International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 4, April - 2022, Pages:53-61

Zariski Topology of fuzzy prime complete $\boldsymbol{\rho}$-filter on $\boldsymbol{\rho}$-algebra

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Abstract: fuzzy c-$\boldsymbol{\rho}$-filter and prime radical of fuzzy complete $\boldsymbol{\rho}$-filter in $\boldsymbol{\rho}$-algebra. Also, we study some topological properties of its spectrum.

Keywords: fuzzy c-$\boldsymbol{\rho}$-filter , fuzzy prime c-$\boldsymbol{\rho}$-filter , spectrum of fuzzy c-$\boldsymbol{\rho}$-filter ,Prime radical of fuzzy complete $\boldsymbol{\rho}$-filter .

1. Introduction

In 2000 a topology on the set of all fuzzy prime ideal of a commutative BCK-algebra Y was defined by A. Hasan khanin [1]. In 1999 [4] J. Naggers Y. B. Jun and H. S. Kim introduce the notion of d-ideal in d-algebra. in 2017 [8] S. M. Khalil and M. Alradha, introduced the notion of characterizations of ρ-algebra and generation permutation topological ρ-algebra using permutation in symmetric group. **[6]** In 2018 Sarinya Sripaeng,Kanlaya Tanamoon &Aiyared Iampan introduced On anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, Journal of Information and Optimization Sciences, In 2019 [3] H .K . Abdullah and A .K Mohammad**,** introduced "Some Types filter of $\boldsymbol{\rho}$-algebra" , [11] 1n 2019 Theeyarat Klinseesook, Sukhontha Bukok & Aiyared Iampan, introdction , Rough set theory applied to UP-algebras. [5] In 2020 Korawit Taboon, Phatchara Butsri & Aiyared Iampan A cubic set theory approach to UP-algebras, Journal of Interdisciplinary Mathematics. [4] in 2020 H .K . Abdullah and A .K Mohammad**,** introduced, Fuzzy $\boldsymbol{\rho}$-filter and fuzzy c-$\boldsymbol{\rho}$-filter in $\boldsymbol{\rho}$-algebra , The aim of this paper is to introduce fuzzy prime c-$\boldsymbol{\rho}$-filter , spectrum of fuzzy c-$\boldsymbol{\rho}$-filter .

2. Preliminaries of ρ-algebra

In this part , we're introducing definition $\boldsymbol{\rho}$-filter , c-$\boldsymbol{\rho}$-filter and fuzzy c-$\boldsymbol{\rho}$-filter in $\boldsymbol{\rho}$-algebra.

Definition (2.1) [8]:

A $\boldsymbol{\rho}$-algebra is a set X with a binary operation " $*$ " and constant "0" which satisfies the following axioms :

1. $x * x = 0$
2. $0 * x = 0$
3. $x * y = 0$ and $y * x = 0$ imply $x = y$, For all $x, y, \in X$
4. For all $x \neq y, x, y \in X - \{0\}$, imply $x * y = y * x \neq 0$

<u>Remark : (2.2)</u>

In $\boldsymbol{\rho}$-algebra X, we denoted $x * 0$ by x^* for every $x \in$ X

<u>Definition (2.3)</u>[3]

A nonempty subset F of a $\boldsymbol{\rho}$-algebra X is said to be $\boldsymbol{\rho}$-*filter if*

1. $0 \in F$
2. $(x^* * y^*)^* \in F$, $y \in F$ implies $x \in F$.

<u>Definition (2.4)[3]</u>

A subset of a $\boldsymbol{\rho}$-algebra X is called complete $\boldsymbol{\rho}$-*filte*r (c-$\boldsymbol{\rho}$-*filte*r) if,

1. $0 \in F$

2. $(x^* * y^*)^* \in F$, $\forall y \in F$, implies $x \in F$

Definition (2.5):[9]

Let X be a non-empty set. A fuzzy set on X is a function $\mu : X \rightarrow [0,1]$. If μ and η be two fuzzy subsets of X, then by $\mu \subseteq \eta$ we mean $\mu(x) \leq \eta(x)$ for all $x \in X$.

Definition (2.6[4])

A fuzzy set μ on -algebra X is said to be fuzzy $\boldsymbol{\rho}$-*filter* of X, if

1. $\mu(0) \geq \mu(x)$, $\forall x \in X$
2. $\mu(x) \geq \min\{\mu((x^* * y^*)^*), \mu(y)\}$, $\forall x, y \in X$

Definition (2.7)[4]

Let F be c-$\boldsymbol{\rho}$-filter of $\boldsymbol{\rho}$-algebra X. A fuzzy subset μ_F of X is said to be fuzzy complete $\boldsymbol{\rho}$-filter(fuzzy c-$\boldsymbol{\rho}$-filter) at F if,

1) $\mu_F(0) \geq \mu_F(x)$, $\forall x \in X$
2) $\mu_F(x) \geq \min\{\mu_F((x^* * y^*)^*), \mu_F(y)\}$, $\forall y \in F$

Remark (2. 8)[4] :

1) The intersection family of fuzzy c-$\boldsymbol{\rho}$-filters at F is fuzzy c-$\boldsymbol{\rho}$-filters.
2) The union of two fuzzy c-$\boldsymbol{\rho}$-filters it is not necessarily fuzzy c-$\boldsymbol{\rho}$-filter.

3. The Prime radical of fuzzy complete ρ-filter

In this part , we define prime, prime radical and introduce some it is properties also we study topological properties of spectrum of prime fuzzy complete $\boldsymbol{\rho}$-filter in $\boldsymbol{\rho}$-algebra.

Definition (3.1):

A non-constant fuzzy c-$\boldsymbol{\rho}$-filter μ_F at a c-$\boldsymbol{\rho}$-filter F of a $\boldsymbol{\rho}$-algebra X is called prime if, for all fuzzy c-$\boldsymbol{\rho}$-filters α_F , β_F at F such that $\alpha_F \cap \beta_F \subseteq \mu_F$, than either $\alpha_F(x) \subseteq \mu_F(x)$, or $\beta_F(x) \subseteq \mu_F(x)$, for all $x \in F$.

Example (3.2):

Let $X = \{0, a, b\}$ and a binary operation $*$ is defined by the following table.

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	a	0

It is clear that (X,$*$, 0) is a $\boldsymbol{\rho}$-algebra and $F = \{0, b\}$ is c-$\boldsymbol{\rho}$-filter in X. Let μ_F be the fuzzy set defined as the following .

1. $\mu_F(x) = \begin{cases} r & if\ \ x = 0, b \\ s & if\ \ x = a \end{cases}$
2. $\mu_F(x) = \begin{cases} 1 & if\ \ x = 0, a \\ s & if\ \ x = b \end{cases}$

3. $\mu_F(x) = \begin{cases} 1 & if \ x = 0 \\ s & if \ x = b, a \end{cases}$
   where $r, s \in [0,1]$ such that $s < r \leq 1$.

if α_F, β_F are two fuzzy c-$\boldsymbol{\rho}$-filter at F in X such that $\alpha_F \cap \beta_F \subseteq \mu_F$, Then μ_F is prime

1. $(\alpha_F \cap \beta_F)(0) \leq \mu_F(0)$, so we have two cases :
   i. If $\alpha_F(0) < \beta_F(0)$ then $\alpha_F(0) \leq \mu_F(0)$
   Since $\alpha_F(b) \leq \mu_F(0)$ and $\mu_F(0) = \mu_F(b)$

   Then $\alpha_F(b) \leq \mu_F(b)$ $imply$ $\alpha_F(x) \subseteq \mu_F(x), \forall x \in F$

   ii. Similarity if $\beta_F(0) < \alpha_F(0)$ then $\beta_F(x) \subseteq \mu_F(x)$, $\forall x \in F$
   Hence $\mu_F$is prime.

2. Since $\mu_F(0) = \mu_F(a) = 1$

Then $\beta_F(0) \leq \mu_F(0)$ and $\alpha_F(0) \leq \mu_F(0)$

Also $\beta_F(a) \leq \mu_F(a)$ and $\alpha_F(a) \leq \mu_F(a)$

Since $(\alpha_F \cap \beta_F)(b) \leq \mu_F(b)$

Then either $(\beta_F)(b) \leq \mu_F(b)$ or $\alpha_F(b) \leq \mu_F(b)$

i. If $(\beta_F)(b) \leq \mu_F(b)$ then $\beta_F(x) \subseteq \mu_F(x), \forall\ x \in F$
ii. If $\alpha_F(b) \leq \mu_F(b)$ then $\alpha_F(x) \subseteq \mu_F(x)$, $\forall\ x \in F$
Hence μ_F is prime .

3. Since $\alpha_F(0) \leq \mu_F(0)$ and $(\beta_F)(0) \leq \mu_F(0)$
either $\alpha_F(b) \leq \mu_F(b)$ or $(\beta_F)(b) \leq \mu_F(b)$

Since $(\alpha_F \cap \beta_F)(0) \leq \mu_F(0)$ and $(\alpha_F \cap \beta_F)(b) \leq \mu_F(b)$

$\alpha_F(x) \subseteq \mu_F(x)$ or $\beta_F(x) \subseteq \mu_F(x), \forall\ x \in F$

<u>Example (3.3):</u>

In Example (3.2)

1. $\mu_F(x) = \begin{cases} 0.7 & if \ x = 0, a \\ 0.1 & if \ x = b \end{cases}$
2. $\mu_F(x) = \begin{cases} 0.6 & if \ x = 0 \\ 0.1 & if \ x = a, b \end{cases}$

   $\beta_F(x) = \begin{cases} 0.6 & if \ x = 0 \\ 0.2 & if \ x = a, b \end{cases}$,

   $\alpha_F(x) = \begin{cases} 0.8 & if \ x = 0 \\ 0.1 & if \ x = a, b \end{cases}$

are fuzzy c- $\boldsymbol{\rho}$-filter at F and $\alpha_F \cap \beta_F \subseteq \mu_F$, but

1. μ_F not prime since

$\alpha_F(0) = 0.8 \nleq \mu_F(0) = 0.7$ and $\beta_F(b) = 0.2 \nleq \mu_F(b) = 0.1$

2. Similarity μ_F is not prime.

Remark (3.4)

In Example (3, 2), every fuzzy prime c-$\boldsymbol{\rho}$-filter at F = { $0, b$ }, takes one the following.

$\mu_F(x) = \begin{cases} r & if\ \ x = 0, b \\ s & if\ \ x = a \end{cases}$ where $r, s \in [0,1]$ and $s < r$.

1. $\mu_F(x) = \begin{cases} 1 & if\ \ x = 0, b \\ s & if\ \ x = a \end{cases}$ where $s < 1$.
2. $\mu_F(x) = \begin{cases} 1 & if\ \ x = 0 \\ s & if\ \ x = b, a \end{cases}$ where $s < 1$

Definition (3.5):

Let F *be a* c-$\boldsymbol{\rho}$-filter of a $\boldsymbol{\rho}$-algebra X and μ_F be a fuzzy c-$\boldsymbol{\rho}$-filter *at* F **.** The prime radical $rad(\mu_F)$ of fuzzy c-$\boldsymbol{\rho}$-filter μ_F at F is the intersection of all fuzzy prime c- $\boldsymbol{\rho}$-filters at F of X containing μ_F. In case there is no such fuzzy prime c-$\boldsymbol{\rho}$-filter containing α_F then $rad\left(\alpha_F\right) = 1$.

Example (3.6)

From Remark (3.4) , $\mu_F(x) = \begin{cases} r\ if & x = 0, b \\ s\ if & x = a \end{cases}$ *are all* fuzzy prime c-$\boldsymbol{\rho}$-filter at F (where $0.7 \le s < r$) containing the fuzzy c-$\boldsymbol{\rho}$-filter

$\alpha_F(x) = \begin{cases} 0.7 & if\ \ x = 0, a \\ 0.6 & if\ \ x = b \end{cases}$

So $rad\ (\alpha_F) = \cap\ \mu_F = 0.7$

Proposition (3.7) :

Let X be a $\boldsymbol{\rho}$-algebra and μ_F , α_F be two fuzzy c-$\boldsymbol{\rho}$-filters at a c-$\boldsymbol{\rho}$-filter of X. Then

1. $\mu_F \subseteq rad(\mu_F)$.
2. $rad\big(rad(\mu_F)\big) = rad(\mu_F)$.
3. If μ_F is a fuzzy prime c-$\boldsymbol{\rho}$-filter, then $rad(\mu_F) = \mu_F$.
4. If $\alpha_F \subseteq \mu_F$ then $rad\ \left(\alpha_F\right) \subseteq rad(\ \mu_F)$.

Proof:

1. By Definition (3.5)
2. Let α_F be fuzzy prime c-$\boldsymbol{\rho}$-filters at F such that $\mu_F \subseteq \alpha_F$ then $rad\ \mu_F \subseteq \alpha_F$ so $rad\ (rad(\mu_F) \subseteq rad\ \mu_F$ but $rad\ \mu_F \subseteq rad\ (rad(\mu_F)$ by (1)

Thus $rad\big(rad(\mu_F)\big) = rad(\mu_F)$.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 4, April - 2022, Pages:53-61

3. Since μ_F is a fuzzy prime c-$\boldsymbol{\rho}$-filter then $rad(\mu_F) \subseteq \mu_F$, this means $\mu_F = rad(\mu_F)$ (by 1) .
4. Since $\mu_F \subseteq rad(\ \mu_F)$ by (1) $and\ \alpha_F \subseteq\ \mu_F$, then $\alpha_F \subseteq rad(\ \mu_F)$, thus $rad\ \left(\alpha_F\right) \subseteq rad\big(rad(\ \mu_F)\big) = rad\ (\mu_F)$ by (2), then $rad\ \left(\alpha_F\right) \subseteq rad(\mu_F)$.

Proposition (3.8)

Let X be a $\boldsymbol{\rho}$-algebra, then for every fuzzy c-$\boldsymbol{\rho}$-filters $\alpha_F$and β_F at a c-$\boldsymbol{\rho}$-filter F of X , the following are held:

1. $V\left(\alpha_F\right) = V(rad\left(\alpha_F\right))$.
2. $V\left(\alpha_F\right) = V(\beta_F)$ if and only if $rad\left(\alpha_F\right) = rad(\beta_F)$.

Proof:

1. Since $\alpha_F \subseteq rad(\alpha_F)$ then $V(rad\left(\alpha_F\right)) \subseteq V\left(\alpha_F\right)$.
   Now, let $\mu_F \in V\left(\alpha_F\right)$, then $\alpha_F \subseteq \mu_F$, so $rad\left(\alpha_F\right) \subseteq \mu_F$.
   Thus $\mu_F \in V(rad\left(\alpha_F\right)$, therefore $V\left(\alpha_F\right) \subseteq V(rad\left(\alpha_F\right))$, then

$V\left(\alpha_F\right) = V(rad\left(\alpha_F\right))$.

2. By Definition (3.5).
   Conversely by (1).

4. Spectrum of a fuzzy ρ-algebra

In this part , we provide the notion of spectrum of fuzzy c-$\boldsymbol{\rho}$-filter and we introduce some of its properties.

Definition (4.1):

Let $\mu\ be$ a fuzzy subset of $\boldsymbol{\rho}$-algebra X and F be c-$\boldsymbol{\rho}$-filter of X. The intersection of all fuzzy c-$\boldsymbol{\rho}$-filter at F conation μ is called the fuzzy set generated by μ at F and denoted by $< \mu >$.

Then by remark ((2,8), 1) then $< \mu >$ is fuzzy c-$\boldsymbol{\rho}$-filter at F .

Definition (4.2):

Let X be a -algebra and F be c-$\boldsymbol{\rho}$-filter, define

1. $spec_F(X) = \{\ \mu_F :\ \mu_F$ is fuzzy prime c-$\boldsymbol{\rho}$-filter at F of $X\}$, it is called the spectrum of fuzzy prime c-$\boldsymbol{\rho}$-filter at F .
2. $V\left(\alpha_F\right) = \{\ \mu_F \in spec_F(X) :\ \alpha_F \subseteq \mu_F\}$, it is called the variety of the fuzzy c-$\boldsymbol{\rho}$-filter $\alpha_F\ at\ F$.
3. $X\left(\alpha_F\right) = spec_F(X)\ \backslash V((\alpha_F)$, the complement of $V\left(\alpha_F\right)$ in $spec_F(X)$.

Proposition (4.3):

Let X be a $\boldsymbol{\rho}$-algebra and F be c-$\boldsymbol{\rho}$-filter, then

1. $X(0) = \emptyset, X(1) = spec_F(X)$.
2. If μ_F , α_F, are fuzzy c-$\boldsymbol{\rho}$-filters at F, such that $\mu_F \subseteq \alpha_F$

then $V(\alpha_F) \subseteq V(\mu_F)$ so $X(\mu_F) \subseteq X(\alpha_F)$.

3. If μ_F, α_F are fuzzy c-$\boldsymbol{\rho}$-filter at F then $X(\mu_F) \cup X\left(\alpha_F\right) = X\left(< \mu_F \cup \alpha_F >\right)$.
4. If μ_F, α_F are fuzzy c-$\boldsymbol{\rho}$-filter at F then $X(\mu_F) \cap X\left(\alpha_F\right) = X\left(\mu_F \cap \alpha_F\right)$

Proof:

1. Since $0 \subseteq \mu_F, 1 \nsubseteq \mu_F$ for every fuzzy prime c-$\boldsymbol{\rho}$-filter μ_F of X then $X(1) = spec_F(X), X(0) = \emptyset$
2. Let $\beta_F \in V(\alpha_F)$, then $\alpha_F \subseteq \beta_F$, since $\mu_F \subseteq \alpha_F$, then $\mu_F \subseteq \beta_F$, thus $\beta_F \in V(\mu_F)$, then $V\left(\alpha_F\right) \subseteq V(\mu_F)$.
3. Let $\sigma_F \in X(\mu_F) \cup X(\alpha_F)$, then $\sigma_F \in X(\mu_F)$ or $\sigma_F \in X(\alpha_F)$, thus $\mu_F \nsubseteq \sigma_F$ or $\alpha_F \nsubseteq \sigma_F$, then $\mu_F \cup \alpha_F \nsubseteq \sigma_F$, so $< \mu_F \cup \alpha_F > \nsubseteq \sigma_F$

$\sigma_F \in X\left(< \mu_F \cup \alpha_F >\right)$, then $X(\mu_F) \cup X\left(\alpha_F\right) \subseteq X\left(< \mu_F \cup \alpha_F >\right)$. Now let $\sigma_F \in X\left(< \mu_F \cup \alpha_F >\right)$, then $\mu_F \cup \alpha_F \nsubseteq \sigma_F$, thus $\mu_F \nsubseteq \sigma_F$ or $\alpha_F \nsubseteq \sigma_F$, then $\sigma_F \in X(\mu_F)$ or $\sigma_F \in X\left(\alpha_F\right)$, thus $\sigma_F \in X(\mu_F) \cup X\left(\alpha_F\right)$ then $X\left(< \mu_F \cup \alpha_F >\right) \subseteq X(\mu_F) \cup X\left(\alpha_F\right)$. Hence $X(\mu_F) \cup X\left(\alpha_F\right) = X\left(< \mu_F \cup \alpha_F >\right)$.

4. If $\sigma_F \in X(\mu_F) \cap X(\alpha_F)$ then $\sigma_F \in X(\mu_F) \cap \sigma_F \in X(\alpha_F)$

$so\ \ \mu_F \nsubseteq \sigma_F\ and\ \ \alpha_F \nsubseteq \sigma_F$, then $\mu_F \cap \alpha_F \nsubseteq \sigma_F$ imply $\sigma_F \in X\left(\mu_F \cap \alpha_F\right)$

then $X(\mu_F) \cap X\left(\alpha_F\right) \subseteq X\left(\mu_F \cap \alpha_F\right)$.

Conversely let $\sigma_F \in X\left(\mu_F \cap \alpha_F\right)$, then $\mu_F \cap \alpha_F \nsubseteq \sigma_F$, thus $\mu_F \nsubseteq \sigma_F$ & $\alpha_F \nsubseteq \sigma_F$, then $\sigma_F \in X(\mu_F)$ and $\sigma_F \in X\left(\alpha_F\right)$, so $\sigma_F \in X(\mu_F) \cap X\left(\alpha_F\right)$

then $X\left(\mu_F \cap \alpha_F\right) \subseteq X(\mu_F) \cap X\left(\alpha_F\right)$ Hence $X(\mu_F) \cap X\left(\alpha_F\right) = X\left(\mu_F \cap \alpha_F\right)$.

Proposition (4.4):

Let X be a $\boldsymbol{\rho}$-algebra, such that μ_F , α_F , and $\overset{i}{\alpha}_F$, $i \in \Delta$, are fuzzy c-$\boldsymbol{\rho}$-filter at c-$\boldsymbol{\rho}$-filter F Then.

1. $V\left(\alpha_F\right) = V(< \alpha_F >)$.
2. $V(\mu_F) \cap V\left(\alpha_F\right) \subseteq V(\mu_F \cap \alpha_F)$.
3. $\bigcap_{i\in\Delta} V\left(\overset{i}{\alpha}_F\right) = V(< \cup_{i\in\Delta} \overset{i}{\alpha}_F >)$.

Proof:

1. Let $\mu_F \in V\left(\alpha_F\right)$, Then $\alpha_F \subseteq \mu_F$, thus $< \alpha_F > \subseteq \mu_F$, then $\mu_F \in V\left(< \alpha_F >\right)$.

Conversely, let $\mu_F \in V\left(< \alpha_F >\right)$, then $< \alpha_F > \subseteq \mu_F$, note that $\alpha_F \subseteq$

$< \alpha_F > \subseteq \mu_F$, then $\mu_F \in V\left(\alpha_F\right)$, thus $V\left(\alpha_F\right) = V\left(< \alpha_F >\right)$.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 4, April - 2022, Pages:53-61

2. Let $\beta_F \in V(\mu_F) \cap V\left(\alpha_F\right)$ .Then $\beta_F \in V(\mu_F)$ and $\beta_F \in V(\alpha_F)$, thus $\mu_F \subseteq \beta_F$ and $\alpha_F \subseteq \beta_F$, then $\mu_F \cap \alpha_F \subseteq \beta_F$, $V(\mu_F) \cap V\left(\alpha_F\right) \subseteq V(\mu_F \cap \alpha_F)$.
3. Let $\mu_F \in \bigcap_{i\in\Delta} V\left(\overset{i}{\alpha}_F\right)$, then $\mu_F \in V\left(\overset{i}{\alpha}_F\right)$, $\forall\, i \in \Delta$, thus $\overset{i}{\alpha}_F \subseteq \mu_F$,$\forall\, i \in \Delta$, then $\cup_{i\in\Delta} \overset{i}{\alpha}_F \subseteq \mu_F$, so $\mu_F \in V(< \cup_{i\in\Delta} \overset{i}{\alpha}_F >)$.
   Now, let $\sigma_F \in V(<\cup_{i\in\Delta} \overset{i}{\alpha}_F >)$,
   then $\cup_{i\in\Delta} \overset{i}{\alpha}_F \subseteq \sigma_F$, thus $\forall\, i \in \Delta$, $\overset{i}{\alpha}_{Fi} \subseteq \cup_{i\in\Delta} \overset{i}{\alpha}_F \subseteq \sigma_F$, then $\sigma_F \in V\left(\overset{i}{\alpha}_F\right)$, $\forall\, i \in \Delta$. Thus $\sigma_F \in \bigcap_{i\in\Delta} V\left(\overset{i}{\alpha}_F\right)$, then $\bigcap_{i\in\Delta} V\left(\overset{i}{\alpha}_F\right) = V(<\cup_{i\in\Delta} \overset{i}{\alpha}_F >)$.

Proposition (4.5):

Let F be a fixed c-$\boldsymbol{\rho}$-filter of a $\boldsymbol{\rho}$-algebra X and let $\tau = \{X(\mu_F) : \mu_F$ is fuzzy c-$\boldsymbol{\rho}$-filter at F of X } then the pair (τ , $spec_F(X)$) is topological space, and it is called Zariski topology on X.

Proof:

1. $since\ X(0) = \emptyset$, $X(1) = spec_F(X)$ (by Proposition.((4.3), 1)

then $\emptyset$, $spec_F(X) \in \tau$.

2. Let $X(\mu_F)$, $X\left(\alpha_F\right) \in \tau$, such that μ_F, α_F are fuzzy c-$\boldsymbol{\rho}$-filter at F.

then by Proposition ((4.3), 4) , $X(\mu_F) \cap X\left(\alpha_F\right) \in \tau$

3.Let $\left\{X\left(\overset{i}{\alpha}_F\right), i \in \Delta\right\} \subseteq \tau$, where $\overset{i}{\alpha}_F$ are fuzzy c-$\boldsymbol{\rho}$-filters at F in a $\boldsymbol{\rho}$-algebra X. Then $\bigcup_{i\in\Delta} X\left(\overset{i}{\alpha}_F\right) \in \tau$ (by Proposition.(4.3), 3)

Thus τ is topology space .

5. Some Topological Properties of $spec_F(X)$

In this section, we will introduce some topological properties of $spec_F(X)$.

Definition (5.1)[10]:

A topological space X is said to be disconnected if X can be expressed as the union of two disjoint non-empty open subsets of X, otherwise X is said to be connected.

Proposition(5.2):

Let X be a $\boldsymbol{\rho}$-algebra, and F $be\ a$ c-$\boldsymbol{\rho}$-filter. If $spec_F(X)$ is disconnected, then there exist two proper fuzzy c-$\boldsymbol{\rho}$-filters α_F, σ_F at F such that $rad\left(\alpha_F \cup \beta_F\right) = 1$ and $rad\left(\alpha_F \cap \beta_F\right) = rad(0)$.

Proof:

Let X be a -algebra, and let $spec_F(X)$ be disconnected. Then there exist two proper fuzzy c-$\boldsymbol{\rho}$-filters, α_F, β_F in X such that $X\left(\alpha_F\right) \neq \emptyset, X(\beta_F) \neq \emptyset$, $X\left(\alpha_F\right) \cap X(\beta_F) = X(0)$, $X\left(\alpha_F\right) \cup X(\beta_F) = X(1)$. Thus $X\left(\alpha_F \cap \beta_F\right) = X(0)$ and $X\left(< \alpha_F \cup \beta_F >\right) = X(1) = spec_F(X)$, so by (Proposition (3.8) ,2) we have $rad\left(\alpha_F \cap \beta_F\right) = rad\ (0)$ and $rad\left(\alpha_F \cup \beta_F\right) = 1$.

Definition (5. 3) :

A -algebra X is said to be prime $\boldsymbol{\rho}$-algebra if every proper fuzzy c-$\boldsymbol{\rho}$-filter contained in a fuzzy prime c-$\boldsymbol{\rho}$-filter.

Theorem (5.4) :

Let X be a prime $\boldsymbol{\rho}$-algebra. Then $spec_F(X)$ is disconnected if and only if there exist two proper fuzzy c-$\boldsymbol{\rho}$-filters α_F, β_F such that $rad\left(\alpha_F \cup \beta_F\right) = 1$ and $rad\ (\alpha_F \cap \beta_F) = rad(0)$.

Proof: Let $spec_F(X)$ be disconnected, by Proposition (5.2) there exist two proper fuzzy c-$\boldsymbol{\rho}$-filters α_F,β_F in X such that $rad\left(\alpha_F \cap \beta_F\right) = rad(0)$ and $rad\left(\alpha_F \cup \beta_F\right) = 1$.

Conversely, let β_F, α_F be two proper fuzzy c-$\boldsymbol{\rho}$-filters at F in X such that $rad(\alpha_F \cup \beta_F) = 1$ and $rad\left(\alpha_F \cap \beta_F\right) = rad(0)$, then by Definition (4.3). we have $V(\alpha_F) \neq \emptyset$ and $V(\beta_F) \neq \emptyset$. Now $\emptyset = V(1) = V(\alpha_F) \cap V(\beta_F)$ and $spec_F(X) = V(0) = V(rad(0))$ (by Proposition (3.8),1) then$s\, pec_F(X) = V\left(\left(\alpha_F \cap \beta_F\right)\right) = V\left(\alpha_F \cap \beta_F\right) = V\left(\alpha_F\right) \cup V(\beta_F)$ thus $spec_F(X)$ is a disconnected.

Definition (5.5)[10]:

A topological space X is said to be a $T_0 - space$ if given any two distinct points x_1 and x_2 of X, there exists an open subset of X which contains at least one of the points but does not contain the other.

Theorem (5.6):

Let X be a $\boldsymbol{\rho}$-algebra and $F\ be\ a$ c-$\boldsymbol{\rho}$-filters on X . Then $spec_F$(X) is a $\mathrm{T}_0 -$ space.

Proof:

Let $\mu_F, \beta_F \in spec_F(X)$ and $\mu_F \neq \beta_F$. Then $\mu_F \nsubseteq \beta_F$ or $\beta_F \nsubseteq \mu_F$. If $\mu_F \nsubseteq \beta_F$, then $\beta_F \notin V(\mu_F)$, but $\mu_F \in V(\mu_F)$, Then $\beta_F \in X(\mu_F)$, and $\mu_F \notin X(\mu_F)$. If $\beta_F \nsubseteq \mu_F$, similarly we have $\mu_F \in X(\beta_F)$ but $\beta_F \notin X(\beta_F)$ then $spec_F(X)$ is a $T_0 - space$.

Definition (5.7):[10]

A topological space (X, T) is said to be a $T_1 - space$ if for any two distinct points x_1 and x_2 of X, there exist two open subsets Y_1 and Y_2 of X such that $x_1 \in Y_1$, $x_2 \notin Y_1$ and $x_2 \in Y_2$, $x_1 \notin Y_2$.

Proposition (5.8):[10]

A topological space X is a $T_1 - space$ if and only if every singleton set of X is closed.

Proposition (5.9):

Let X be a $\boldsymbol{\rho}$-algebra. and F c-$\boldsymbol{\rho}$-filter. Then $spec_F(X)$ is a $T_1 - space$ if and only if for every fuzzy prime c-$\boldsymbol{\rho}$-filter μ_F at F of X, there exists a fuzzy c-$\boldsymbol{\rho}$-filter α_F such that $\alpha_F \subseteq \mu_F$, and $\alpha_F \nsubseteq \sigma_F$ for every $\sigma_F \in spec_F(X)$, such that $\mu_F \neq \sigma_F$.

Proof: Let $spec_F(X)$ be a $T_1 -$ space and $\mu_F \in spec\ _F(X)$ there exist a fuzzy c-$\boldsymbol{\rho}$-filter $\alpha_F\ at\ F$ in X such that $\{\mu_F\} = V\left(\alpha_F\right)$. Hence, $\alpha_F \subseteq \mu_F$, and $\alpha_F \nsubseteq \sigma_F \forall\ \sigma_F \in spec\ _F(X)$.

Conversely let $\mu_F \in spec_{\ F}(X)$, then there exists a fuzzy c-$\boldsymbol{\rho}$-filter $\alpha_{_F} \subseteq \mu_F$ and $\alpha_{_F} \nsubseteq \sigma_F$, $\forall\ \sigma_F \in spec_F(X)$, and $\mu_F \nsubseteq \sigma_F$.Hence $V\left(\alpha_{_F}\right) = \{\mu_F\}$. Thus $spec(X)$ is a $T_1 - space$.

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