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Derivation of a complex method for calculating the triple integrals with continuous integrals numerically

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Abstract *The main objective of this research is to derive a rule for numerically calculating the values of triple integrals with continuous integrals in the region of integration using the trapezoidal rule on the two dimensions y, x, the midpoint rule on the dimension and how to find the correction limits for it (error formula) and improve these results using the Romberg acceleration method Through the correction limits that we found, when the number of partial periods into which the integration period is divided on the inner dimension x is equal to twice the number of partial periods into which the integration period is divided on the middle dimension Y and on the outer dimension Z. We will symbolize this method with the symbol rttm and it is reliable because it has given high accuracy through the integrations we reviewed in the results compared to the analytical values of the integrals with a few partial periods.*

Keywords—Numerical integration 65D30 ; double integers 32A55 ; Romberg Accelerating 65B99 ; Taylor series 30K05;

1. Introduction

The importance of the subject of numerical analysis lies in devising certain methods that contribute to finding approximate solutions to problems in mathematics, including the integrals that constitute an important part of this topic, as this importance is more evident in the practical applications practiced by engineers and physicists, and finding the approximate value of the integration came as a result:

1. The impossibility of finding the analytical value of the integration.

2. When the process of finding the analytical value of the integration is possible, but with difficulty and requires a long time.

3. The analytic integration value may be approximate mainly because it contains terms that take their values from tables (such as the logarithm or the inverse tangent).

4. The problem may be to find an area under a curve defined by a table of values (that is, the function is defined at a few points in the integration period), as is the case when analyzing the results of experiments.

The process of finding a numerical value for the triple integration constitutes a more complex issue than the problem of finding the value of the unitary and binary integration, since the integrator here depends on three variables, and the issue of continuity or defectiveness in the integrator or impairment in the partial derivatives of the integrator poses great difficulties, as well as here we will deal with regions of integration (regions) or surfaces (Surfaces) and not with integration intervals as in the case of unary integration.

Therefore, finding the values of integrals of this kind is not an easy matter for some cases. Therefore, there has become an urgent need to find approximate values for these integrals. The importance of triple integrals lies in finding volumes, average centers, and inertial inertia of volumes, which prompted many researchers to work in the field of triple integrals and researchers who shed light on the calculation of integrals. Continuous integrals of the form $f(x,y,z)=f_1(x)f_2(y)f_3(z)$

Hilal [1], Hassan [2], Muhammad [3] and others.

In this paper, we present a theorem with proof to derive a new rule for calculating approximate values of triple integrals whose integrals are continuous functions and the error formula for them. This rule results from the application of the middle point rule to the dimension z the base of the trapezoid on the middle two dimensions y inner dimension x when (n is the number of sub-intervals into which the integral period breaks down on the internal dimension $[a,b]$ and n_1 the number of partial periods into which the integration period is broken into on the middle dimension $[c,d]$ and n_2 the number of partial periods into which the integration period is divided on the outer dimension $[e,g]$ from the partial periods and take a special case n = 2n1 and we will symbolize this method with the symbol, and in order to improve the results we use the method of accelerating Ro Mubarak, then we symbolize this rule with the symbol and we have obtained good results in terms of accuracy and speed of approach and with a relatively small number of partial periods.

2. Calculating numerically continuous triple integrals

Theorem:- Let the $f(x,y,z)$ function be continuous and differentiable at every point in the region $[a,b]\times[c,d]\times[e,g]$ the approximate value of the triple integral $I=\int_e^g\int_c^d\int_a^b f(x,y,z)dxdydz$ can be calculated from the following rule:

$$ttm=\frac{h_1h_2h_3}{4}\sum_{i=0}^{n-1}[f(x_i+\frac{h_1}{2},c,e)+f(x_i+\frac{h_1}{2},c,g)+f(x_i+\frac{h_1}{2},d,e)+f(x_i+\frac{h_1}{2},d,g)+$$
$$+2\sum_{k=1}^{n_1-1}(f(x_i+\frac{h_1}{2},c,z_k)+f(x_i+\frac{h_1}{2},d,z_k)+2\sum_{i=0}^{n_1-1}\sum_{j=1}^{n_2-1}(f(x_i+\frac{h_1}{2},y_j,e)+f(x_i+\frac{h_1}{2},y_j,g)$$
$$+\sum_{k=1}^{n-1}f(x_i+\frac{h_1}{2},y_j,z_k))]$$

the error formula is

$$E_{ttm}(h_1,h_2,h_3)=I-ttm(h_1,h_2,h_3)=A_{ttm}h_1^{\,2}+A_{ttm}h_2^{\,2}+A_{ttm}h_3^{\,2}+$$
$$B_{ttm}h_1^{\,4}+B_{ttm}h_2^{\,4}+B_{ttm}h_3^{\,4}+C_{ttm}h_1^{\,6}+C_{ttm}h_2^{\,6}\cdots$$

where $A_{ttm},B_{ttm},C_{ttm},\ldots$ Constants that depend on the partial derivatives of a function f

Proof: for a monointegral $\int_a^b f(x,y,z)dx$ It can be calculated numerically using the base of the trapezoid over the dimension χ and (dealing with x and y as constants) and its value:

$$t=\int_a^b f(x,y,z)dx=\frac{h_1}{2}\left[f(a,y,z)+f(b,y,z)+2\sum_{i=1}^{n-1}f(x_i,y,z)]\right]+$$
$$\frac{(b-a)}{-12}h_1^{\,2}\frac{\partial^2 f(\mu_1,y,z)}{\partial x^2}+\frac{(b-a)}{720}h_1^{\,4}\frac{\partial^4 f(\mu_2,y,z)}{\partial x^4}-\frac{(b-a)}{30240}h_1^{\,6}\frac{\partial^6 f(\mu_3,y,z)}{\partial x^6}+\cdots$$
$$\mu_1,\mu_2,\ \ldots\in(a,b)\quad,\qquad,\quad i=1,2,\ldots,n\ ,\ xi=a+ih_1$$

By integrating the above formula numerically over the period $[c,d]$ using the trapezoidal rule over the dimension, we get:

$$+\int_c^d\left[\frac{(b-a)}{-12}h_1^{\,2}\frac{\partial^2 f(\mu_1,y,z)}{\partial x^2}+\frac{(b-a)}{720}h_1^{\,4}\frac{\partial^4 f(\mu_2,y,z)}{\partial x^4}-\frac{(b-a)}{30240}h_1^{\,6}\frac{\partial^6 f(\mu_3,y,z)}{\partial x^6}+\cdots\right]dy$$

$$\frac{h_2}{2}\left[\frac{-(d-c)}{180}h_2^{\,4}\frac{\partial^4 f(a,\xi_1,z)}{\partial y^4}+\frac{(d-c)}{1512}h_2^{\,6}\frac{\partial^6 f(a,\xi_2,z)}{\partial y^6}+\cdots\right.$$
$$-\frac{(d-c)}{180}h2^4\frac{\partial^4 f(b,\xi_1,z)}{\partial y^4}+\frac{(d-c)}{1512}h_2^{\,6}\frac{\partial^6 f(b,\xi_2,z)}{\partial y^6}+\ldots$$

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$$+2\sum_{i=1}^{n_1-1}\left(\frac{-(d-c)}{180}h_2^{\ 4}\frac{\partial^4 f(x_{2i},\xi_{1i},z)}{\partial y^4}+\frac{(d-c)}{1512}h_2^{\ 6}\frac{\partial^6 f(x_{2i},\xi_{2i},z)}{\partial y^6}+\cdots\right)\Bigg]+$$

$$\frac{h_2}{2}\left[\frac{-(d-c)}{180}h_2^{\ 4}\frac{\partial^4 f(a,\xi_1,z)}{\partial y^4}+\frac{(d-c)}{1512}h_2^{\ 6}\frac{\partial^6 f(a,\xi_2,z)}{\partial y^6}+\cdots\right.$$

$$-\frac{(d-c)}{180}h_2^{\ 4}\frac{\partial^4 f(b,\xi_1,z)}{\partial y^4}+\frac{(d-c)}{1512}h_2^{\ 6}\frac{\partial^6 f(b,\xi_2,z)}{\partial y^6}+\ldots$$

And by integrating formula (7) numerically over the period $[e,g]$ using the centroid rule over the dimension z, we get

$$+\int_e^g\int_c^d\left[\frac{(b-a)}{-12}h_1^{\ 2}\frac{\partial^2 f(\mu_1,y,z)}{\partial x^2}+\frac{(b-a)}{720}h_1^{\ 4}\frac{\partial^4 f(\mu_2,y,z)}{\partial x^4}-\frac{(b-a)}{30240}h_1^{\ 6}\frac{\partial^6 f(\mu_3,y,z)}{\partial x^6}+\cdots\right]dydz$$

$$+\frac{h_2}{2}\int_e^g\left[\frac{-(d-c)}{12}h_2^{\ 2}\frac{\partial^2 f(a,\xi_1,z)}{\partial y^2}+\frac{(d-c)}{720}h_2^{\ 4}\frac{\partial^4 f(a,\xi_2,z)}{\partial y^4}+\cdots\right.$$

$$-\frac{(d-c)}{12}h_2^{\ 2}\frac{\partial^2 f(b,\xi_1,z)}{\partial y^2}+\frac{(d-c)}{720}h_2^{\ 4}\frac{\partial^4 f(b,\xi_2,z)}{\partial y^4}+\ldots$$

$$+2\sum_{i=1}^{n_1-1}\left(\frac{-(d-c)}{12}h_2^{\ 2}\frac{\partial^2 f(x_{2i},\xi_{1i},z)}{\partial y^2}+\frac{(d-c)}{720}h_2^{\ 2}\frac{\partial^2 f(x_{2i},\xi_{2i},z)}{\partial y^2}+\cdots\right)\Bigg]dz$$

$$+\frac{h_1h_2}{6}\left[\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(a,c,\lambda_1)}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(a,c,\lambda_2)}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(a,c,\lambda_3)}{\partial z^6}-\cdots\right.$$

$$+\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(a,d,\lambda_1)}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(a,d,\lambda_2)}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(a,d,\lambda_3)}{\partial z^6}-\cdots$$

$$+\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(b,c,\lambda_1)}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(b,c,\lambda_2)}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(b,c,\lambda_3)}{\partial z^6}-\cdots$$

$$+\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(b,d,\lambda_1)}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(b,d,\lambda_2)}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(b,d,\lambda_3)}{\partial z^6}-\cdots$$

$$+2\sum_{j=1}^{n_2-1}\left[\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(a,y_{2j},\lambda_{1j})}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(a,y_{2j},\lambda_{2j})}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(a,y_{2j},\lambda_{3j})}{\partial z^6}-\cdots\right.$$

$$\left.+\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(b,y_{2j},\lambda_{1j})}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(b,y_{2j},\lambda_{2j})}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(b,y_{2j},\lambda_{3j})}{\partial z^6}-\cdots\right]$$

$$+2\sum_{i=1}^{n_2-1}\left[\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(x_{2i},c,\lambda_{1i})}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(x_{2i},c,\lambda_{2i})}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(x_{2i},c,\lambda_{3i})}{\partial z^6}-\cdots\right.$$

$$+\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(x_{2i},d,\lambda_{1i})}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(x_{2i},d,\lambda_{2i})}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(x_{2i},d,\lambda_{3i})}{\partial z^6}-\cdots$$

$$+2\sum_{j=1}^{n_2-1}\left(\frac{(g-e)}{6}h_3^{\ 2}\frac{\partial^2 f(x_{(2i)},y_{(2j)},\lambda_{1ij})}{\partial z^2}-\frac{7(g-e)}{360}h_3^{\ 4}\frac{\partial^4 f(x_{(2i)},y_{(2j)},\lambda_{2ij})}{\partial z^4}+\frac{31(g-e)}{15120}h_3^{\ 6}\frac{\partial^6 f(x_{(2i)},y_{(2j)},\lambda_{3ij})}{\partial z^6}-\cdots\right)\Bigg]\Bigg]$$

By applying the two theorems (the middle value theorem and the mean value theorem in integration),

we get

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$$E_{ttm}(h)=(g-e)(d-c)(b-a)\left(\frac{-h_1^2}{12}\frac{\partial^2 f\left(\overline{\overline{\overline{n_1}}},\overline{\overline{\overline{\mu_1}}},\overline{\overline{\overline{\kappa_1}}}\right)}{\partial x^2}+\frac{h_1^4}{720}\frac{\partial^4 f\left(\overline{\overline{\overline{n_2}}},\overline{\overline{\overline{\mu_2}}},\overline{\overline{\overline{\kappa_2}}}\right)}{\partial x^4}-\cdots\right)+$$

$$+\frac{\mathrm{h}_2}{2}(g-e)(d-c)\left(\frac{-h_1^2}{12}\frac{\partial^2 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial y^2}+\frac{h_2^4}{720}\frac{\partial^4 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial y^4}-\cdots\right.$$

$$\left.\frac{-h_1^2}{12}\frac{\partial^2 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial y^2}+\frac{h_1^4}{720}\frac{\partial^4 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial y^4}-\cdots+2n_2-2\left(\frac{-h_2^2}{12}\frac{\partial^4 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial y^2}+\frac{h_2^4}{720}\frac{\partial^4 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial y^4}-\right)\right)$$

$$+\frac{h_1h_2}{4}(g-e)\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right.$$

$$+\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots$$

$$+\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots$$

$$+\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots$$

$$+4n_1\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right.$$

$$\left.+\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right)$$

$$+(2n_2-2)\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right.$$

$$\left.+\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right)$$

$$+(2n_2-2)\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right.$$

$$\left.+\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right)$$

$$+(8n_1^2-8n_1)\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right.$$

$$+(4n_1^2-8n_1+4)\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right.$$

adding the above equations, we get

Since the $\frac{\partial^2 f}{\partial x^2},\frac{\partial^4 f}{\partial x^4},\ldots$, $\frac{\partial^4 f}{\partial y^4},\frac{\partial^6 f}{\partial y^6},\ldots$, $\frac{\partial^2 f}{\partial z^2},\frac{\partial^4 f}{\partial z^4},\ldots$ Continuous at every point in the area $[a,b]\times[c,d]\times[e,g]$ the formula for correction limits for a triple integral I base ttm becomes:

$$E_{ttm}(h)=(g-e)(d-c)(b-a)\left(\frac{-h_1^2}{12}\frac{\partial^2 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial x^2}+\frac{h_1^4}{720}\frac{\partial^4 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial x^4}-\frac{h_1^6}{30240}\frac{\partial^6 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial x^6}\cdots\right)$$

$$+(g-e)(d-c)(b-a)\left(\frac{-h_2^2}{12}\frac{\partial^2 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial y^2}+\frac{h_2^4}{720}\frac{\partial^4 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial y^4}-\cdots\right)$$

$$+(g-e)(d-c)(b-a)\left(\frac{h_3^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h_3^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h_3^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right)$$

$\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right),\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right),\ldots\in[a,b]\times[c,d]\times[e,g]$, $\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right),\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right),\ldots\in[a,b]\times[c,d]\times[e,g]$
$\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right),\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right),\ldots\in[a,b]\times[c,d]\times[e,g]$

So if the integral is a continuous function and its partial derivatives exist Exist at each point of the region of integration $[a,b]\times[c,d]\times[e,g]$ the error formula for the mentioned rule can be written as follows:

$$E_{ttm}(h_1,h_2,h_3)=I-ttm(h_1,h_2,h_3)=A_{ttm}h_1^2+A_{ttm}h_2^2+A_{ttm}h_3^2+$$
$$B_{ttm}h_1^4+B_{ttm}h_2^4+B_{ttm}h_3^4+C_{ttm}h_1^6+C_{ttm}h_2^6\cdots$$

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where $A, B, \ldots$ Constants that depend on the partial derivatives of the function in the region of integration, and thus the proof ends.

3. Examples

$\int_2^3\int_1^2\int_0^1 xe^{-(x+y+z)}dxdydz$	It has an analytical value 0.0052567434550 rounded to thirteen decimal places
$\int_0^1\int_0^1\int_0^1 \sin(\frac{\pi}{2}(x+y+z))dxdydz$	It has an analytical value 0.5160245509312 rounded to thirteen decimal places
$\int_1^2\int_1^2\int_1^2 1/\sqrt{x^4+y^4+z^4}dxdydz$	analytical value unknown

Note: We use the correction bounds formula in the ttm base in all the examples

$$E_{ttm}(h_1,h_2,h_3) = I - ttm(h_1,h_2,h_3) = A_{ttm}h_1^{\ 2} + A_{ttm}h_2^{\ 2} + A_{ttm}h_3^{\ 2} +$$
$$B_{ttm}h_1^{\ 4} + B_{ttm}h_2^{\ 4} + B_{ttm}h_3^{\ 4} + C_{ttm}h_1^{\ 6} + C_{ttm}h_2^{\ 6}\cdots$$

4. Results

1- Integral function $\int_2^3\int_1^2\int_0^1 xe^{-(x+y+z)}dxdydz$ continuous and differentiable for each $(x,y,z)\in[0,1]\times[1,2]\times[2,3]$

From table (1) we notice that the value of integration using the rule is true for five decimal places $n=16$, and when using the Rumberck acceleration with the mentioned rule, it is true for thirteen decimal places (partial period 2^{12}), which is identical to the real value rounded to thirteen decimal places.

2- Integral function $\int_0^1\int_0^1\int_0^1 \sin(\frac{\pi}{2}(x+y+z))dxdydz$ continuous and differentiable for each $(x,y,z)\in[0,1]\times[0,1]\times[0,1]$

from table (2), we conclude when $n=32$ the value of the integration using the rule ttm is true to two decimal places, and when using the Rumberck acceleration with the mentioned rule, the value is identical to the real value, rounded to thirteen decimal places with (partial period 2^{15}).

3-- Integral function $\int_1^2\int_1^2\int_1^2 1/\sqrt{x^4+y^4+z^4}dxdydz$ continuous and differentiable for each $(x,y,z)\in[1,2]\times[1,2]\times[1,2]$ the integral has an unknown analytic value and when noticing table (3) when $n=128$, k=8,10,12,14 the value is constant (for fifteen decimal places) horizontally and also when $n=64$, k = 8,10,12 that the value is fixed (for fourteen decimal places) horizontally, so we can say that the value is true at least for fifteen decimal places when applying this method ttm, (which means that convergence is correctly towards analytic value).

n-n1	n2	ttm	K=2	K=4	K=6	K=8
1	2	0.0037989419941				
2	4	0.0048955533469	0.0052610904646			

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4	8	0.0051667211074	0.0052571103609	0.0052568450207		
8	16	0.0052342562678	0.0052567679879	0.0052567451631	0.0052567435780	
16	32	0.0052511228273	0.0052567450138	0.0052567434822	0.0052567434555	0.0052567434550

Table 1 $\int_{2}^{3}\int_{1}^{2}\int_{0}^{1} xe^{-(x+y+z)}dxdydz$

n=n1	n2	ttm	k=2	k=4	k=6	k=8	k=10
1	2	0.3535533905933					
2	4	0.4759504390421	0.5167494551918				
4	8	0.5060585148918	0.5160945401751	0.5160508791739			
8	16	0.5135365615189	0.5160292437279	0.5160248906314	0.5160244781149		
16	32	0.5154027771112	0.5160248489753	0.5160245559918	0.5160245506801	0.5160245509647	
32	64	0.5158691215020	0.5160245696322	0.5160245510093	0.5160245509302	0.5160245509312	0.5160245509312

Table 2 $\int_{0}^{1}\int_{0}^{1}\int_{0}^{1} \sin(\frac{\pi}{2}(x+y+z))dxdydz$

n=n1	n2	ttm	k=2	k=4	k=6	k=8	k=10	k=12
1	2	0.241586424038489						
2	4	0.247298865095850	0.249203012114971					
4	8	0.248480427651436	0.248874281836631	0.248852366484741				
8	16	0.248770864208748	0.248867676394519	0.248867236031711	0.248867472056266			
16	32	0.248843336544849	0.248867493990216	0.248867481829930	0.248867485731489	0.248867485785117		
32	64	0.248861448431672	0.248867485727280	0.248867485176418	0.248867485229537	0.248867485227568	0.248867485227023	
64	128	0.248865976051509	0.248867485258121	0.248867485226843	0.248867485227644	0.248867485227636	0.248867485227636	0.248867485227636
128	256	0.248867107935005	0.248867485229503	0.248867485227596	0.248867485227608	0.248867485227607	0.248867485227607	0.248867485227607

Table 3 $\int_{1}^{2}\int_{1}^{2}\int_{1}^{2} 1/\sqrt{x^4+y^4+z^4}\,dxdydz$

5. Conclusion

It is clear from the results of the tables of this research that when calculating the approximate values of triple integrals with continuous integrals of the compound base from the bases of the middle point on the trapezoidal dimension on the two dimensions when the number of partial periods to which the period is divided into the internal dimension is equal to twice the number of partial periods to which the period is divided into the middle dimension and equal to twice the number of partial periods into which the period of the outer dimension is divided. This rule (base) gives integer values (for several decimal places) compared with the real values of the integrals, and by using a number of partial periods without using the process of external adjustment on them, for example, in the first and second integrations, we get On the integer value of five decimal places and two decimal places respectively and in the third integral the real value is unknown.

However, when using the Romberg acceleration method with the mentioned rule, it gave better results in terms of the speed of approaching with a relatively small number of partial periods to the values of the real integrals, as they were identical to the real value in the first and second integrals when at $n = 16$ and $n = 128$. And in the third integral

When using the rule ttm with the acceleration of Romberg, we got a fixed value for thirteen decimal places at $k = 8,10,12$ by taking the common value in Table 3, so it can be said that the value of this integral is 0.2488674852276 rounded to thirteen decimal places, so it is possible to rely on rttm in calculating triple integrals Continuous calls.

References

[1] Hilal,Rana Hassan,"A methods SIM&M and M&SIM to Evaluate double integrals with Continuous Integrands on the two dimensions are not equal", Journal of Engineering and Sciencs **13(10):33371-3776,2018.SSN:1816**-949X**.**

[2] Hassan, Zainab Falih Hassan, "Calculating binary and triple integrals numerically when the numbers of partial periods on the dimensions are not equal using the middle point rule with an accelerated method", Master's thesis submitted to the University of Kufa, 2015.

[3] Mohammed A. H. , Hayder A. K. and Hassen A. F. " On The Numerical Integration ", an article accepted by scientific conference of Morocco ,2009