0.9 \nleq max\{\nu((1 * 1) * 2), \nu(1)\} = 0.8$.

Coro. 4.13.

If a FSμ of BZA $(X; *, 0)$ is a DABZI $\Longrightarrow$ for every $t \in Im(\mu)$, $L(\mu,t)$is an BZI of X.

Prop. 4.14.

Let I be an BZI of an BZA $(X,*,0)$, $\Longrightarrow \forall$ fixed number t in an open interval $(0,1)$, $\exists$ a DABZI μ of $X \ni L(\mu,t)$=I.

Prf.:

Define $\mu: X \rightarrow [0,1]$ by μ (x) =$\begin{cases} 0 & if x \in I \\ t & if x \notin I \end{cases}$. Where t is a fixed number in (0,1).

Clearly, μ (0) $\leq \mu$ (x) and we have level sets $L(\mu, 0) = I$ or $L(\mu, t) = X$, which are BZIs of X, $\Longrightarrow$ from Coro. (4.13), μ is a DABZI of X. ■

Th.4.15.

A homo. pre-image of DABZ-ideal is also a DABZI.

Prf.:

Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a homo.of BZAs, β is a DABZI of Y and μ the $pre-image$ of β under f , $\Longrightarrow$ $\beta(f(x)) = \mu(x), \forall x \in X$.

Let $x \in X, \Longrightarrow \mu(0) = \beta(f(0)) \leq \beta(f(x)) = \mu(x)$.

Now, let $x, y, z \in X \Longrightarrow$

$$\mu(x * z) = \beta(f(x*z)) = \beta(f(x) *' f(z))$$
$$\leq max\{\beta((f(x) *' f(y))*'f(z)), \beta(f(y))\}$$
$$= \max\{\beta(f((x * y)*z), \beta(f(y))\}$$
$$= \max\{\mu((x*y)* z), \mu(y)\}. ■$$

Th.4.16.

Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be an epimo. between BZAs X and Y resp. and f has $inf\ property$. For every DABZI of X, $f(\mu)$ is a DABZ-ideal of Y.

Prf.:

By Def.(2.15), $\beta(y') = f(\mu)(y') = \inf_{x \in f^{-1}(y')} \mu(x), \forall y' \in Y$.

We have to prove that $\beta(x' *' z') \leq \max\{\beta((x' *' y') *' z'), \beta(y')\}, \forall x', y', z' \in Y$.

Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be an epimo. of BZAs, μ is a DABZI of X with inf property and β the image of μ under f , since μ is DABZI of X, we have $\mu(0) \leq \mu(x)\ \forall x \in X$.

Note that $0 \in f^{-1}(0')$, where 0, 0'are the zero of X and Y, resp..

$\Longrightarrow \beta(0') = \inf_{t \in f^{-1}(x')} \mu(t) = \beta(x'), \forall x \in X$, which $\Longrightarrow$ that $\beta(0') \leq \inf_{t \in f^{-1}(x')} \mu(t) = \beta(x'), \forall x' \in Y$.

$\forall x', y', z' \in Y$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ and $z_0 \in f^{-1}(z')$ be $\ni$

$\mu((x_0 * y_0) * z_0) = \inf_{t \in f^{-1}((x' *' y') *' z')} \mu(t)$, $\mu(y_0) = \inf_{t \in f^{-1}(y')} \mu(t)$ and $\mu(x * z) = \inf_{t \in f^{-1}(x' *' z')} \mu(t) \Longrightarrow$

$$\beta(x' *' z') = \inf_{t \in f^{-1}(x' *' z')} \mu(t) = \mu(x_0 * z_0)$$

$$\leq \max\{\mu((x_0 * y_0) * z_0), \mu(y_0)\}$$

$$= \max\{\inf_{t \in f^{-1}((x' *' y') *' z')} \mu(t), \inf_{t \in f^{-1}(y')} \mu(t)\}$$

$$= \max\{\beta((x' *' y') *' z'), \beta(y')\}.$$

$\Longrightarrow \beta$ is a DABZI of Y. ■

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