International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 12, December - 2022, Pages: 97-101

Review of Bounded Linear Functionals and Dual Spaces

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Abstract— *This article is review about bounded linear operator and functionals. We study some facts about them.*

Keywords— bounded linear functional and operator , Banach space , normed linear spaces in $\mathbb{X} \times \mathbb{X}$, normed linear spaces $\mathbb{m}$ -times in $\mathbb{X} \times \mathbb{X} \times \ldots \times \mathbb{X}$, the dual space.

1. Introduction and Primilinaries:

S.Gahler was studied the concepts of normed space in $\mathbb{X} \times \mathbb{X}$ and normed spaces $\mathbb{m}$ times $\mathbb{X} \times \mathbb{X} \times \ldots \times \mathbb{X}$ in about 1964 ,see [2] .Gahler offered that $\mathbb{X}$ in $\mathbb{m}$ -times with $\mathbb{d} \geq \mathbb{m}$ be able to showed as normed space in $\mathbb{m}$ – times by taking the Gahler norm in $\mathbb{m}$-times , which is symbolize by $\|.,\ldots,.\|_{\mathbb{G}}$.Many scientists have investigated the operators and functionals on spaces of norms in 2- and $\mathbb{m}$-times ,see [2,6,7] . Pangalela and Gunawan showed in [7,8] the idea of $\mathbb{m}$-dual spaces.
We need the following definitions.

1.1 Definition [9] : For a linear space of real numbers $\mathbb{X}$ of $\mathbb{d} > \mathbf{1}$ with $\|.,.\|$ is a real -valued -function in $\mathbb{X} \times \mathbb{X}$ times having 4 properties:

- $\|\mathfrak{v}, \mathfrak{s}\| = 0 \ iff\ \mathfrak{v}, \mathfrak{s}$ are linearly independent ;
- $\|\mathfrak{v}, \mathfrak{s}\| = \|\mathfrak{s}, \mathfrak{v}\|$;
- $\|\mathfrak{v}, \alpha\mathfrak{s}\| = |\alpha|\|\mathfrak{v}, \mathfrak{s}\|,\ \forall \alpha \in \mathbb{R}$;
- $\|\mathfrak{v}, \mathfrak{s} + \mathfrak{f}\| \leq \|\mathfrak{v}, \mathfrak{s}\| + \|\mathfrak{v}, \mathfrak{f}\|$, $\forall\ \mathfrak{v}, \mathfrak{s}, \mathfrak{f} \in \mathbb{X}$.

The $\|.,.\|$ defined a norm on $\mathbb{X} \times \mathbb{X}$ (2-times) with $(\mathbb{X}, \|.,.\|)$ a linear space of norm in 2-times.

1.2 Definition [9]: When $\mathbb{X}$ and $\mathbb{Y}$ are linear spaces of real numbers . Symbolize by $G \neq \emptyset \subset \mathbb{X} \times \mathbb{Y}, s.t. \forall \mathfrak{v} \in \mathbb{X}, \mathfrak{s} \in \mathbb{Y}$,the sets $G_{\mathfrak{v}} = \{\mathfrak{s} \in \mathbb{Y}; (\mathfrak{v}, \mathfrak{s}) \in G\}$ and $G^{\mathfrak{s}} = \{\mathfrak{v} \in \mathbb{X}; (\mathfrak{v}, \mathfrak{s}) \in G\}$ are linear subspace of $\mathbb{Y}$ and $\mathbb{X}$, in the order . For a mapping $\|.,.\|: G \to [0, \infty)$ defines a generalized norm in 2- times on G if it is accept the 3 conditions:

- $\|\mathfrak{v}, \alpha\mathfrak{s}\| = |\alpha|\|\mathfrak{v}, \mathfrak{s}\| = \|\alpha\mathfrak{v}, \mathfrak{s}\|,\ \forall\ \alpha \in \mathbb{R}, \forall(\mathfrak{v}, \mathfrak{s}) \in G$;
- $\|\mathfrak{v}, \mathfrak{s} + \mathfrak{f}\| \leq \|\mathfrak{v}, \mathfrak{s}\| + \|\mathfrak{v}, \mathfrak{f}\|$, $\mathfrak{v} \in \mathbb{X}, \mathfrak{s}, \mathfrak{f} \in \mathbb{Y}, s.t. (\mathfrak{v}, \mathfrak{s}), (\mathfrak{v}, \mathfrak{f}) \in G$;
- $\|\mathfrak{v} + \mathfrak{s}, \mathfrak{f}\| \leq \|\mathfrak{v}, \mathfrak{f}\| + \|\mathfrak{s}, \mathfrak{f}\|$, $for\ \mathfrak{v}, \mathfrak{s} \in \mathbb{X}$, $\mathfrak{f} \in \mathbb{Y}, s.t. (\mathfrak{v}, \mathfrak{f}), (\mathfrak{s}, \mathfrak{f}) \in G$.

The system G is defined a 2-normed set .

1.3 Definition [9]: When a linear space $\mathbb{X}$ of real numbers . Symbolize by $\varkappa \neq \emptyset \subset \mathbb{X} \times \mathbb{X}$ with the assets $\varkappa = \varkappa^{-1}$ and s.t. the linear system $\varkappa^{\mathfrak{s}} = \{\mathfrak{v} \in \mathbb{X}; (\mathfrak{v}, \mathfrak{s}) \in \varkappa\}$ is a subspace of $\mathbb{X}$,$\forall\ \mathfrak{s} \in \mathbb{X}$. Let $\|.,.\|: \varkappa \to [0, \infty)$ be a function accept 3 conditions:

- $\|\mathfrak{v}, \mathfrak{s}\| = \|\mathfrak{s}, \mathfrak{v}\|, \forall(\mathfrak{v}, \mathfrak{s}) \in \varkappa$;
- $\|\mathfrak{v}, \alpha\mathfrak{s}\| = |\alpha|\|\mathfrak{v}, \mathfrak{s}\|, \forall\alpha \in \mathbb{R}, , \forall(\mathfrak{v}, \mathfrak{s}) \in \varkappa$;
- $\|\mathfrak{v}, \mathfrak{s} + \mathfrak{f}\| \leq \|\mathfrak{v}, \mathfrak{s}\| + \|\mathfrak{v}, \mathfrak{f}\|$, $\mathfrak{v}, \mathfrak{s}, \mathfrak{f} \in \mathbb{X}, s.t (\mathfrak{v}, \mathfrak{s}), (\mathfrak{v}, \mathfrak{f}) \in \varkappa$;

defines a generalized symmetric norm on $\varkappa$ in 2-times .For the system $\varkappa$ is defined a symmetric normed set in 2-times.

1.4 Definition [10]: Assume a space $\mathbb{X}$ which satisfy the condition for all Cauchy sequence lie in the space $\mathbb{X}$ which is convergent to a point in $\mathbb{X}$ then is defined sequentially complete.
1.5 Theorem [10] : Banach space is said the space of norm $(\mathbb{X}, \|.\|)$ iff the symmetric space of norm in 2-times with norm in 2-times defined by $\|\mathfrak{v}, \mathfrak{s}\| = \|\mathfrak{v}\|.\|\mathfrak{s}\|$, $\forall \mathfrak{v}, \mathfrak{s} \in \mathbb{X}$ is sequentially complete.
1.6 Definition [3,4,5] : A mapping $\|.,\ldots,.\|: \mathbb{X}^{\mathbb{m}} \to \mathbb{R}$, satisfying the following 4 properties with $\mathbb{m} > 0$ and a vector space $\mathbb{X}$ of real numbers with $\mathbb{d} \geq \mathbb{m}$,

1. $\|\mathfrak{v}_1, \ldots\ldots, \mathfrak{v}_{\mathbb{m}}\| = 0 \leftrightarrow \mathfrak{v}_1, \mathfrak{v}_2, \ldots\ldots, \mathfrak{v}_{\mathbb{m}}$ not all equal zero ;

2. $\|\mathfrak{v}_1, \dots \dots, \mathfrak{v}_{\mathfrak{m}}\|$ is invariant under permutation ;

3. $\|\alpha\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}\| = |\alpha|\|\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}\|\ for\ any\ \alpha \in \mathbb{R}$;

4. $\|\mathfrak{v} + \mathfrak{s}, \mathfrak{v}_2, \dots, \mathfrak{v}_{\mathfrak{m}}\| \le \|\mathfrak{v}, \mathfrak{v}_2, \dots, \mathfrak{v}_{\mathfrak{m}}\| + \|\mathfrak{s}, \mathfrak{v}_2, \dots, \mathfrak{v}_{\mathfrak{m}}\|$,

the space with 4 conditions is defined a norm on $\mathbb{X}$ in $\mathfrak{m}$ -times, whereas the symbol $(\mathbb{X}, \|.,\dots,.\|)$ is defined a space of norm in $\mathfrak{m}$ -times.

1.7 Definition [3,4,5]: The norm

$\|\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}\|_{\mathbb{G}} := \underbrace{\sup}_{\substack{\mathbb{f}_{i \in \mathbb{X}^{(1)}, \|\mathbb{f}_i\| \le 1} \\ 1 \le i \le \mathfrak{m}}} abs\left(det\left[\mathbb{f}_j(\mathfrak{v}_i)\right]_{i,j}\right)$, $\forall \mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}} \in \mathbb{X}$, is defined on $\mathbb{X}$ in $\mathfrak{m}$ – times, where $\mathbb{X}^{(1)}$ the dual space of $\mathbb{X}$.

1.8 Definition [7]: A functional $\mathbb{f}$ is multi-linear in $\mathfrak{m}$ – times if it is satisfies 2 conditions:

- ❖ $\mathbb{f}(\mathfrak{v}_1 + \mathfrak{s}_1, \dots, \mathfrak{v}_{\mathfrak{m}} + \mathfrak{s}_{\mathfrak{m}}) = \sum_{z_{i \in \{n_i, m_i\}, 1 \le i \le \mathfrak{m}}} \mathbb{f}(z_1, \dots, z_{\mathfrak{m}})$
- ❖ $\mathbb{f}(\alpha_1\mathfrak{v}_1, \dots, \alpha_{\mathfrak{m}}\mathfrak{v}_{\mathfrak{m}}) = \alpha_1 \dots \alpha_{\mathfrak{m}-1}\mathbb{f}(\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}})$

for all $\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}, \mathfrak{s}_1, \dots, \mathfrak{s}_{\mathfrak{m}} \in \mathbb{X}\ and\ \alpha_1 \dots \alpha_{\mathfrak{m}} \in \mathbb{R}$.

1.9 Definition [7]: Assume that $\mathbb{f}, \mathbb{j}$ are multi-linear- functionals on $\mathbb{X}$ in $\mathfrak{m}$ – times defined by the formula $\mathbb{f} + \mathbb{j}$ as shown below $(\mathbb{f} + \mathbb{j})(n_1, \dots, n_{\mathfrak{m}}) := \mathbb{f}(\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}) + \mathbb{j}(\mathfrak{v}, \dots, \mathfrak{v}_{\mathfrak{m}})for\ \mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}} \in \mathbb{X}$. Then $\mathbb{f} + \mathbb{j}$ is again multilinear .

1.10 Definition [7]: We say a functional $\mathbb{f}$ in $\mathfrak{m}$ - times bounded on $(\mathbb{X}, \|.,\dots,.\|)$ if $\exists \mathcal{C} > 0,\ s.t.\ |\mathbb{f}(\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}})| \le \mathcal{C}\|\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}\|, \forall \mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}} \in \mathbb{X}$,
also in space of norm $(\mathbb{X}, \|.\|)$ in $\mathfrak{m}$- times which is a bounded if $\exists \mathcal{C} > 0,\ s.t. |\mathbb{f}(\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}})| \le \mathcal{C}\|\mathfrak{v}_1\| \dots \|\mathfrak{v}_{\mathfrak{m}}\|$.

1.11 Definition [7] : The collection of per-mutations f $(1, \dots, \mathfrak{m})$ denoted by $\mathcal{G}_{\mathfrak{m}}$ .Reworded from that every multi-linear- functional $\mathbb{f}$ on $(\mathbb{X}, \|.,\dots,.\|)$ which is bounded in $\mathfrak{m}$-times is anti-symmetric means that: $\mathbb{f}(\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}) = sign\ (\sigma)\mathbb{f}(\mathfrak{v}_{\sigma(1)}, \dots, \mathfrak{v}_{\sigma(\mathfrak{m})})$, for $\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}} \in \mathbb{X}$ and $\sigma \in \mathcal{G}_{\mathfrak{m}}$. Now $sign(\sigma) = 1\ if\ \sigma$ is 2,4,… (even per-mutation) and $sign(\sigma) = -1\ if\ \sigma$ is 1,3,… (odd per-mutation). If the condition $\mathbb{f}(\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}}) = 0$ hold for any linearly dependent $\mathfrak{v}_1, \dots, \mathfrak{v}_{\mathfrak{m}} \in \mathbb{X}$,then $\mathbb{f}$ is anti-symmetric .

1.12 Definition [7] : The space of dual $(\mathbb{X}, \|.\|)$ in $\mathfrak{m}$ –times denoted by $\mathbb{X}^{(\mathfrak{m})}$ is called the space of multi-linear - functionals which is bounded on $(\mathbb{X}, \|.\|)\ in$ $\mathfrak{m}$ –times .When $\mathfrak{m}$=0 , we set $\mathbb{X}^{(0)}$ as $\mathbb{R}$.The norm $\|.\|_{\mathfrak{m},1}$ on $X^{(\mathfrak{m})}$,where $\|\mathbb{f}\|_{\mathfrak{m},1} := \sup_{\mathfrak{v}_1,\dots,\mathfrak{v}_n \ne 0} \frac{|\mathbb{f}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathfrak{m}})|}{\|\mathfrak{v}_1\|\dots\|\mathfrak{v}_{\mathfrak{m}}\|}$,for $\mathbb{f} \in \mathbb{X}^{(\mathfrak{m})}$, sets a norm on $\mathbb{X}^{(\mathfrak{m})}$ .Whereas , the space of all multi-linear- functional $\mathbb{f}$ on $(\mathbb{X}, \|.,\dots,.\|)$ which is bounded in $\mathfrak{m}$-times is known the dual of $(\mathbb{X}, \|.,\dots,.\|)$ in $\mathfrak{m}$ –times .It is again space of norm with $\|\mathbb{f}\|_{\mathfrak{m},\mathfrak{m}} := \sup_{\|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathfrak{m}}\| \ne 0} \frac{|\mathbb{f}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathfrak{m}})|}{\|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathfrak{m}}\|}$.

1.13 Definition[7]: Assume that $\mathbb{X}, \mathbb{Y}$ are spaces of norms for real numbers .The symbol $\mathcal{T}(\mathbb{X}, \mathbb{Y})$ is define the set linear of operators which is bounded from $\mathbb{X}$ into $\mathbb{Y}$. The map $\|.\|_{op}$ where $\|\mathcal{w}\|_{op} := \sup_{n \ne 0} \frac{\|\mathcal{w}(\mathfrak{v})\|}{\|\mathfrak{v}\|}$, $\forall\ \mathcal{w} \in \mathcal{T}(\mathbb{X}, \mathbb{Y})$, is a usual norm on $\mathcal{T}(\mathbb{X}, \mathbb{Y})$.

1.14 Definition [10] : Let $\mathcal{T}: G \to \mathbb{Y}$, $G \ne \emptyset \subseteq \mathbb{X} \times \mathbb{Y}$ is operator defined to be linear in 2-times if it is accept the two condition below:

- ❖ $\mathcal{T}(\mathfrak{v} + \mathfrak{s}, \mathcal{f} + \mathcal{g}) = \mathcal{T}(\mathfrak{v}, \mathfrak{s}) + \mathcal{T}(\mathcal{f}, \mathcal{g})\ for\ \mathfrak{v}, \mathfrak{s}, \mathcal{f}, \mathcal{g} \in \mathbb{X}, s.t. \mathfrak{v}, \mathcal{f} \in G^{\mathfrak{s}} \cap G^{\mathcal{g}}$
- ❖ $\mathcal{T}(\alpha\mathfrak{v}, \beta\mathfrak{s}) = \alpha.\beta.\mathcal{T}(\mathfrak{v}, \mathfrak{s})for\ \alpha, \beta \in \mathbb{R}, (\mathfrak{v}, \mathfrak{s}) \in G$.

1.15 Definition[10]: A normed operator $\mathcal{T}$ in 2- times is defined to become bounded if there exists a number $\mathcal{C} > 0$, s.t. $\|\mathcal{T}(\mathfrak{v}, \mathfrak{s})\| \le \mathcal{C}.\|\mathfrak{v}, \mathfrak{s}\|\ \ for\ all\ (\mathfrak{v}, \mathfrak{s}) \in G$.

1.16 Definition [10]: Suppose that $\mathcal{T}$ is an operator which is bounded ,subsequently the numeral $\|\mathcal{T}\| = \inf\{\ \mathcal{C} > 0\ ; \|\mathcal{T}(\mathfrak{v}, \mathfrak{s})\| \le \mathcal{C}.\|\mathfrak{v}, \mathfrak{s}\|\ \ for\ (\mathfrak{v}, \mathfrak{s}) \in G\}$ is defined the linear operator of norm in 2-times $\mathcal{T}$.

1.17 Definition[10] : Suppose that $\mathbb{Y}$ a space of norm and $G \subseteq \mathbb{X} \times \mathbb{X}$ is a set of norm in 2-times . The Symbol by $\mathcal{T}_2(G, \mathbb{Y})$ is denoted the system of all linear operators which is bounded in 2-times form G at $\mathbb{Y}$. Specially ,the system $\mathcal{T}_2(\mathbb{X}, \mathbb{Y})$,whether $\mathbb{X}$ is a generalized space of norm in 2-times and $G = \mathbb{X} \times \mathbb{X}$. See $\mathbb{L}, \mathbb{K} \in \mathcal{T}_2(G, \mathbb{Y})$ and satisfy two conditions:

- ❖ $(\mathbb{L} + \mathbb{K})(\mathfrak{v}, \mathfrak{s}) = \mathbb{L}(\mathfrak{v}, \mathfrak{s}) + \mathbb{K}(\mathfrak{v}, \mathfrak{s}), \forall (\mathfrak{v}, \mathfrak{s}) \in G$;
- ❖ $(\alpha.\mathbb{L})(\mathfrak{v}, \mathfrak{s}) = \alpha.\mathbb{L}(\mathfrak{v}, \mathfrak{s})for\ \alpha \in \mathbb{R}, (\mathfrak{v}, \mathfrak{s}) \in G$.

1.18 Definition [1] :We will symbolize $l^p = l^p_{\mathbb{N}}(\mathbb{R}), 1 \le p < \infty$:the space of real numbers of p-summable sequence. Reworded that $\mathcal{u} := \{\mathcal{u}_k\}_{k=1}^{\infty}$ forms a sequence of real numbers $\in l^p$, we defined $\|\mathcal{u}\|_p := (\sum_{k=1}^{\infty} |\mathcal{u}_k|^p)^{1/p} < \infty$. The dual space of l^p is $l^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1.There$ are several n-norms on l^p explained below

$$\|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}\|_p^{\mathbb{G}} := sup_{\mathfrak{s}_i\in l^{p'},\|\mathfrak{s}_i\|_{p'}\le 1}\begin{vmatrix}\sum_{k=1}^{\infty}\mathfrak{v}_{1k}\mathfrak{s}_{1k} & \cdots & \sum_{k=1}^{\infty}\mathfrak{v}_{1k}\mathfrak{s}_{\mathbb{m}k}\\ \vdots & \ddots & \vdots\\ \sum_{k=1}^{\infty}\mathfrak{v}_{\mathbb{m}k}\mathfrak{s}_{1k} & \cdots & \sum_{k=1}^{\infty}\mathfrak{v}_{\mathbb{m}k}\mathfrak{s}_{\mathbb{m}k}\end{vmatrix}$$

$$\|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}\|_p^{H} := \left[\frac{1}{\mathbb{m}!}\ \sum_{k=1}^{\infty}\dots\sum_{k=1}^{\infty} abs\begin{vmatrix}\mathfrak{v}_{1k_1} & \cdots & \mathfrak{v}_{1k_{\mathbb{m}}}\\ \vdots & \ddots & \vdots\\ \mathfrak{v}_{\mathbb{m}k_1} & \cdots & \mathfrak{v}_{\mathbb{m}k_{\mathbb{m}}}\end{vmatrix}^p\right]^{\frac{1}{p}}$$

$$\|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}\|_p^{I} := sup_{\mathfrak{s}_i\in l^{p'},\|\mathfrak{s}_1,\dots,\mathfrak{s}_{\mathbb{m}}\|_{p'}^{H}\le 1}\begin{vmatrix}\sum_{k=1}^{\infty}\mathfrak{v}_{1k}\mathfrak{s}_{1k} & \cdots & \sum_{k=1}^{\infty}\mathfrak{v}_{1k}\mathfrak{s}_{\mathbb{m}k}\\ \vdots & \ddots & \vdots\\ \sum_{k=1}^{\infty}\mathfrak{v}_{\mathbb{m}k}\mathfrak{s}_{1k} & \cdots & \sum_{k=1}^{\infty}\mathfrak{v}_{\mathbb{m}k}\mathfrak{s}_{\mathbb{m}k}\end{vmatrix}$$

Also let $:= \{\mathfrak{s}_1,\dots,\mathfrak{s}_{\mathbb{m}}\}\in l^{p'}$, we have

$\mathbb{f}_{\mathbb{Y}}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}) := \frac{1}{\mathbb{m}!}\sum_{k1=1}^{\infty}\dots\sum_{k\mathbb{m}=1}^{\infty}\begin{vmatrix}\mathfrak{v}_{1k_1} & \cdots & \mathfrak{v}_{1k_{\mathbb{m}}}\\ \vdots & \ddots & \vdots\\ \mathfrak{v}_{\mathbb{m}k_1} & \cdots & \mathfrak{v}_{\mathbb{m}k_{\mathbb{m}}}\end{vmatrix}\begin{vmatrix}\mathfrak{s}_{1k_1} & \cdots & \mathfrak{s}_{1k_{\mathbb{m}}}\\ \vdots & \ddots & \vdots\\ \mathfrak{s}_{\mathbb{m}k_1} & \cdots & \mathfrak{s}_{\mathbb{m}k_{\mathbb{m}}}\end{vmatrix}$,for $(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}})\in l^p$ .Further, we have $|\mathbb{f}_{\mathbb{Y}}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}})|\le \|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}\|_p^H\|\mathfrak{s}_1,\dots,\mathfrak{s}_{\mathbb{m}}\|_{p'}^H$, and $\mathbb{f}_{\mathbb{Y}}$ is bounded on $\left(l^p,\|.,\dots,.\|_p^H\right)$ for $\|\mathbb{f}_{\mathbb{Y}}\|\le\|\mathfrak{s}_1,\dots,\mathfrak{s}_{\mathbb{m}}\|_{p'}^H$.

1.19 Definition [1]**:** The system $\mathcal{X}^*$ of all multilinear functionals on $(\mathbb{X},\|.,\dots,.\|)$ in $\mathbb{m}$- times which it is bounded. Defined by $\|\mathbb{f}\| := \inf\{\mathcal{C}>0\}$ or equivalently $\|\mathbb{f}\| := \sup\{|\mathbb{f}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}})|:\|\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}\|\le 1\}$,determines a norm on $\mathcal{X}^*$.

2. Linear Operators and Functionals which are Bounded on Normed Sets in 2-times and on Normed Spaces in $\mathbb{m}$ - times

2. 1 Theorem [10]: Suppose that $\mathcal{T}$ linear operator in 2- times which it is bounded .Then we have

a.$\|\mathcal{T}\|\le\mathcal{C}$ for $\mathcal{C}\in\mathcal{P}^{(\mathcal{T})}=\{\mathcal{C}'>0;\|\mathcal{T}(\mathfrak{v},\mathfrak{s})\|\le\ \mathcal{C}'.\|\mathfrak{v},\mathfrak{s}\|\ for\ (\mathfrak{v},\mathfrak{s})\in G\}$;

b. $\|\mathcal{T}(\mathfrak{v},\mathfrak{s})\|\le\|\mathcal{T}\|.\|\mathfrak{v},\mathfrak{s}\|\ \ for\ each\ (\mathfrak{v},\mathfrak{s})\in G$;

c. $\|\mathcal{T}\|=\sup\{\ \|\mathcal{T}(\mathfrak{v},\mathfrak{s})\|;(\mathfrak{v},\mathfrak{s})\in G,\|\mathfrak{v},\mathfrak{s}\|=1\}$
$=\sup\{\ \|\mathcal{T}(\mathfrak{v},\mathfrak{s})\|;(\mathfrak{v},\mathfrak{s})\in G,\|\mathfrak{v},\mathfrak{s}\|\le 1\}$
$=\sup\{\ \frac{\|\mathcal{T}(\mathfrak{v},\mathfrak{s})\|}{\|\mathfrak{v},\mathfrak{s}\|};(\mathfrak{v},\mathfrak{s})\in G,\|\mathfrak{v},\mathfrak{s}\|\ne 0\}$

2.2 Theorem [10]: Suppose that G is a normed system in 2-times with $\mathbb{Y}$ which is normed space , then $(\mathcal{T}_2(G,\mathbb{Y}),\|.\|)$ is the space of norm.

2.3 Theorem [10]:Suppose that $\mathbb{Y}$ is a space of Banach with G is system of norm in 2-times ,then $\mathcal{T}_2(G,\mathbb{Y})$ defines a space of Banach.

2.4 Corollary [10] :Suppose that $\mathbb{Y}$ is a space of Banach with $\varkappa$ is a symmetric system of norm in 2-times, then $\mathcal{T}_2(\varkappa,\mathbb{Y})$ is a space of norm which is symmetric sequentially complete in 2-times and take the property : $\|\mathbb{L},\mathbb{K}\|=\|\mathbb{L}\|.\|\mathbb{K}\|$ for $\mathbb{L},\mathbb{K}\in\mathcal{T}_2(\varkappa,\mathbb{Y})$.

2.5 Proposition [10]**:** If G be a system of norm in 2-times, the set $\{\|\mathbb{F}_{\mathbb{m}}\|;\mathbb{m}\in N\}$ which is bounded , $\mathbb{Y}$ a space of norm and the set $\{\mathbb{F}_{\mathbb{m}};\mathbb{m}\in N\}\subset\mathcal{T}_2(G,\mathbb{Y})$, then $\forall(\mathfrak{v},\mathfrak{s})\in G$ the system $\{\|\mathbb{F}_{\mathbb{m}}(\mathfrak{v},\mathfrak{s})\|;\mathbb{m}\in N\}$ is- bounded.

2.6 Theorem [10]**:** Let $\mathbb{Y}$ a space of norm and $\mathbb{X}$ is a generalized space of norm in 2-times . When $\{\mathbb{F}_{\mathbb{m}};\mathbb{m}\in N\}\ is\ bounded\ set\ and\subset\mathcal{T}_2(\mathbb{X},\mathbb{Y})$ which is pointwise- convergent to $\mathcal{F}$, then $\mathbb{F}\in\mathcal{T}_2(\mathbb{X},\mathbb{Y})$.

2.7 Theorem [1] : The three $\mathbb{m}$- norms on l^p , viz
$\|.,\dots,.\|_p^I,\|.,\dots,.\|_p^H, and\|.,\dots,.\|_p^{\mathbb{G}}$, are equivalent .

2.8 Proposition [1]: Suppose that $\mathbb{f}$ is a multi-linear- functional in $\mathbb{m}$ –times on $(\mathbb{X},\|.,\dots,.\|)$ Which is bounded, then $\mathbb{f}$ is anti-symmetric , in order to $\mathbb{f}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}})=sign(\sigma)\mathbb{f}(\mathfrak{v}_{\sigma(1)},\dots,\mathfrak{v}_{\sigma(\mathbb{m})})$, where $\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}}\in\mathbb{X}$.

2.9 Fact [1]**:** Let $(l^2,\|.,\dots,.\|_2^H)$ be normed space in $\mathbb{m}$ -times. For fixed linearly independent $\mathbb{Y}:=\{\mathfrak{s}_1,\dots,\mathfrak{s}_{\mathbb{m}}\}\in l^2, the\ set\ \ \mathbb{f}_{\mathbb{Y}}$ is the multi-linear- functional in $\mathbb{m}$ -times explained in definition 1.18 .Then $\mathbb{f}_{\mathbb{Y}}$ is bounded on $(l^2,\|.,\dots,.\|_2^H)$ with $\|\mathbb{f}_{\mathbb{Y}}\|=\|\mathfrak{s}_1,\dots,\mathfrak{s}_{\mathbb{m}}\|_2^H$.

2.10 Proposition[7] : Suppose that $(\mathbb{X},\|.\|)$ is a space of norm for real numbers ,where $\mathbb{d}\ge\mathbb{m}$, $\mathbb{f}$ is a multi-linear- functional in $\mathbb{m}$-times which is bounded .Then $\exists\mathcal{w}_{\mathbb{f}}\in\mathcal{T}(\mathbb{X},\mathbb{X}^{\mathbb{m}-1}), s.t.\ for\ (\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}-1},\mathfrak{v})\in\mathbb{X}$.
$\mathbb{f}(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}-1},\mathfrak{v})=(\mathcal{w}_{\mathbb{f}}(\mathfrak{v}))(\mathfrak{v}_1,\dots,\mathfrak{v}_{\mathbb{m}-1})$. Moreover ,$\|\mathbb{f}\|_{\mathbb{m},1}=|\mathcal{w}_{\mathbb{f}}|_{op}$.

2.11 Theorem [7]: Suppose that $\mathbb{X}$ is a space of norm for real numbers ,where $\mathbb{d}\ge\mathbb{m}$ .Then the $(\mathbb{X},\|.\|)$ space of dual in $\mathbb{m}$ – times is $\mathcal{T}(\mathbb{X},\mathbb{X}^{\mathbb{m}-1})$.

2.12 Theorem [7]: Suppose that $\mathbb{X}$ is a space of norm for real numbers ,where $\mathbb{d}\ge\mathbb{m}$ .Then the $(\mathbb{X},\|.\|)$ space of dual in $\mathbb{m}$-times is – space of Banach.

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International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 12, December - 2022, Pages: 97-101

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