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Some Variants of Regular Closure Spaces and the Continuity on Regularity Space

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Abstract— *New generalizations of regularity space in closure space such as regular space, semi-regular space and semi generalized-regular space will be introduced, and we will denote briefly by (scl -regular space) for semi-regular space and (sgcl -regular space) for semi generalized-regular space. Also, this paper describes several notions of its basic properties. In addition, the concept of the continuity on regularity is studied by using of scl -regular space and prove some important propositions and theorems.*

Keywords: Closure space, sg^{cl}-closed set, g^{cl}-closed set, regular space, s^{cl}-regular space and sg^{cl}-regular space.

1. Introduction

Topological space of generalized closed sets was first analyzed by N. Levin's [1] in 1970. These types of sets were introduced to expand some of the main properties of closed sets to a wider family. In addition, identified the notion of (g^{cl}-closed set) and (sg^{cl} -closed set) by using s^{cl} -open set see [1], [2]. This work aims to examining the description of some variants of regular closure spaces and the continuity on regularity space. In 1966 E. Cech [3], introducing and shedding some Light on how they were examined by different mathematicians see [4], [5], [6]. Some notion and essential facts will be listed as we will need them for Identifying the properties of s^{cl}-regular space and sg^{cl}-regular space and to compare them with the properties of the closed sets. At the end, the continuity on regularity space will be studied in closure space.

2. Preliminaries

In this paper, we will represent for closure spaces by (V, Cl) and (U, Cl*).

Definition 2.1[7] Suppose Cl: P(V) → P(V) be a function defined on the power set P (V) of a non-empty set V satisfying the following axioms, Cl will be the closure operator over V and pair a (V, Cl) is called the closure space:

- Cl (∅) = ∅,
- N⊆ Cl (N) for each N ⊆ V,
- N ⊆ M ⇒ Cl (N) ⊆ Cl (M) for each N, M ⊆ V.

Definition 2.2 [7] A closure operator Cl upon a set V is called idempotent if Cl (N) = Cl Cl (N), and we say Cl called additive when N, M are subset of V, then Cl (N) ∪ Cl (M)= Cl (N ∪ M).

Definition 2.3 [7] An interior operator Int^{op} is a function from power set of V to itself, for all N ⊆ V, Int^{op}N = V \ Cl (V \N). A set Int^{op} N is named the interior of N in closure space (V, Cl).

Definition 2.4 [7] Let a closure space (V, Cl), a subset N of V is

- Closed, if N = Cl (N).
- Open, if N = Int^{op} (N).
- Semi open set (briefly, s^{cl}-open), if N ⊆ Cl Int^{op} (N) and semi closed set (briefly, s^{cl}-closed), if Cl (N) ⊆ N, or N is said to be semi closed set, if complement of N is semi open set in (V, Cl).

Definition 2.5 [7] A closure space (U, Cl*) is called a subspace of (V, Cl), if U ⊆ V and Cl*(N) = Cl (N) ∩ U for every subset N ⊆ U.

Definition 2.6 [7] A closure space (U, Cl*) is called closed subspace of (V, Cl), if F is a closed subset of (U, Cl*), then F is a closed subset of (V, Cl).

Definition 2.7 The subset F of closure space (V, Cl) is called a semi generalized closed set in the closure space (briefly, sg^{cl}-closed set), if s^{cl}-closed (F) ⊆ D while F ⊆ D and D is s^{cl}-open set on V.

Definition 2.8 The subset F of closure space (V, Cl) is called a generalized closed set in the closure space (briefly, g^{cl}-closed set), if Cl (F) ⊆ D while F ⊆ D and D is open set on V.

Definition 2.9 [8] A function $\Re$: (V, Cl) → (V, Cl*) is called continuous, if $\Re$ (Cl (X)) ⊆ Cl* ($\Re$ (X)) for every X ⊆ V, where (V, Cl) and (V, Cl*) are closure spaces.

Definition 2.10 [8] Let (V, Cl) and (U, Cl*) are closure spaces. A function $\Re$: (V, Cl) →(U, Cl*) is called:

- Closed function if for every closed subset N of V, then $\Re$ (N) is closed subset on U.
- Open function if for every open subset N of V, then $\Re$ (N) is an open subset on U.
- Semi closed function if for every closed subset N of V, then $\Re$ (N) is semi -closed subset on U.
- Semi open function if for every open subset N of V, then $\Re$ (N) is semi -open subset on U.

Definition 2.11 [8] The subset F of closure space (V, Cl) is called a semi clopen set its semi open and semi closed at the same time.

Definition 2.12 [9] The subset F of closure space (V, Cl) is called a regular-closed set (briefly, r^{cl} -closed set), if F = Cl Int°p(F).

Definition 2.13 [9] The subset N of closure space (V, Cl) is called a regular-open set (briefly, r^{cl}- open), if N= Int°p Cl (N). The complement of r^{cl}-closed set is called r^{cl}-open set in (V, Cl).

Example 2.14 Let V = {v_1, v_2, v_3, v_4} and define a closure operator Cl on V by Cl (∅) = ∅, Cl {v_1, v_2} = Cl {v_1, v_2, v_4} = {v_1, v_2, v_4}, Cl{v_3} = Cl {v_3, v_4} = {v_3, v_4}, Cl (N) = Cl (V) = V for any N ⊆ V. Clearly, {v_1, v_2, v_4} and {v_3, v_4} are r^{cl}-closed sets. Also, {v_1, v_2} and {v_3} are r^{cl}-open sets.

Definition 2.15 Let the closure space (V, Cl), a subset N of V is:

- Semi regular set (briefly, s^{cl}-regular set), if N is a semi open set and semi closed set in (V, Cl).
- Regular semi generalized closed (briefly, rsg^{cl}-closed set), if Cl (N) ⊂ X such that N ⊂ X for every semi regular set X in (V, Cl).
- Regular semi generalized open set (briefly, rsg^{cl} -open set), if V \ N is rsg^{cl} -closed set.

Definition 2.16 A closure space (V, Cl) is a semi $T_{1\backslash 2}$ -space (briefly, s^{cl}- $T_{1\backslash 2}$ space) if, and only if, every singleton set is either semi-open set or semi- closed set in closure space (V, Cl).

3. Some Variants Of Regular Closure Spaces

In this section, we discuss the basic properties of some variants of regular closure spaces.

Definition 3.1 [7] A closure space (V, Cl) is said to be regular space, if for a closed set Cl (X) = X and a point a ∉ Cl (X), there exist disjoint open sets U and O such that a ∈ U and Cl (X) ⊆ O.

Definition 3.2 A closure space (V, Cl) is said to be semi regular (briefly, s^{cl}-regular space), if for each closed set F and any point x ∈ V \ F, there exist disjoint semi open sets U and D in (V, Cl) such that x ∈ U and F ⊂ D.

Definition 3.3 A closure space (V, Cl) is said to be semi generalized regular space (briefly, sg^{cl}-regular space), if for each sg^{cl}-closed set F and any point x ∈ V \ F, there exist disjoint semi open sets U and D in (V, Cl) such that x ∈ U and F ⊂ D.

Example 3.4 Suppose V = {v_1, v_2, v_3} and define a closure operator Cl on V by Cl (∅) = ∅, Cl{v_1} = {v_1}, Cl {v_2, v_3} = {v_2, v_3}, Cl (N) = Cl (V) = V for any N ⊆ V, such that{v_1} and {v_2, v_3} are disjoint open sets. Also, {v_1} and {v_2, v_3} are disjoint closed sets. Hence, (V, Cl) is a regular closure space.

Definition 3.5 A closure space (V, Cl) is said to be regular semi generalized - regular space (briefly, rsg^{cl}- regular space), if for every rsg^{cl} -closed set F and any point x ∈ V \ F, there exist disjoint open sets U and D in (V, Cl) such that x ∈ U and F ⊂ D.

4. Properties Of Regular Space, Semi-Regular Space And Semi Generalized-Regular Space In Closure Spaces

In this section, we introduce some basic properties of (s^{cl} -regular) and (sg^{cl} -regular) in the closure space.

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Remark 4.1 Every *rsg*^{cl}-regular closure space is *s*^{cl}-regular closure space, but the converse is not true in general, as seen in the following example.

Example 4.2 Let V = {v_1, v_2, v_3, v_4} and define a closure operator Cl on V by Cl(∅) = ∅, Cl {v_1} = Cl {v_1, v_2} = {v_1, v_2, v_3}, Cl {v_4} = {v_2, v_3, v_4}, Cl (N) = Cl (V) = V for any N ⊆ V. {v_1, v_2, v_3} and {v_3, v_4} are semi open sets. Also, {v_1, v_2} and {v_4} are semi open sets. Then, a closure space (V, Cl) is said a *s*^{cl}-regular space. But a closure space (V, Cl) is not *rsg*^{cl} -regular space.

Remark 4.3 Every *rsg*^{cl}-regular closure space is regular closure space. The converse is not always true, as it can be seen in the following example.

Example 4.4 From the (Example 3.4). So, (V, Cl) is said to be regular space. But a closure space (V, Cl) is not *rsg*^{cl}-regular space.

Lemma 4.5 A closure space (V, Cl) is *sg*^{cl} -regular space if, and only if, (V, Cl) is *s*^{cl}-regular space and *s*^{cl} $T_{1\backslash 2}$ -space.
Proof. Suppose that (V, Cl) is *sg*^{cl} - regular space. Then, clearly (V, Cl) is *s*^{cl} -regular space. Now, let F be *sg*^{cl}-closed set in (V, Cl). For each x ∉ F, there exists a *s*^{cl} -open set G in (V, Cl) containing x such that G ∩ F = ∅ . If G = ∪{G_x: x ∈ V \ F}. Then, G is *s*^{cl}-open set and G = V \ F. Hence, F is *s*^{cl} -closed set. Then, (V, Cl) is *s*^{cl} $T_{1\backslash 2}$ -space.
Conversely, it is clear.♦

Theorem 4.6 For a closure space (V, Cl), the following are equivalent:

- (V, Cl) is *rsg*^{cl} -regular.
- For every *rsg*^{cl} -open set U containing x ∈ V, there exists an open set D in (V, Cl) such that x ∈ G ⊂ Cl (G) ⊂ U.

Proof. 1) ⇒ 2) Let U be any *rsg*^{cl} -open set containing x ∈ V. Then, x ∉ V \ U, where V \ U is *rsg*^{cl} -closed set in (V, Cl). Hence, there exist disjoint open sets G and H such that x ∈ G and V \ U ⊂ H or x ∈ G ⊂ Cl (G) ⊂ V \ H ⊂ U.
2) ⇒ 1) Let F be a *rsg*^{cl} -closed set and a point x ∈ V \ F. So, there exists an open set G in (V, Cl) such that x ∈ G ⊂ Cl (G) ⊂ V \ F or x ∈ G and F ⊂V \ Cl (G) where G ∩ (V \ Cl (G)) =∅. This proves that (V, Cl) is *rsg*^{cl} - regular.♦

Lemma 4.7 Let a closure space (V, Cl) is *rsg*^{cl}-regular space, then every *rsg*^{cl}-open set is the union of open sets.
Proof. Let U be a *rsg*^{cl} -open subset of *rsg*^{cl}-regular closure space such that x ∈ U. If F = V \U.
Then, F is a *rsg*^{cl}-closed set and x ∈ V \ F. So, there exist disjoint open sets G and H of (V, Cl) such that x ∈ G and F ⊂ H. It follows that x ∈ G ⊂ U. ♦

Corollary 4.8 Let a closure space (V, Cl) is *rsg*^{cl} -regular space, then every *rsg*^{cl} -closed set is the intersection of closed sets.
Proof: The proof is according to the above Lemma (4.7), by taking the complement.♦

Theorem 4.9 For a closure space (V, Cl), the following are equivalent:

- (V, Cl) is *sg*^{cl}-regular space.
- Every *sg*^{cl} -open set U is a union of semi clopen sets.
- Every *sg*^{cl} -closed set F is an intersection of semi clopen sets.

Proof. 1) ⇒ 2): Let U be *sg*^{cl} -open set in (V, Cl). Let x ∈ U. If F = V \ U. Then, F is *sg*^{cl} -closed. It is assumed that there are disjoint semi open subsets O and D of (V, Cl) such that x ∈ O and F ⊆ D. If G = Cl (O). Then, G is a semi clopen and G ∩ F ⊆ G ∩ D = ∅. It follows that x ∈ G ⊆ U. Thus, U is a union of semi clopen sets.
2) ⇒ 3): This is obvious. 3) ⇒1): Let F is *sg*^{cl} -closed of (V, Cl) and x ∉ F. Let G a semi clopen set such that F ⊂ G and x ∉ G. If U = V \ G. Then, U is a semi open set containing x and U ∩ G = ∅. Thus, (V, Cl) *sg*^{cl}- regular space♦

5. The Continuity On Regular Closure Space

In this section, we give sufficient conditions on $\Re$:(V, Cl) → (U, Cl*) so that $\Re$ preserved *s*^{cl} -normality space.

Definition 5.1 Let $\Re$: (V, Cl)→ (U, Cl*) be a function, (V, Cl) and (U, Cl*) are two closure spaces. Then, $\Re$ is called *g*^{cl}–closed function if $\Re$ (N) is *g*^{cl}-closed of U for every closed subset N of V.

Example 5.2 Let (V, Cl) and (V, Cl*) are two closure spaces where V = {v_1, v_2} by define a closure operator Cl on V by Cl(∅) = ∅, Cl {v_1} = {v_1}, Cl {v_2} = Cl (V) = V. U = {u_1, u_2, u_3}, then identified a closure operator Cl* on U by Cl* (∅) = ∅, Cl*{u_1} = {u_1, u_2}, Cl*{v_3} = {u_3}, Cl* (M) = Cl* {U} = {U} for any M ⊆ U. Let $\Re$: (V, Cl) → (U, Cl*) is defined by $\Re$ {v_1} ={u_1}. Since {u_1} is *g*^{cl}-closed set of U. Then, $\Re$ is called *g*^{cl}–closed function.

Definition 5.3 Let $\Re$: (V, Cl) → (U, Cl*) be a function and (V, Cl), (U, Cl*) are two closure spaces. Then, $\Re$ is called *sg*^{cl}–closed function, if $\Re$ (N) is *sg*^{cl} - closed of U for every closed subset N of V.

Theorem 5.4 Let (V, Cl) and (U, Cl*) are two closure spaces. A function $\Re$: (V, Cl) → (U, Cl*) is *sg*^{cl} -closed if, and only if, for each subset B of U and for each open set D containing $\Re^{-1}$(B), there exists a *sg*^{cl} -open set O of U containing B and $\Re^{-1}$(O) ⊂ D.
Proof. Let B be subset of (U, Cl*) and D be an open set of (V, Cl), such that $\Re^{-1}$(B) ⊂ D.
Then, U \\$\Re$ (V \ D) = O is a *sg*^{cl} -open set containing B, such that $\Re^{-1}$ (O) ⊂ D.
Conversely, Let F be a closed set of V. So, $\Re$ (F) is *sg*^{cl}-closed set in U. Then, $\Re^{-1}$(U \ $\Re$ (F)) ⊂ V \ F. By taking B = U \ $\Re$ (F) and D = V \ F. So, exists a *sg*^{cl}-open set O of U containing U \ $\Re$(F) and $\Re^{-1}$ (O) ⊂ V \ F. Then, we have F ⊂ V \ $\Re^{-1}$ (O) and U \ O = $\Re$ (F). Since U \ O is *sg*^{cl} - closed set in U, $\Re$ (F) is *sg*^{cl}-closed set in U. Thus, $\Re$ is *sg*^{cl}-closed function.♦

Theorem 5.5 Let (V, Cl) and (U, Cl*) are two closure spaces and $\Re$: (V, Cl) → (U, Cl*) be a continuous, *s*^{cl} -open and *sg*^{cl}-closed onto function from a regular space (V, Cl) to a space (U, Cl*), then (U, Cl*) is *s*^{cl} -regular space.
Proof. Let u ∈ U. So, O be an open set containing u in U, $\Re$ is onto, then there exists x ∈ V such that $\Re$ (x) = u. Now, $\Re^{-1}$(O) is an open set in (V, Cl) containing x. But (V, Cl) is regular space. Then, there exist an open set D such that, x ∈ D ⊆ Cl (D) ⊆ $\Re^{-1}$(O), u ∈ $\Re$ (D) ⊆ $\Re$ (Cl (D)) ⊆ O. But $\Re$ (Cl (D)) is *sg*^{cl} -closed set. Then, we have Cl ($\Re$ (Cl (D))) ⊆ O. Therefore, u ∈ $\Re$ (D) ⊆ Cl ($\Re$ (D)) ⊆ O and $\Re$ (D) is *s*^{cl}-open set in (U, Cl*). Because $\Re$ is *s*^{cl}-open function. Hence, (U, Cl*) is *s*^{cl}-regular space.♦

Corollary 5.6 Let (V, Cl) and (U, Cl*) are two closure spaces. If $\Re$: (V, Cl) → (U, Cl*) is a continuous, open and *sg*^{cl} -closed onto function from a regular space (V, Cl) to a space (U, Cl*), then (U, Cl*) is *s*^{cl}-regular space.

Proof. Since $\Re$ is *sg*^{cl} -closed function. Hence, we get that (U, Cl*) is *s*^{cl}-regular space (By depend on (Theorem 5.5)).♦

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REFERNCES

[1] N. Levine, ‘Generalized closed sets in topology’, *Rend. del Circ. Mat. di Palermo*, vol. 19, no. 1, pp. 89–96, 1970.

[2] J. Cao, M. Ganster, and I. Reilly, ‘On generalized closed sets’, *Topol. Appl.*, vol. 123, no. 1, pp. 37–46, 2002.

[3] E. ÄCech, ‘Topological Spaces (revised by Z. FrolÃ k, M. KatÄetov)’, *Acad. Prague*, 1966.

[4] A. Journal and O. F. Mathematics, ‘On regular semi generalized closed sets’, vol. 4, no. 2, pp. 49–56, 2010.

[5] Z. I. Salman, ‘Semi-Generalized Closed Mappings and Generalized-Semi Closed Mappings and Its Relationships with Semi-Normal and Semi-Regular Spaces’, *J. Al-Nahrain Univ. Sci.*, vol. 14, no. 3, pp. 125–128, 2011, doi: 10.22401/jnus.14.3.19.

[6] D. Mathematica, ‘M. Ganster, S. Jafari, G. B. Navalagi ON S E M I - g - R E G U L A R A N D S E M I - g - N O R M A L SPACES’, vol. XXXV, no. 2, 2002.

[7] S. Problems and I. N. Topology, ‘a Study of ~ Ech Closure Spaces’, no. July, 1994.

[8] J. Khampakdee and R. J. ŠLAPAL, ‘Semi open sets in closure spaces’. Ph. D. thesis, Bruno University, 2009.

[9] R. Gupta and A. K. Das, ‘Some Variants of Normal Čech Closure Spaces via Canonically Closed Sets’, *Mathematics*, vol. 9, no. 11, p. 1225, 2021.