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$\mathcal{G}$ −Common Fixed Points

May A.A. AL-Yaseen , Zainab Zuhair Knaish AL-Ghurabiu, Noor Razak Jafar Majed

Mathematics Department ,University of Babylon ,Hilla , Babylon, Iraq
E-mail: pure.may.alaa@uobabylon.edu.iq
E-mail:zainabzuhair1982@gmail.com
E-mail:noorrazak1993@gmail.com

Abstract *This a review's paper is about common fixed points in* $\mathcal{G}$ *−spaces. We see the conditions where the map has common fixed points in* $\mathcal{G}$ *−spaces.*

Keywords : common fixed point, unique common fixed point , $\mathcal{G}$ −space.

1. Introduction

Let $(\mathfrak{N}, \mathcal{G})$ be a metric space and let Ξ, Θ be two self mappings of $\mathfrak{N}$. let $\mathcal{G} : \mathfrak{N} \times \mathfrak{N} \times \mathfrak{N} \to R^+$ be a mapping with : $(\mathcal{G}(\varsigma, \xi, z) = 0$ if $\varsigma = \xi = \zeta$, & $0 < \mathcal{G}(\varsigma, \varsigma, \xi)$ for all $\varsigma, \xi \in \mathfrak{N}$ with $\varsigma \neq \xi$ & $\mathcal{G}(\varsigma, \varsigma, \xi) \leq \mathcal{G}(\varsigma, \xi, \zeta)$ for all $\varsigma, \xi, \zeta \in \mathfrak{N}$ with $z \neq \xi$ & $\mathcal{G}(\varsigma, \xi, \zeta) = \mathcal{G}(\varsigma, \zeta, \xi) = \mathcal{G}(\xi, \zeta, \varsigma)$ & $\mathcal{G}(\varsigma, \xi, \zeta) \leq \mathcal{G}(\varsigma, a, a) + \mathcal{G}(a, \xi, \zeta)$ for all $\varsigma, \xi, \zeta, a \in \mathfrak{N}$.

Mappings Ξ and Θ on a metric space $(\mathfrak{N}, \mathcal{G})$ are said to be compatible (CM.)if $\lim_{n\to\infty} \mathcal{G}(\Xi\Theta\varsigma_n, \Theta\Xi\varsigma_n) = 0$ Whenever ς_n is a sequence in $\mathfrak{N}$ such that $\lim_{n\to\infty} \Xi\varsigma_n = \lim_{n\to\infty} \Theta\varsigma_n = t$ for some $t \in \mathfrak{N}$. Let Ξ and Θ be two self mappings of a metric space $(\mathfrak{N}, \mathcal{G})$. We say that Ξ and Θ satisfy $(E{:}A)$ − property if there exists a sequence (ς_n) in $\mathfrak{N}$ such that l$\lim_{n\to\infty} \Xi\varsigma_n = \lim_{n\to\infty} \Theta\varsigma_n = t$ for some $t \in \mathfrak{N}$. Two self mappings Ξ and Θ on a metric space $(\mathfrak{N}, \mathcal{G})$ are said to be weakly CM. if they commute at coincidence points. CM. maps are weakly CM. but the converse is not true.

Let $(\mathfrak{N}, \preccurlyeq)$ be a partially ordered set and A, B be two nonempty subsets of $\mathfrak{N}$ with $\mathfrak{N} = A \cup B$. Let $\Xi, \Theta : \mathfrak{N} \to \mathfrak{N}$ be two mappings. Then the pair (Ξ, Θ) is said to be (A, B) −weakly increasing if $\Xi\varsigma \preccurlyeq \Theta\Xi\varsigma$ for all $\varsigma \in A$ and $\Theta\varsigma \preccurlyeq \Xi\Theta\varsigma$ for all $\varsigma \in B$. [3] [4]. [5].

2.Review

Proposition2.1 **[1] [5][3]**

Let $(\mathfrak{N}, \mathcal{G})$ be a complete $\mathcal{G}$ −metric space, and let Ξ, Θ be mappings from $\mathfrak{N}$ into itself have a common fixed point in $\mathfrak{N}$ if one of the following satisfying :

1. $\max\{\mathcal{G}(\Xi(\varsigma), \Theta(\Xi(\varsigma)), \Theta(\Xi(\varsigma))), \mathcal{G}(\Theta(\varsigma), \Xi(\Theta(\varsigma)), \Xi(\Theta(\varsigma)))\} \leq r \min\{\mathcal{G}(\varsigma, \Xi(\varsigma), \Xi(\varsigma)), \mathcal{G}(\varsigma, \Theta(\varsigma), \Theta(\varsigma))\}$ for every $\varsigma \in \mathfrak{N}$, where $0 \leq r < 1$ and that $\inf[\mathcal{G}(\varsigma, \xi, \xi) + \min\{\mathcal{G}(\varsigma, \Xi(\varsigma), \Xi(\varsigma)), \mathcal{G}(\varsigma, \Theta(\varsigma), \Theta(\varsigma))\} : \varsigma \in \mathfrak{N}] > 0$ for every $\xi \in \mathfrak{N}$ with y is not a common fixed point of Ξ and Θ.

2. $\max\{\mathcal{G}(\Xi(\varsigma), \Xi(\varsigma), \Theta(\Xi(\varsigma))), \mathcal{G}(\Theta(\varsigma), \Theta(\varsigma), \Xi(\Theta(\varsigma)))\}$
$\leq r \min\{\mathcal{G}(\varsigma, \varsigma, \Xi(\varsigma)), \mathcal{G}(\varsigma, \varsigma, \Theta(\varsigma))\}$ for every $\varsigma \in \mathfrak{N}$, where $0 \leq r < 1$ and that
$\inf[\mathcal{G}(\varsigma, \varsigma, \xi) + \min\{\mathcal{G}(\varsigma, \varsigma, \Xi(\varsigma)), \mathcal{G}(\varsigma, \varsigma, \Theta(\varsigma))\} : \varsigma \in \mathfrak{N}] > 0$ for every $\xi \in \mathfrak{N}$ with ξ is not a common fixed point of Ξ and Θ.

3. $\max\{\mathcal{G}(\Xi(\varsigma), \Theta(\Xi(\varsigma)), \Theta(\Xi(\varsigma))), \mathcal{G}(\Theta(\varsigma), \Xi(\Theta(\varsigma)), \Xi(\Theta(\varsigma)))\}$
$\leq r \min\{\mathcal{G}(\varsigma, \Xi(\varsigma), \Xi(\varsigma)), \mathcal{G}(\varsigma, \Theta(\varsigma), \Theta(\varsigma))\}$ for every $\varsigma \in \mathfrak{N}$, where $0 \leq r < 1$ and that $\inf[\mathcal{G}(\varsigma, \xi, \xi) + \min\{\mathcal{G}(\varsigma, \Xi(\varsigma), \Xi(\varsigma)), \mathcal{G}(\varsigma, \Theta(\varsigma), \Theta(\varsigma))\} : \varsigma \in \mathfrak{N}] > 0$
for every $\xi \in \mathfrak{N}$ with ξ is not a common fixed point of Ξ andΘ.
Or $\max\{\mathcal{G}(\Xi(\varsigma), \Xi(\varsigma), \Theta(\Xi(\varsigma))), \mathcal{G}(\Theta(\varsigma), \Theta(\varsigma), \Xi(\Theta(\varsigma)))\}$
$\leq r \min\{\mathcal{G}(\varsigma, \varsigma, \Xi(\varsigma)), \mathcal{G}(\varsigma, \varsigma, \Theta(\varsigma))\}$ for every $\varsigma \in \mathfrak{N}$, where $0 \leq r < 1$ and that
$\inf[\mathcal{G}(\varsigma, \varsigma, \xi) + \min\{\mathcal{G}(\varsigma, \varsigma, \Xi(\varsigma)), \mathcal{G}(\varsigma, \varsigma, \Theta(\varsigma))\} : \varsigma \in \mathfrak{N}] > 0$ for every $\xi \in \mathfrak{N}$ with ξ is not a common fixed point of Ξ and Θ.

4. $\mathcal{G}(\Theta\Xi\varsigma, \Xi\varsigma) \geq a\mathcal{G}(\Xi\varsigma, \varsigma)$ $\mathcal{G}(\Xi\Theta\varsigma, \Theta\varsigma) \geq b\mathcal{G}(\Theta\varsigma, \varsigma)$ for all ς in $\mathfrak{N}$, where $a, b > 1$. If either Ξ or Θ is continuous.
5. $\varpi(\mathcal{G}(\Theta\Xi\varsigma, \Xi\varsigma, \Xi\varsigma) \geq \mathcal{G}(\Xi\varsigma, \varsigma, \varsigma)$ and $\varpi(\mathcal{G}(\Xi\Theta\varsigma, \Theta\varsigma, \Theta\varsigma) \geq \mathcal{G}(\Xi\varsigma, \varsigma, \varsigma)$

for all $\varsigma \in \mathfrak{N}$, where $\varpi \in \vartheta$. If either Ξ or Θ is continuous.

6. *The pair (Ξ, Θ) is (A,B) −weakly increasing. $\mathfrak{N} = A \cup B$., & $\Xi(A) \subseteq B$ and $\Theta(B) \subseteq A$. &. There exist two altering distance functions ϑ and ψ such that $\vartheta\mathcal{G}(\Xi\varsigma, \Theta\Xi\varsigma, \Theta\xi) \leq \vartheta\mathcal{G}(\varsigma, \Xi\varsigma, \xi) - \psi\mathcal{G}(\varsigma, \Xi\varsigma, \xi)$ holds for all comparative elements $\varsigma, \xi \in \mathfrak{N}$ with$\varsigma \in A$ and $\xi \in B$ and $\vartheta\mathcal{G}(\Theta\varsigma, \Xi\Theta\varsigma, \Xi\xi) \leq \vartheta\mathcal{G}(\varsigma, \Theta\varsigma, \xi) - \psi\mathcal{G}(\varsigma, \Theta\varsigma, \xi)$ holds for all comparative elements $\varsigma, \xi \in \mathfrak{N}$ with $\varsigma \in B$ and $\xi \in A$ & Ξ or Θ is continuous.*

7. *The pair (Ξ, Θ) is (A,B) −weakly increasing. & $\Xi(A) \subseteq B$ and $\Theta(B) \subseteq A$.& There exists $r \in [0,1)$ such that $\mathcal{G}(\Xi\varsigma, \Theta\Xi\varsigma, \Theta\xi) \leq r\mathcal{G}(\varsigma, \Xi\varsigma, \xi)$holds for all comparative elements* $\varsigma, \xi \in \mathfrak{N}$ with $\varsigma \in A, \xi \in B$, *and* $\mathcal{G}(\Theta\varsigma, \Xi\Theta\varsigma, \Xi\xi) \leq r\,\mathcal{G}(\varsigma, \Theta\varsigma, \xi)$

holds for all comparative elements $\varsigma, \xi \in \mathfrak{N}$ with $\varsigma \in B, \xi \in A$. &. Ξ or Θ is continuous.

8. $\mathcal{G}(\Theta\Xi\varsigma, \Xi\varsigma, \Xi\varsigma) \geq a\mathcal{G}(\Xi\varsigma, \varsigma, \varsigma)$ & $\mathcal{G}(\Xi\Theta\varsigma, \Theta\varsigma, \Theta\varsigma) \geq b\mathcal{G}(\Theta\varsigma, \varsigma, \varsigma)$ for all ς in , where $a, b > 1$. If either Ξ or Θ is continuous.

Proposition 2.2 [4][1] [5]

Let $(\mathfrak{N}, \mathcal{G})$ be a $\mathcal{G}$ −metric space and suppose mappings $\Xi, \Theta : \mathfrak{N} \to \mathfrak{N}$ have a unique common fixed point If one satisfy one of the following conditions:

1. Ξ and Θ are $\mathcal{G}$ − weakly commuting of type $\mathcal{G}_{\Xi}$, & $\Xi(\mathfrak{N}) \subseteq \Theta(\mathfrak{N})$, & $\Theta(\mathfrak{N})$ is a $\mathcal{G}$ −complete subspace of $\mathfrak{N}$, & $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \vartheta(M(\varsigma, \xi, \zeta))$, for all $\varsigma, \xi, \zeta \in \mathfrak{N}$, where,
$$M(\varsigma, \xi, \zeta) = max\left\{\begin{matrix}\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Theta(\zeta)), \mathcal{G}(\Theta(\varsigma), \Xi(\xi), \Theta(\varsigma)), \mathcal{G}(\Theta(\xi), \Xi(\varsigma), \Theta(\xi)), \mathcal{G}(\Theta(\zeta), \Xi(\varsigma), \Theta(\zeta)), \\ \mathcal{G}(\Theta(\zeta), \Xi(\xi), \Theta(\zeta)), \mathcal{G}(\Theta(\xi), \Xi(\zeta), \Theta(\xi)), \mathcal{G}(\Theta(\varsigma), \Xi(\zeta), \Theta(\varsigma))\end{matrix}\right\}$$

If there exists $\varsigma_0 \in \mathfrak{N}$ such that $(O(\varsigma_0, \Xi, \infty)) < \infty$.

2. $\Xi(\mathfrak{N}) \subseteq \Theta(\mathfrak{N})$, & Ξ or Θ is continuous, & $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \alpha\mathcal{G}(\Xi(\varsigma), \Theta(\xi), \Theta(\zeta)) + \beta\mathcal{G}(\Theta(\varsigma), \Xi(\xi), \Theta(\zeta)) + \gamma\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Xi(\zeta))$, for every $\varsigma, \xi, \zeta \in \mathfrak{N}$ and $\alpha, \beta, \gamma \geq 0$with $0 \leq \alpha + 3\beta + 3\gamma < 1$. provided Ξ and Θ are CM. maps.

3. the pairs $\{\Xi, \Theta\}$ and $\{\Lambda, \mho\}$ are CM. mappings. If there exists $\varpi \in \vartheta$ such that the inequality $\varpi(\mathcal{G}(\Xi\varsigma, \Lambda\xi, a) \geq \mathcal{G}(\Theta\varsigma, \Lambda\xi, a)$ holds.
4. the pairs $\{\Xi, \Theta\}$ and $\{\Lambda, \Lambda\}$ are weakly CM. mappings. If there exists $\varpi \in \vartheta$ such that the inequality (2) holds.
5. the pairs $\{\Xi, \Theta\}$ and $\{\Lambda, \Lambda\}$ are weakly CM. mappings. Assume that there exists $h > 1$ such that $\mathcal{G}(\Xi\varsigma, \Lambda\xi, \Lambda\xi) \geq h\mathcal{G}(\Theta\varsigma, \Lambda\xi, \Lambda\xi)$ for all $\varsigma, \xi \in \mathfrak{N}$. Then Ξ, Λ, Θ and Λ.
6. If there exists $\varpi \in \vartheta$ such that the inequality $\varpi(\mathcal{G}(\Xi\varsigma, \boldsymbol{B}\xi, \Lambda\xi)) \geq \mathcal{G}(\varsigma, \xi, \xi)$ Holds.
7. Assume that there exists $h > 1$ s.t $\mathcal{G}(\Xi\varsigma, \Lambda\xi, \Lambda\xi) \geq h\mathcal{G}(\varsigma, \xi, \xi)\ \forall \varsigma, \xi \in \mathfrak{N}$ Then.

Proposition 2.3 [1]

Let $(\mathfrak{N}, \mathcal{G})$ be a complete $\mathcal{G}$ −metric space and Ξ, Θ be two CM. self mappings on $(\mathfrak{N}, \mathcal{G})$ have a unique common fixed point in $\mathfrak{N}$ If satisfies one of the following :

1. $\Xi(\mathfrak{N}) \subseteq \Theta(\mathfrak{N})$, & Ξ or Θ is continuous, &$\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq q\mathcal{G}(\varsigma, \xi, \zeta)$

for every $\varsigma, \xi, \zeta \in \mathfrak{N}$ and $0 < q < 1$.

2. $\Xi(\mathfrak{N}) \subseteq \Theta(\mathfrak{N})$, & $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \alpha\mathcal{G}(\Xi(\varsigma), \Theta(\xi), \Theta(\zeta)) + \beta\mathcal{G}(\Theta(\varsigma), \Xi(\xi), \Theta(\zeta)) + \gamma\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Xi(\zeta))$, and any one of the subspace $\Xi(\mathfrak{N})$ or $\Theta(\mathfrak{N})$ is complete.

Proposition 2.4 [1][2] [4]

Let $(\mathfrak{N}, \mathcal{G})$ be a complete $\mathcal{G}$ −metric space and Ξ, Θ be two self mappings on $(\mathfrak{N}, \mathcal{G})$ satisfy property (E.A.), Then Ξ and Θ have a unique common fixed point in $\mathfrak{N}$ *if* satisfying one of the following :

1. $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \alpha\mathcal{G}(\Xi(\varsigma), \Theta(\xi), \Theta(\zeta)) + \beta\mathcal{G}(\Theta(\varsigma), \Xi(\xi), \Theta(\zeta)) + \gamma\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Xi(\zeta))$, &$\Theta(\mathfrak{N})$ is a closed subspace of $\mathfrak{N}$, provided Ξ and Θ are weakly CM. self maps.

2. $\Theta(\mathfrak{N})$ is a closed subspace of $\mathfrak{N}$. &$\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq q\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Theta(\zeta))$

for every $\varsigma, \xi, \zeta \in \mathfrak{N}$ and $0 < q < 1$. provided Ξ and Θ are weakly CM. self maps.

3. $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \alpha\mathcal{G}(\Xi(\varsigma), \Theta(\xi), \Theta(\zeta)) + \beta\mathcal{G}(\Theta(\varsigma), \Xi(\xi), \Theta(\zeta)) + \gamma\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Xi(\zeta))$ for every $\varsigma, \xi, z \in \mathfrak{N}$ and $\alpha, \beta, \gamma \geq 0$with $\alpha + 3\beta + 3\gamma < 1$,

& $\Xi(\mathfrak{N})$ is a closed subspace of .

4. $\Theta(\mathfrak{N})$ is a closed subspace of $\mathfrak{N}$, & $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \vartheta(M(\varsigma, \xi, \zeta))$, where,

$$M(\varsigma, \xi, \zeta) = \max\left\{\begin{matrix}\mathcal{G}\big(\Theta(\varsigma), \Xi(\xi), \Xi(\xi)\big), \mathcal{G}\big(\Theta(\varsigma), \Xi(\zeta), \Xi(\zeta)\big), \mathcal{G}\big(\Theta(\xi), \Xi(\varsigma), \Xi(\varsigma)\big), \mathcal{G}\big(\Theta(\zeta), \Xi(\varsigma), \Xi(\varsigma)\big), \\ \mathcal{G}(\Theta(\zeta), \Xi(\xi), \Xi(\xi)), \mathcal{G}(\Theta(\xi), \Xi(\zeta), \Xi(\zeta))\end{matrix}\right\}$$

for all $\varsigma, y, \zeta \in \mathfrak{N}$.

5. $\Theta(\mathfrak{N})$ is a closed subspace of $\mathfrak{N}$, & $\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta))$
$$\leq \vartheta\,(\max\{\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Theta(\zeta)), \mathcal{G}(\Theta(\varsigma), \Xi(\varsigma), \Theta(\zeta)), \mathcal{G}(\Theta(\zeta), \Xi(\zeta), \Theta(\zeta)), \mathcal{G}(\Theta(\xi), \Xi(\xi), \Theta(\zeta))\})$$

for all $\varsigma, \xi, \zeta \in \mathfrak{N}$..

6. $\Theta(\mathfrak{N})$ is a closed subspace of $\mathfrak{N}$, & there exist nonnegative real constants α and β with $0 \leq \alpha + 2\beta < 1$ such that for all $\varsigma, \xi, \zeta \in \mathfrak{N}$,

$$\mathcal{G}(\Xi(\varsigma), \Xi(\xi), \Xi(\zeta)) \leq \alpha\mathcal{G}(\Theta(\varsigma), \Theta(\xi), \Theta(\zeta)) + \beta\mathcal{G}(\Theta(\xi), \Xi(\xi), \Xi(\xi)) + \mathcal{G}(\Theta(\zeta), \Xi(\zeta), \Xi(\zeta)) + \mathcal{G}(\Theta(\varsigma), \Xi(\varsigma), \Xi(\varsigma))$$

References

[1] *S. C. Binayak, K. Sanjay, Asha and D. Krishnapada*, 2012, Some fixed point theorems in $\mathcal{G}$ -metric spaces, Mathematical Sciences Letters . *An International Journal*, 1, No. 1, 25-31.

[2] S.K.Mohanta and M. Srikanta, 2012, A Common Fixed Point Theorem in $\mathcal{G}$ -Metric Spaces, CUBO A Mathematical Journal, Vol.14, No. 03, (85–101).

[3] W. Shatanawi and K. Abodayeh, 2020, Some Fixed and Common Fixed Point Results in $\mathcal{G}$ -Metric Spaces
Which Can't be Obtained From Metric Spaces, Bol. Soc. Paran. Mat., 38 , 6 ,43–51.

[4] V. Popa and A. M. Patriciu, 2015, On Some Common Fixed Point Theorems in $\mathcal{G}$ -Metric Spaces Using $\mathcal{G}$ -(E.A)-Property, Palestine Journal of Mathematics, 4, 65-72.

[5] S. Priya, K. Jatinderdeep and S. S. Bhatia, 2014, Common Fixed Points of Expansive Mappings in Generalized Metric Spaces, Journal of Applied Research and Technology, 12, 607-614.