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# Interval-Valued Fuzzy Ideal of BZ-algebra

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***Abstract***—*In this paper, the notion of interval − valued fuzzy ideals in BZ − algebras is introduced. Several theorems are stated and proved. The image and inverse image of interval-valued fuzzy ideals are defined and how the homomorphic images and inverse images of interval − valued fuzzy ideals become interval − valued fuzzy ideals in BZ − algebras is studied as well.*

**Keywords**—component; BZ-algebras, fuzzy ideals, interval − valued fuzzy subalgebras , interval-valued fuzzy ideals .

## 1. Introduction

In this paper, using thenotion of interval-valued fuzzyset,we introduce the concept of an interval-valued fuzzy idealsof aBZ-algebra, and studysome of their properties. Usingan interval-valuedlevel set of aninterval-valued fuzzyset,we statea characterization of n interval-valuedfuzzy ideals. We prove that every ideals of a BZ-algebra X can be realizedas an interval-valued level idealsof an interval-valued fuzzy ideals of X.In connection with thenotion of homomorphism,we study howtheimages andinverse imagesof interval-valuedfuzzy idealsbecome interval-valuedfuzzyideals.

## 2. Preliminaries

*In this section* we *introduced* an *algebraic structure called a BZ − algebra*

***Definition* 2.1**([**18, 19**]). *Let* $(X;*,0)$ *be an algebra with operation* $(+)$ and *constant* $(0)$. $X$ *is called* a $BZ-algebra$ if it satisfies the following identities: for any $x, y, z \in X$ ,

(BZ-1) $((x * z) * (y * z)) * (x * y) = 0$;

(BZ-2) $x * 0 = x$;

(BZ-3) $x * y = 0$ *and* $y * x = 0$ implies that $x = y$.

**Re.2.2. ([18,19])**.

*On BZ − algebra* $(X,*,0)$, we *defined a binary relation* $\leq$ *on* X by *putting* $x \leq y$ *if and only if* $x * y = 0$.

**Def.2.3. ([19])**. A *subset* $S$ of a $BZ$-algebra X is *called* **subal*gebra* of *X* (SA.)** if $x * y \in S$ whenever $x, y \in S$.

**Def.2.4. ([1-3])**. A non − *empty subset* $I$ of a $BZ$-al*gebr*a $(X,*,0)$ is *called* ***BZ*-ideal of *X* (BZI)** *if it satisfies the following conditions*: for any $x, y, z \in X$

$$(I-1)\ \ 0 \in I$$

$$(I-2)\ \ (x * y) * z \in I \text{ and } y \in I \implies x * z \in I.$$

**Prop. 2.5 ([18,19]).** Every $BZI$ of $BZ$-algebra $(X,*,0)$ is a SA. of $X$.

**Prop. 2.6 ([18,19]).** Let $\{I_i \mid i \in \Lambda\}$ be a fa*mily of id*eals of $BZ$-al*g*ebra $(X,*,0)$. The inter*section of a*ny set of $BZ$Is of $X$ *is also* an $BZI$ of $X$.

***Th.* 2.7** ([**1 − 3**]). *Let* $f : (X;*,0) \to (Y;*',0')$ *be a* homo*morphism* of a $BZ-alg$ebra $X$ into an $BZ-algebra$ Y, th*en*:

A.

B. jective *if* and *only* if $ker\ f = \{0\}$.

C. mp*lies* $f(x) \leq f(y)$.

***Th.* 2.8**([**1 − 3**]). Let $f : (X;*,0) \to (Y;*',0')$be *a homomorphism of an BZ − algebra X into an BZ − algebra Y, then* :

$(F1)$ *If S is a SA. of X*, $\Rightarrow f(S)$ *is a SA. of Y* .

$(F2)$ *If I is BZI of X*, then $f(I)$ *is BZI in Y* .

$(F3)$ *If D is a SA.* of Y $\Rightarrow f^{-1}(D)$ *is a SA. of X* .

$(F4)$ *If J is BZI in Y*, *then* $f^{-1}(J)$ *is BZIin X* .

$(F5)$ *ker* f is *BZI of X*.

$(F6)$ $Im(f)$ is a *SA. of Y*.

***Def.* 2.9**([**23**]). *Let* $(X,*,0)$ *be a nonempty* set, *a fuzzy* sub*set* $\mu$ *of* $X$ *is* a *mapping* $\mu: X \to [0,1]$.

***Def.* 2.12.** [**23**] *Let* μ *be a fuzzy subset of a set* X . $\forall t \in [0,1]$, *the set* $\mu_t = U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$, *is called* up*per level cut* (level su*bset*) *of* $\mu$ and the *set* $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ *is ca*lled low*er* level *cut of* μ.

***Def*.2.10** ([**9**]).

*A fuzzy subset* $\mu$ *of a set* X ***has sup property*** *if for any subset* $T$ *of* $X$, *there exist* $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t)/t \in T\}$.

***Def*.2.11**([**1,18**]). Let $(X,*,0)$ *be an* $BZ-algebra$, a fuzzy subset $\mu$ *of* $X$ *is called a* ***fuzzy SA. of X*** (FSA)*if* $\forall\, x, y \in X$, $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$

***Prop*.2.12**([**1,19**]). *Let* $\mu$ *be* a fuzzy subset of $BZ-$algebra $(X,*,0)$. If $\mu$ is a *fuzzy SA. of* $X$, then *for any* $t \in [0,1]$, $\mu_t$ is a $SA.$ *of* $X$.

***Def*.2.13.**[**5**]. Let $(X;*,0)$ *be an* BZ $-$ *algebra*. A fuzzy subset $\mu$ *of* $X$ *is called* ***a fuzzy BZ*** $-$ ***ideal of X*** *if it satisfies the following conditions*: for all $x, y \in X$,

(1) $\mu(0) \geq \mu(x)$.

(2) $\mu(x*z) \geq \min\{\mu((x*y)*z), \mu(y)\}$.

***Prop*.2.14**[**5**]. *Every fuzzy BZ* $-$ *ideal of BZ* $-$ *algebra is fuzzy SA.*.

***Prop*.2.21** ([**5**]). *Let* $f:(X;\ *,0) \rightarrow (Y;\ *`,0`)$ *be a homomorphism between BZ* $-$ *algebras X and Y respectively.*

1 $-$ *For every fuzzy SA.* $\beta$ *of* $Y$, $f^{-1}(\beta)$ *is a fuzzy SA. of* $X$.

2 $-$ *For every fuzzy SA.* $\mu$ *of* $X$, $f(\mu)$ *is a fuzzy SA. of* $Y$.

3 $-$ *For every* fuzzy *BZI* $\beta$ *of* $Y$, $f^{-1}(\beta)$*is a fuzzy BZI of* $X$.

4 $-$ *For every fuzzy BZI* $\mu$ *of* $X$ *with sup* property, $f(\mu)$ *is a fuzzy BZI f* $Y$, *where f is onto.*

## 3. Interval-valued fuzzy subalgebras of BZ-algebra

**Re. 3.1([8,11,23,26]).**

An interval number is $\tilde{a}=[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $I$ be a closed unit interval, (i.e., I = [0, 1]).

Let $D[0,1]$ denote the family of all closed subintervals of I = [0, 1], that is, D[0, 1] = { $\tilde{a}=[a^-, a^+] \mid a^- \leq a^+$, for $a^-, a^+ \in$ I}.

*Now*, we *define* what is *known* as refined minimum (*briefly, rmin*) *of two* element in D[0,1].

**Def. 3.2([8,11,23,26]).**

We also define the symbols (≽), (≼), (=), "rmin" and "rmax" in case of two elements in D[0, 1]. Consider two interval numbers (elements numbers)

ã=[ a^-, a^+], b̃=[ b^-, b^+]in D[0, 1] : ⟹

(1) ã ≽ b̃ ⟺, a^-≥b^- and a^+≥b^+,

(2) ã ≼ b̃ ⟺ a^-≤b^- and a^+≤b^+,

(3) ã = b̃ ⟺ a^-=b^- and a^+=b^+,

(4) rmin {ã, b̃}= [min {a^-, b^-}, min {a^+, b^+}],

(5) rmax {ã, b̃}= [max {a^-, b^-}, max {a^+, b^+}],

**Re. 3.3([8,11,23,26]).**

It is obvious that (D[0, 1], ≼, ∨, ∧) is a complete lattice with $\tilde{0}$ = [0, 0] as its least element and $\tilde{1}$ = [1, 1] as its greatest element. Let $\tilde{a}_i \in$D[0, 1] where $i \in \Lambda$.

We define $r\,inf_{i\in\Lambda}\tilde{a} = [r\,inf_{i\in\Lambda}a^-, r\,inf_{i\in\Lambda}a^+]$, $r\,sup_{i\in\Lambda}\tilde{a} = [r\,sup_{i\in\Lambda}a^-, r\,sup_{i\in\Lambda}a^+]$.

**Def. 3.4([8,11,23,26]).**

*An* ***interval*** $-$ ***valued fuzzy subset*** $\tilde{\mu}_A$ ***on X*** (***i*** $-$ ***vFS***) *is defined as*

$\tilde{\mu}_A = \{< x, [\mu_A^-(x), \mu_A^+(x)] > / x \in X\}$. *Where* $\mu_A^-(x) \leq \mu_A^+(x), \forall x \in X$.

$\Rightarrow$ *the ordinary fuzzy subsets* $\mu_A^-: X \rightarrow [0,1]$ *and*

$\mu_A^+: X \rightarrow [0,1]$ *are called a* ***lower fuzzy subset and an upper fuzzy subset*** *of* $\tilde{\mu}_A$ *resp.*. *Let* $\tilde{\mu}_A(x)$

$$= [\mu_A^-(x), \mu_A^+(x)], \tilde{\mu}_A: X \rightarrow D[0,1], \Rightarrow A = \{< x, \tilde{\mu}_A(x) > / x \in X\}.$$

**Def. 3.5.**

An i-vFS A in BZ-algebra $(X;*,0)$ is called an **interval** $-$ **valued fuzzy subalgebra of** $X$ (**i-vFSA**) if

$\tilde{\mu}_A(x*y) \succcurlyeq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}, \forall\ x, y \in X$.

**Ex.3.6.**

Let $X = \{0, 1, 2, 3\}$ in which the operation be define by the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |

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| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 1 | 0 |

$\Rightarrow (X;*,0)$ is an BZ-algebra. Define a FS. $\mu: X\to [0,1]$ by: $\mu(x) = \begin{cases} 0.7 & if\ x \in \{0,1\} \\ 0.3 & otherwise \end{cases}$.

$I_1=\{0,1\}$ is an ideal of $X$. Routine calculation given that $\mu$ is a FSA. of $X$. Define $\tilde{\mu}_A$ (x) as follows:

$\tilde{\mu}_A(x) = \begin{cases} [0.3,0.9] & if\ x \in \{0,1\} \\ [0.1,0.6] & otherwise \end{cases}$.

It is easy to check that $A$ is i-vFSA. .

**Prop.3.7.**

If A is an i-vFSA of X, $\Rightarrow \tilde{\mu}_A(0) \succcurlyeq \tilde{\mu}_A(x), \forall\ x\in X$.

**Proof.**

$\forall\ x\in X$, we have

$\tilde{\mu}_A(0) = \tilde{\mu}_A(0*x) \succcurlyeq \mathrm{rmin}\{\tilde{\mu}_A(0), \tilde{\mu}_A(x)\}$

$= \mathrm{rmin}\{[\mu_A^-(0), \mu_A^+(0)], [\mu_A^-(x), \mu_A^+(x)]\}$

$= \mathrm{rmin}\{[\mu_A^-(x), \mu_A^+(x)]\} = \tilde{\mu}_A(x)$. ⌂

**Prop.3.8.**

Let A be i-vFSA of $X$, if there exist a sequence $\{X_n\}$ in $X$ such that $\lim_{n\to\infty} \tilde{\mu}_A(x_n) = [1,1]$, $\Rightarrow \tilde{\mu}_A(0) = [1,1]$.

**Pf.**

By Prop.(3.7), we have $\tilde{\mu}_A(0) \succcurlyeq \tilde{\mu}_A(x), \forall\ x\in X. \Rightarrow$

$\tilde{\mu}_A(0) \succcurlyeq \tilde{\mu}_A(x_n)$ for every positive integer n,

Consider the inequality $[1,1] \geq \tilde{\mu}_A(0) \geq \lim_{n\to\infty} \tilde{\mu}_A(x_n) = [1,1]$. Hence $\tilde{\mu}_A(0) = [1,1]$ .⌂

**Th. 3.9.** *An i-v FS.* $A = [\mu_A^-, \mu_A^+]$ of BZ-algebra $(X;*,0)$ is i-vFSA of $X \Leftrightarrow \mu_A^-$ and $\mu_A^+$ are FSA. of $X$.

**Pf.**

If $\mu_A^-$ and $\mu_A^+$ are fuzzy $BZ$-subalgebras of $X$, $\forall\ x,y\in X$. Observe

$\mu_A^-(x*y) \geq \min\{\mu_A^-(x), \mu_A^-(y)\}$ and

$\mu_A^+(x*y) \geq \min\{\mu_A^+(x), \mu_A^+(y)\}$, $\Rightarrow$

$\tilde{\mu}_A(x*y) = [\mu_A^-(x*y), \mu_A^+(x*y)]$

$\geq [\min\{\mu_A^-(x), \mu_A^-(y)\}, \min\{\mu_A^+(x), \mu_A^+(y)\}]$

$= [\min\{\mu_A^-(x), \mu_A^+(x)\}, \min\{\mu_A^-(y), \mu_A^+(y)\}]$

$\succcurlyeq \mathrm{rmin}\{[\mu_A^-(x), \mu_A^+(x)], [\mu_A^-(y), \mu_A^+(y)]\}$

$= \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$.

From what was mentioned above we can conclude that $A$ is an i-vFSA of $X$.

Conversely, suppose that $A$ is an $i-vFSA$ of $X. \forall\ x,y \in X$, we have

$[\mu_A^-(x*y), \mu_A^+(x*y)] = \tilde{\mu}_A(x*y) \succcurlyeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

$= [\min\{\mu_A^-(x), \mu_A^+(x)\}, \min\{\mu_A^-(y), \mu_A^+(y)\}]$

$= [\min\{\mu_A^-(x), \mu_A^-(y)\}, \min\{\mu_A^+(x), \mu_A^+(y)\}]$

Therefore, $\mu_A^-(x*y) \geq \min\{\mu_A^-(x), \mu_A^-(y)\}$ and

$\mu_A^+(x*y) \geq \min\{\mu_A^+(x, \mu_A^+(y)\}$.

Hence, we get that $\mu_A^-$ and $\mu_A^+$ are FSA. s of $X$. ⌂

**Prop.3.10.**

Let $A_1$ and $A_2$ be i-v FSA. s of BZ-algebra, $\Rightarrow A_1\cap A_2$ is an i-v FSA. .

**Pf.**

Since $A_1$ and $A_2$ be i-v FSA. s of BZ-algebra$(X;*,0)$,

Suppose $x,y \in X$ such that $x \in A_1\cap A_2$ and $y \in A_1\cap A_2$ .

Since $A_1$ and $A_2$ are i-v FSA. s of $X$, $\Rightarrow$ by the Th. (3.9), we get

$\tilde{\mu}_{A_1\cap A_2}(x*y) = [\mu_{(A1\cap A2)}^-(x*y), \mu_{(A1\cap A2)}^+(x*y)]$

$\succcurlyeq \mathrm{rmin}\{\min\{\mu_{(A1\cap A2)}^-(x), \mu_{(A1\cap A2)}^-(y)\}, \min\{\mu_{(A1\cap A2)}^+(x), \mu_{(A1\cap A2)}^+(y)\}\}$

$= \mathrm{rmin}\{\min\{\mu_{(A1\cap A2)}^-(x), \mu_{(A1\cap A2)}^+(x)\}, \min\{\mu_{(A1\cap A2)}^-(y), \mu_{(A1\cap A2)}^+(y)\}\}$

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= rmin{μ˜_(A_1∩A_2 ) (x), μ˜_(A_1∩A_2 ) (y)}} . ⌂

**Coro. 3.11.**

*Let* $\{A_i \mid i\in\Lambda\}$ be *a family of* i-$v$ $FSA$ on BZ-al$gebra(X;*,0)$, $\Longrightarrow$ $\bigcap_{i\in\wedge} A_i$ is also an i-vFSA *of* $X$ .

**Th. 3.12.**

Let $(X;*,0)$ be an BZ-alg*e*bra and $A$ be an $i$-v FS. of $X$. A is $i$-vFSA of $X \Leftrightarrow$, the nonempty set $\widetilde{U}$ (A;[$\delta_1$,$\delta_2$]) = { $x\in X$ $|\tilde{\mu}_A$ (x) $\succcurlyeq$ [$\delta_1$,$\delta_2$]} is an sub*al*gebra of $X$, for every [$\delta_1$, $\delta_2$]$\in D[0,1]$ . $W$e call $\widetilde{U}$ (A ; [$\delta_1$, $\delta_2$] ) the $i$-v level sub*alg*ebra of A.

**Pf.**

Assume that A is an i-vFSA o$f$ $X$ and let [$\delta_1$, $\delta_2$] $\in D[0,1]$ be ∃x, y $\in\widetilde{U}$ (A ; [$\delta_1$, $\delta_2$] ), $\Longrightarrow$
$\tilde{\mu}_A(x*y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x) , $\tilde{\mu}_A$ (y)} $\succcurlyeq$ rmin{[$\delta_1$, $\delta_2$] , [$\delta_1$, $\delta_2$] } = [$\delta_1$, $\delta_2$] *and so* $(x*y) \in \widetilde{U}$ (A ; [$\delta_1$, $\delta_2$] ) .
$\Longrightarrow$ $\widetilde{U}$ (A ; [$\delta_1$, $\delta_2$] ) *the i − v level subalgebra of A.*

*Conversely, assume that* $\widetilde{U}(A;[\delta1,\delta2])\neq\emptyset$ *is an subalgebra of X, for every* $[\delta1,\delta2]\in D[0,1]$.
*In the contrary, suppose that there exist* $x0$ , $y0 \in X$ , ∋

$$\tilde{\mu}_A(x0*y0) \prec rmin\{\tilde{\mu}_A(x0),\tilde{\mu}_A(y0)\}.$$

$$Let\ \tilde{\mu}_A(x0) = [\gamma1,\gamma2], \tilde{\mu}_A(y0) = [\gamma3,\gamma4]\ and\ \tilde{\mu}_A(x0*y0) = [\delta1,\delta2].$$

$$If$$

$$[\delta1,\delta2] \prec rmin\{[\gamma1,\gamma2],[\gamma3,\gamma4]\} = min\{min\{\gamma1,\gamma3\}, min\{\gamma2,\gamma4\}\}.$$

$$So\ \ \delta1 < min\{\gamma1,\gamma3\}\ and\ \ \delta2 < min\{\gamma2,\gamma4\}.\ Consider$$

$$[\lambda1,\lambda2] = \frac{1}{2}\{\tilde{\mu}_A(x0*y0) + r\,min\{\tilde{\mu}_A(x0),\tilde{\mu}_A(y0)\}\}$$

*We find that*

$$[\lambda1,\lambda2] = \frac{1}{2}\{[\delta1,\delta2] + r\,min\{[\gamma1,\gamma2],[\gamma3,\gamma4]\}\}$$

$$= \frac{1}{2}[(\delta1 + min\{\gamma1,\gamma3\}),(\delta2 + min\{\gamma2,\gamma4\})].$$

Therefore $min\{\gamma_1,\gamma_3\} > \lambda_1 = \frac{1}{2}(\delta_1 + min\{\gamma_1,\gamma_3\}) > \delta_1$ ,

$$min\{\gamma2,\gamma4\} > \lambda2 = \frac{1}{2}(\delta2 + min\{\gamma2,\gamma4\}) > \delta2 .$$

$$Hence\ \ [min\{\gamma1,\gamma3\}, min\{\gamma2,\gamma4\}] \succ [\lambda1,\lambda2] \succ [\delta1,\delta2] = \tilde{\mu}_A(x0*y0),$$

$$so\ that,\ (x0*y0) \notin \widetilde{U}(A;[\lambda1,\lambda2]).\ which\ is\ C!.\ ,since$$

$$\tilde{\mu}_A(x0*y0) = [\gamma1,\gamma2] \geq [min\{\gamma1,\gamma3\}, min\{\gamma2,\gamma4\}] > [\lambda1,\lambda2].$$

$\tilde{\mu}_A(y0) = [\gamma3,\gamma4] \geq [min\{\gamma1,\gamma3\}, min\{\gamma2,\gamma4\}] \succ [\lambda1,\lambda2]$ , $\Longrightarrow$ *that*
$x_0 * y_0 \in\widetilde{U}$ (A ; [$\lambda_1$,$\lambda_2$] ). $\Longrightarrow$
$\tilde{\mu}_A(x*y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x) , $\tilde{\mu}_A$ (y)},$\forall\, x,y\in X$. ⌂

**Th. 3.13.**

Every BZ-al$gebra$ $(X;*,0)$ ca$n$ $b$e real*iz*ed as i-$v$ l*eve*l sub*alge*bra of an i-vFSA o$f$ $X$.

**Pf.**

Let $Y$ be a s*ubalge*bra of $X$ and let A be i-v FS. on $X$ de*fined* by $\tilde{\mu}_A(x) = \begin{cases}[\alpha_1,\alpha_2] & if x\in Y\\ [0,0] & otherwise\end{cases}$ .

Where $\alpha_1,\alpha_2 \in (0,1]$ *with* $\alpha_1 < \alpha_2$ .

It i*s clea*r that $\widetilde{U}$ (A ; [ $\alpha_1$ , $\alpha_2$]) = Y . W*e sh*ow that A *is i*-vFSA o$f$ $X$. *L*et $x,y,z\in X$,

If x,y $\in Y \Longrightarrow (x*y) \in Y$ , and *the*refore
$\tilde{\mu}_A(x*y)$ = [$\alpha_1$ , $\alpha_2$] $\succcurlyeq$ *rmin*{[ $\alpha_1$ ,$\alpha_2$ ] , [$\alpha_1$,$\alpha_2$]}
=rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}.

I$f$ $x$ ,y $\notin$Y , $\Longrightarrow$ $\tilde{\mu}_A$ (x) = [0,0] =$\tilde{\mu}_A$ (y) *and* so
$\tilde{\mu}_A(x*y) \geq$ [0,0] = rmin{[0,0],[0,0]} $\succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)} ,

If x$\in Y$ *and* y $\notin Y \Longrightarrow$ $\tilde{\mu}_A$ (x) =[ $\alpha_1$,$\alpha_2$ ] and μ˜_A (y)= [0,0] , $\Longrightarrow$ μ˜_A (x*y) $\succcurlyeq$ [0,0] =rmin{[ $\alpha_1$ ,$\alpha_2$ ],[0,0]} = rmin{μ˜_A (x) , μ˜_A (y)}.

Similarly for the case $x \notin Y$ and $y \in Y$ we get

$\tilde{\mu}_A(x*y) \succcurlyeq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$.

Therefore A is an i-vFSA of $X$.⌂

**Th. 3.14.**

If A is i-vFSA of BZ-algebra $(X;*,0) \Longrightarrow$ the set $X_{\tilde{M}_A} = \{x \in X \mid \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$ is an subalgebra of $X$.

**Pf.**

Let $x, y \in X_{\tilde{M}_A}$, $\Longrightarrow \tilde{\mu}_A(x) = \tilde{\mu}_A(0) = \tilde{\mu}_A(y)$, and so

$\tilde{\mu}_A(x*y) \succcurlyeq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(0)$.

By Prop.(3.7), we get $\tilde{\mu}_A(x*y) = \tilde{\mu}_A(0)$, that is $(x*y) \in X_{\tilde{M}_A}$.

Hence $X_{\tilde{M}_A}$ is a subalgebra of $X$. ⌂

## 4. I-v fuzzy ideal of BZ-algebra

*In this section, we will introduce a new notion called $i-v$ ideal of $BZ-$algebras and study several properties of it.*

**Def. 4.1.**

An i-v FS. $A = \{< x, \tilde{\mu}_A(x) > / x \in X\}$ of BZ-algebra $(X;*,0)$ is called **an i-v fuzzy ideal** (i-v fuzzy ideal, in short) if it satisfies the following conditions:

($A_1$) $\tilde{\mu}_A(0) \succcurlyeq \tilde{\mu}_A(x)$,

($A_2$) $\tilde{\mu}_A(y) \succcurlyeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$, $\forall\ x, y \in X$.

**Ex.4.2.**

Let $X = \{0, 1, 2, 3\}$ as in Ex.(3.6). Define $\tilde{\mu}_A(x)$ as follows:

$$\tilde{\mu}_A(x) = \begin{cases} [0.3, 0.9] & if\ x \in \{0,1\} \\ [0.1, 0.6] & otherwise \end{cases}.$$

It is easy to check that A is an i-v FI of $X$.

**Th. 4.3.**

An i-v FS. $A = [\mu_A^-, \mu_A^+]$ of BZ-algebra $(X;*,0)$ is an i-v fuzzy ideal of $X \Leftrightarrow \mu_A^-$ and $\mu_A^+$ are fuzzy ideals of $X$.

**Pf.**

Suppose that A is an i-v fuzzy ideal of $X$. $\forall\ x, y \in X$ we have

$[\mu_A^-(y), \mu_A^+(y)] = \tilde{\mu}_A(y)$

$\succcurlyeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$

$= [min\{\mu_A^-(x*y), \mu_A^+(x*y)\}, min\{\mu_A^-(x), \mu_A^+(x)\}]$

$= [min\{\mu_A^-(x*y), \mu_A^-(x)\}, min\{\mu_A^+(x*y), \mu_A^+(x)\}]$.

*Therefore*, $\mu_A^-(y) \geq min\{\mu_A^-(x*y), \mu_A^-(x)\}$ *and*

$\mu_A^+(y) \geq min\{\mu_A^+(x*y), \mu_A^+(x)\}$.

Hence, we get that $\mu_A^-$ and $\mu_A^+$ are fuzzy ideals of $X$.

Conversely, $\mu_A^-$ and $\mu_A^+$ are fuzzy ideals of $X$, for any $x, y \in X$. Observe $\mu_A^-(y) \geq min\{\mu_A^-(x*y), \mu_A^-(x)\}$ *and* $\mu_A^+(y) \geq min\{\mu_A^+(x*y), \mu_A^+(x)\}$. $\Longrightarrow$

$\tilde{\mu}_A(y) = [\mu_A^-(y), \mu_A^+(y)]$

$\succcurlyeq [min\{\mu_A^-(x*y), \mu_A^-(x)\}, min\{\mu_A^+(x*y), \mu_A^+(x)\}]$

$\succcurlyeq rmin\{min\{\mu_A^-(x*y), \mu_A^+(x*y)\}, min\{\mu_A^-(x), \mu_A^+(x)\}\}$

$= rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$.

From what was mentioned above we can conclude that A is an i-v fuzzy ideal of $X$. ⌂

**Prop.4.4.**

Let $A_1$ and $A_2$ be i-v fuzzy ideals of an BZ-algebra $\Longrightarrow A_1 \cap A_2$ is an $i-$v fuzzy ideal.

**Pf.**

Since $A_1$ and $A_2$ be i$-v$ fuzzy ideals of BZ-algebra $(X;*,0) \Longrightarrow$

$\tilde{\mu}_{A_1 \cap A_2}(0) = [\mu_{A1\cap A2}^-(0), \mu_{A1\cap A2}^+(0)] \succcurlyeq [\mu_{A1\cap A2}^-(x), \mu_{A1\cap A2}^+(x)] = \tilde{\mu}_{A_1 \cap A_2}(x)$.

Suppose $x, y \in X \ni (x*y) \in A_1 \cap A_2$ *and* $x \in A_1 \cap A_2$.

Since $A_1$ and $A_2$ are i-vFIs of $X \Longrightarrow$ by the Th. (4.3), we get

μ˜_(A_1∩A_2 ) (y) = [μ_(A1∩A2)^- (y) , μ_(A1∩A2)^+ (y) ]
⩾ [min{ μ_(A1∩A2)^- (x*y) , μ_(A1∩A2)^- (x)}, min{μ_(A1∩A2)^+ (x*y) , μ_(A1∩A2)^+ (x)}]
= rmin{μ˜_(A_1∩A_2 ) (x*y), μ˜_(A_1∩A_2 ) (x)}] . ⌂

**Coro. 4.5.**

Let $\{A_i \mid i\in\Lambda\}$ be a family of i-vFI of BZ-algebra $(X;*,0)$, $\Longrightarrow$ $\bigcap_{i\in\Lambda} A_i$ is also i-vFI of $X$ .

**Th. 4.6.**

Let $(X;*,0)$ be BZ-algebra and $A$ be an i-v FS. of $X$. A is an i-v fuzzy ideal of $X \Longleftrightarrow$, the nonempty set $\widetilde{U}$ (A;[δ1,δ2]):={ x∈ X $|\tilde{\mu}_A$ (x) ≥ [δ1,δ2]} is an ideal of $X$, for every [δ1, δ2]∈ D[0, 1] . We call $\widetilde{U}$ (A ; [δ1, δ2] ) the i-v level ideal of A.

**Pf.**

Assume that A is i-vFI of $X$ and let
[δ1, δ2] ∈ $D[0, 1]$ be ∋ $(x*y), x \in \widetilde{U}$ (A ; [δ1, δ2] ), ⟹
$\tilde{\mu}_A(y) \succcurlyeq$ rmin{$\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)$} ⩾ rmin{[δ1, δ2] , [δ1, δ2] } = [δ1,δ2] and so $(y) \in \widetilde{U}$ (A ; [δ1,δ2] ) .
⟹ $\widetilde{U}$ (A ; [δ1,δ2] ) the i-v level ideal of A.

Conversely, assume that $\widetilde{U}$ (A ; [δ1, δ2] ) ≠ ∅ is ideal of $X$, for every
[δ1, δ2] ∈ D[0, 1] .

In the contrary, suppose that there exist $x_0$ , $y_0 \in X$ , ∋

$\tilde{\mu}_A(y0) \prec rmin\{\tilde{\mu}_A(x0*y0), \tilde{\mu}_A(x0)\}$.

Let $\tilde{\mu}_A(x0*y0) = [\gamma1,\gamma2]$ , $\tilde{\mu}_A(x0) = [\gamma3,\gamma4]$ and $\tilde{\mu}_A(y0) = [\delta1,\delta2]$.

If [δ1, δ2] ≺ rmin{ [γ1, γ2], [γ3, γ4]} = min { min {γ1, γ3}, min {γ2, γ4}} .

So $\delta_1 < \min\{\gamma_1, \gamma_3\}$ and $\delta_2 < \min\{\gamma_2, \gamma_4\}$ .

Consider $[\lambda_1, \lambda_2] = \frac{1}{2}\{\tilde{\mu}_A(y_0) + r\min\{\tilde{\mu}_A(x_0*y_0), \tilde{\mu}_A(x_0)\}\}$

We find that

$$[\lambda_1, \lambda_2] = \frac{1}{2}\{[\delta1,\delta2] + r\,min\{[\gamma1,\gamma2],[\gamma3,\gamma4]\}\}$$

$$= \frac{1}{2}[(\delta1 + min\{\gamma1,\gamma3\}), (\delta2 + min\{\gamma2,\gamma4\})].$$

$$Therefore\ min\{\gamma1,\gamma3\} > \lambda1 = \frac{1}{2}(\delta1 + min\{\gamma1,\gamma3\}) > \delta1 ,$$

$$min\{\gamma2,\gamma4\} > \lambda2 = \frac{1}{2}(\delta2 + min\{\gamma2,\gamma4\}) > \delta2 .$$

$$Hence\ [min\{\gamma1,\gamma3\}, min\{\gamma2,\gamma4\}] > [\lambda1,\lambda2] > [\delta1,\delta2] = \tilde{\mu}_A(y0) ,$$

so that, $(y0) \notin \widetilde{U}(A; [\lambda1,\lambda2])$ , which is a C!. , since

$\tilde{\mu}_A(x_0*y_0) = [\gamma_1,\gamma_2] \geq [\min\{\gamma_1,\gamma_3\}, \min\{\gamma_2,\gamma_4\}] > [\lambda_1,\lambda_2]$ .
$\tilde{\mu}_A(x_0) = [\gamma_3,\gamma_4] \geq [\min\{\gamma_1,\gamma_3\}, \min\{\gamma_2,\gamma_4\}] > [\lambda_1,\lambda_2]$ , ⟹ that
$(x_0*y_0)$ ,$(x_0) \in \widetilde{U}(A; [\lambda_1,\lambda_2])$. ⟹
$\tilde{\mu}_A(y) \succcurlyeq$ rmin{$\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)$},∀ $x, y \in X$. ⌂

**Th. 4.7.**

Every ideal of BZ-algebra $(X;*,0)$ can be realized as an i-v level ideal of i-vFI of $X$.

**Pf.**

Let $Y$ be an ideal of $X$ and let A be i-vFS. of $X$ defined by $\tilde{\mu}_A(x) = \begin{cases}[\alpha_1,\alpha_2] & if\,x \in Y\\ [0,0] & otherwise\end{cases}$.

Where $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 < \alpha_2$ .

It is clear that $\widetilde{U}$ (A ; [ α1 , α2]) = Y . We show that A is an i-vFI of $X$. Let $x, y \in X$,
If $(x*y)$ , $x \in Y$ ⟹ $(y) \in Y$ , and therefore
$\tilde{\mu}_A(y) = [\alpha_1,\alpha_2] \succcurlyeq$ rmin{[ α1 , α2 ],[α1 , α2]}
= rmin{$\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)$}.
If $(x*y)$ , $x \notin Y$ ,⟹ $\tilde{\mu}_A(x*y) = [0,0] = \tilde{\mu}_A(x)$ and so
$\tilde{\mu}_A(y) \geq [0,0] = rmin\{[0,0],[0,0]\} \succcurlyeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$ ,

If $(x*y) \in Y$ and $x \notin Y$, $\Longrightarrow$ $\tilde{\mu}_A(x*y) = [\alpha_1, \alpha_2]$ and
$\tilde{\mu}_A(x) = [0,0]$, $\Longrightarrow$
$\tilde{\mu}_A(y) \succcurlyeq [0,0] = rmin\{[\alpha_1, \alpha_2], [0,0]\}$
$= rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$.
Similarly for the case $(x*y) \notin Y$ and $x \in Y$ we get
$\tilde{\mu}_A(y) \succcurlyeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\}$.
Therefore A is an i-vFI of $X$. ⌂

**Coro. 4.8.**

Let $(X;*,0)$ be an BZ-algebra, B be a FS. of $X$ and let A be i-vFS. of $X$ defined by
$\tilde{\mu}_A(x) = \begin{cases}[\alpha_1, \alpha_2] & if x \in Y \\ [0,0] & otherwise\end{cases}$, where $\alpha_1, \alpha_2 \in (0, 1]$ with $\alpha_1 < \alpha_2$.
If A is an i-vFSA of $X$, $\Longrightarrow$ B is a FSA. of $X$.

**Th. 4.9.**

*If A* is i-vFI of BZ-algebra $(X;*,0)$ $\Longrightarrow$ the set $X_{\tilde{M}_A}$ is ideal of $X$.

**Pf.**

Let $(x*y), x \in X_{\tilde{M}_A}$, $\Longrightarrow$ $\tilde{\mu}_A(x*y) = \tilde{\mu}_A(0) = \tilde{\mu}_A(x)$, and so
$\tilde{\mu}_A(y) \succcurlyeq rmin\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\} = rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(0)$.
Combining this with condition (1) of Def. (4.1), we get
$\tilde{\mu}_A(y) = \tilde{\mu}_A(0)$, that is $(y) \in X_{\tilde{M}_A}$. Hence $X_{\tilde{M}_A}$ is an ideal of $X$. ⌂

## 5. Homomorphism of BZ-algebra

**Def. 5.1 ([26]).**

Let $f : (X; *, 0) \to (Y; *', 0')$ be a mapping from set $X$ into a set $Y$. let B be an i-vFS. of $Y \Longrightarrow$ the inverse image of B, denoted by $f^{-1}(B)$, is an i-vFS. of $X$ with the membership function given by
$\mu_{f^{-1}(B)}(x) = \tilde{\mu}_B(f(x)), \forall x \in X$.

**Prop.5.2 ([26]).**

Let $f : (X; *, 0) \to (Y; *', 0')$ be a mapping from set $X$ into a set $Y$, let $\tilde{m} = [m^-, m^+]$, and $\tilde{n} = [n-, n+]$ be i-vFSs of $X$ and $Y$ resp.. $\Longrightarrow$

(1) $f^{-1}(\tilde{n}) = [f^{-1}(n^-), f^{-1}(n^+)]$,

(2) $f(\tilde{m}) = [f(m^-), f(m^+)]$.

**Th. 5.3.**

Let $f : (X; *, 0) \to (Y; *', 0')$ be homomorphism from BZ-algebra $X$ into BZ-algebra $Y$. If B is i-vFSA of $Y \Longrightarrow$ the pre-image $f^{-1}$ (B) of B is an i-vFSA of $X$.

**Pf.**

Since B= [ $\mu_B^-$, $\mu_B^+$ ] is an i-vFSA of $Y$, it follows that from Th. (3.9), that ( $\mu_B^-$ ) and ( $\mu_B^+$ ) are FSA. s of $Y$.

Using Prop.(2.23(1)), we know $f^{-1}(\mu_B^-)$ and $f^{-1}(\mu_B^+)$ are FSAs of $X$. Hence by Prop.(5.2), we conclude that $f^{-1}$ (B) $= [f^{-1}(\mu_B^-), f^{-1}(\mu_B^+)]$ is an i-vFSA of $X$. ⌂

**Def. 5.4 ([26]).**

Let $f : (X; *, 0) \to (Y; *', 0')$ be a mapping from a set $X$ into a set $Y$. Let A be i-vFS of $X \Longrightarrow$ the image of A, denoted by $f$ (A), is the i-vFS. of $Y$ with membership function denoted by :
$\tilde{\mu}_{f(A)}(x) = \begin{cases} sup_{z \in f^{-1}(y)} \tilde{\mu}_A(z) & if\ f^{-1}(y) \neq \varphi, y \in Y \\ [0,0] & otherwise \end{cases}$,
where $f^{-1}(y) := \{x \in X \mid f(x) = y\}$.

**Th. 5.5.**

Let $f$ be an epimorphism from BZ-algebra $X$ into an BZ-algebra $Y$. If A is i-vFSA of $X$ with sup property $\Longrightarrow f$ (A) of A is i-vFSA of $Y$.

**Pf.**

Assume that A= $[\mu_A^-, \mu_A^+]$ is an i-vFSA of $X$. it follows that from Th. (3.9), that ($\mu_A^-$) and ($\mu_A^+$) are FSAs of $X$.

Using Prop.(2.23(2)), that the images $f$ ($\mu_A^-$) and $f$ ($\mu_A^+$) are FSAs of $Y$.

Hence by Prop.(5.2), we conclude that $f$ (A)= [$f$ ($\mu_A^-$),$f$ ($\mu_A^+$)] is i-vFSA of $Y$ .⌂

**Th. 5.6.**

Let $f : (X; *, 0) \to (Y; *', 0')$ be homomorphism from an BZ-algebra $X$ into an BZ-algebra $Y$. If B is i-vFI of $Y \Longrightarrow$ the pre-image $f^{-1}$ (B) of B is i-vFI of X.

**Pf.**

Since B= [ $\mu_B^-$, $\mu_B^+$ ] is an i-vFI of $Y$, it follows that from Th. (5.3), that ( $\mu_B^-$ ) and ( $\mu_B^+$ ) are fuzzy ideals of $Y$.

Using Th. (2.23(3)), we know $f^{-1}$ ( $\mu_B^-$ ) and $f^{-1}$ ( $\mu_B^+$ ) are fuzzy ideals of $X$.

Hence by Prop.(5.2), we conclude that $f^{-1}$ (B) = [$f^{-1}$ ( $\mu_B^-$ ) , $f^{-1}$ ( $\mu_B^+$ )] is an i-vFI of $X$. ⌂

**Th. 5.7.**

Let $f$ be an epimorphism from BZ-algebra $X$ into BZ$-$algebra $Y$ . If A is an i-vFI of $X$ with sup property, $\Longrightarrow f$ (A) of A is an i-Vfi of $Y$.

**Pf.**

Assume that A= $[\mu_A^-, \mu_A^+]$ is an i-v fuzzy ideal of $X$. it follows that from Th. (4.3), that ($\mu_A^-$) and ($\mu_A^+$) are FIs of $X$.

Using Th. (2.23(4)), that the images $f$ ($\mu_A^-$) and $f$ ($\mu_A^+$) are FIs of $Y$.

Hence by Prop.(5.2), we conclude that $f$ (A)= [$f$ ($\mu_A^-$), $f$ ($\mu_A^+$)] is i-vFI of $Y$ .⌂

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