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Some Results on Ultrafilter in $C_{Rc}(\mathcal{W})$

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Abstract. *The primary goal of this research is to define a new class of ultrafilter with respect to the ring of real valued Rc-continuous functions $C_{Rc}(\mathcal{W})$. The fundamental features of on Filter and Ultrafilter in $C_{Rc}(\mathcal{W})$, as well as several essential theorems connected to this sort of Filter and Ultrafilter functions are introduced.*

Keywords: Nearly compact space, Rc- continuous functions, Filter, Ideals, Ultrafilter

1. Introduction

Filters were introduced by H. Cartan in 1937[1], [2] and subsequently used by Bourbaki in their book Topologie Générale as an alternative to the similar notion of a net developed in 1922 by E. H. Moore and H. L. Smith.
N. T. AbdAlameer and A. A. Atiyah [3] introduced the definition of real valued Rc- continuous functions depending on the concept of regular open sets and nearly compact to present new types of super continuous functions that is called Rc – continuous functions.
N. T. Abd Alameer and A. A. Atiyah [4] introduced the definition of Ideal and Filter in $C_{Rc}(\mathcal{W})$.

2. Preliminaries

we will represent by $(\mathcal{W},\tau)$ and $(\mathcal{V},\rho)$for topological spaces on which no separation axioms are assumed unless explicitly stated and they will simply be written as $\mathcal{W}$ and $\mathcal{N}$. Also, we will denote by Z $(\wp)$ to the zero set of $\wp$, where $\wp \in$ C $(\mathcal{W})$ and Z$(\mathcal{W})$ denotes to all zero sets Z$(\wp)$. Moreover, if $\delta \in C_{Rc}(\mathcal{W})$, $Z_{Rc}(\mathcal{W})$ denote to all zero set $Z_{Rc}(\delta)$.

Definition 2.1. [6] A topological space$(\mathcal{W},\tau)$ is called compact if every open cover hasa finite sup cover that covers$(\mathcal{W},\tau)$.

Definition 2.2.[5] An open cover $\{ V_i \mid i \in \Omega\}$ of a space $\mathcal{W}$ is said to be regular open if V_i is regular open sets for each $i \in \Omega$.

Definition 2.3.[5] A space $(\mathcal{W},\tau)$ is defined to be nearly compact if every regular cover has a finite subcover, i.e., if $\{ V_i \mid i \in \Omega\}$ is a regular open cover of $\mathcal{W}$, there exist a finite subset $\acute{\Omega} \subset \Omega$ such that $\mathcal{W} = \cup \{ V_i \mid i \in \acute{\Omega}\}$.

Definition 2.4.[5] A subset $\mathcal{D}$ is said to be nearly compact if it is nearly compact as a subspace of $\mathcal{W}$.
Definition 2.5.[5] Let $\mathcal{D}$be asubspace of a topological space $(\mathcal{W},\tau)$, $\mathcal{D}$ is defined to be nearly compact relative to $\mathcal{W}$ if for each cover$\{V_i \mid i \in \Omega\}$of $\mathcal{D}$ by regular open sets of $(\mathcal{W},\tau)$ has a finite subcover, i.e., $\mathcal{D} \subset \cup\ V_i \mid i \in \acute{\Omega}\}$.

Definition 2.6[4]
For $\wp \in C_{Rc}(\mathcal{W})$, the zero set of $\wp$, denoted by $Z(\wp)$=$Z_w(\wp) = \{ w \in \mathcal{W} : \wp(w) = 0\}$ and denoted by $Z_{Rc}(\mathcal{W})$ to the set of all zero sets $Z(\wp)$.

Definition 2.7.[4]
A nonempty subfamily $\mathfrak{F}$ of $Z_{Rc}(\mathcal{W})$ is called to be $Z_{Rc} - filter$ in $\mathcal{W}$ if the following conditions hold.

- $\emptyset \in \mathfrak{F}$
- If $\mathfrak{j}, \mathfrak{l} \in \mathfrak{F}$, then $\mathfrak{j} \cap \mathfrak{l} \in \mathfrak{F}$, and
- If $\mathfrak{j} \in \mathfrak{F}$, $\mathfrak{l} \in Z_{Rc}(\mathcal{W})$ and $\mathfrak{j} \subseteq \mathfrak{l}$, then $\mathfrak{l} \in \mathfrak{F}$.

Theorem 2.8.[3]
For a function $\wp : \mathcal{W} \to \mathcal{V}$ the following are equivalent:

(1) $\wp$ is Rc - continuous
(2) If F is open in $\mathcal{V}$and $\mathcal{V} - F$ is nearly compact relative to $\mathcal{V}$, then$\wp^{-1}(F)$ is open in $\mathbb{W}$.
(3) If H is closed in $\mathcal{V}$ and nearly compact relative to $\mathbb{W}$, then $\wp^{-1}$(H) is closed in $\mathcal{W}$.

Definitions 2.9.[4]

An ideal j in $C_{Rc}(\mathcal{W})$ is maximal ideal if $j \neq C_{Rc}(\mathcal{W})$ and $C_{Rc}(\mathcal{W})$is the only ideal strictly containing j.

Definitions 2.10.[4] An ideal j in $C_{Rc}(\mathcal{W})$ is said to be a fixed ideal if $\cap Z_{Rc}[j] \neq \emptyset$. And if $\cap Z_{Rc}[j] = \emptyset$, then we call j is free ideal. It is easy to see that $\emptyset, \mathcal{W} \in Z_{Rc}(\mathcal{W})$ $[Z_{Rc}(1) = \emptyset, Z_{Rc}(0) = \mathcal{W}]$ *and* from third part of above definition we can see that $\mathcal{W} \in Z_{Rc}$ −filter.

Remarks 2.11.[4]

1. For a family of zero sets L of $\mathcal{W}$ and they have the finite intersection property, in other hand, $L_j \in L \ni 1 \leq j \leq n$, $n \in \mathbb{N} \Rightarrow \cap_{1\leq j\leq n} L_j \neq \emptyset$. Hence, there is a Z_{Rc} − filter H on $\mathcal{W}$ with L$\subseteq H$.
2. Clearly, if $\aleph$ is the collection of all finite interaction of sets from L, and $H = \{B \in Z_{Rc}(\mathcal{W}) | E \subseteq B\}$ for some $E \subseteq \aleph$, so H is the smallest Z_{Rc} − filter on $\mathcal{W}$ containing L.
3. If L is regular open under finite intersection, then L is a base for H.
4. If $\mathcal{W}$ is any nonempty topological space and $\mathcal{F}$ be any filter on $\mathcal{W}$, so $H = \mathcal{F} \cap Z_{Rc}(\mathcal{W})$ is a Z_{Rc} −filter on $\mathcal{W}$. On other hand, if H is a Z_{Rc} −filter on $\mathcal{W}$, then $\mathcal{F} = \{A \subseteq \mathcal{W} | D \subseteq A\}$ for some D $\in H$ is a filter on $\mathcal{W}$ and it is the smallest filter on $\mathcal{W}$ containing H and satisfies $\mathcal{F} \cap Z_{Rc}(\mathcal{W}) = H$.

Definition 2.12.[3]
A function $\wp: \mathcal{W} \to \mathcal{V}$ is called Rc-continuous if for each w $\in \mathcal{W}$ and each open subset F of $\mathcal{V}$ containing $\wp(w)$ having the complement nearly compact relative to $\mathcal{V}$, there exists an open subset H of $\mathcal{W}$ contains w $\ni \wp(H) \subset$ F.The function $\wp$ is said to be Rc-continuous if it is Rc-continuous at each w $\in \mathcal{W}$.

Definition 2.13. [4]
A function $\wp \in C_{Rc}(\mathcal{W})$ is called to be unit in $C_{Rc}(\mathcal{W})$ if and only if, $\wp(w) \neq 0$ for all $w \in \mathcal{W}$ in this case $\wp^{-1}(w) = \frac{1}{\wp(w)}$ for all $w \in \mathcal{W}$.
In other words, we can say that $\wp \in C_{Rc}(\mathcal{W})$ is a unit $\Leftrightarrow Z_{Rc}(\mathcal{W}) = \emptyset$.

Remark 2.14. [4]
If j_1, j_2,j_3,… are non-empty family of ideals, then $j_1 \cap j_2 \cap j_3 \cap$ ... is an ideal.

Definition 2.15. [4]

- An ideal j in $C_{Rc}(\mathcal{W})$ is maximal ideal if $j \neq C_{Rc}(\mathcal{W})$ and $C_{Rc}(\mathcal{W})$is the only ideal strictly containing j.
- An ideal j in $C_{Rc}(\mathcal{W})$ is said to be a fixed ideal if $\cap Z_{Rc}[j] \neq \emptyset$. And if $\cap Z_{Rc}[j] = \emptyset$, then we call j is free ideal. It is easy to see that $\emptyset, \mathcal{W} \in Z_{Rc}(\mathcal{W})$ $[Z_{Rc}(1) = \emptyset, Z_{Rc}(0) = \mathcal{W}]$ *and* from third part of above definition we can see that $\mathcal{W} \in Z_{Rc}$ −filter.

Remark 2.16. [4]

1. For a family of zero sets L of $\mathcal{W}$ and they have the finite intersection property, in other hand, $L_j \in L \ni 1 \leq j \leq n$, $n \in \mathbb{N} \Rightarrow \cap_{1\leq j\leq n} L_j \neq \emptyset$. Hence, there is a Z_{Rc} − filter H on $\mathcal{W}$ with L$\subseteq H$.
2. Clearly, if $\aleph$ is the collection of all finite interaction of sets from L, and $H = \{B \in Z_{Rc}(\mathcal{W}) | E \subseteq B\}$ for some $E \subseteq \aleph$, so H is the smallest Z_{Rc} −filter on $\mathcal{W}$ containing L.
3. If L is regular open under finite intersection, then L is a base for H.
4. If $\mathcal{W}$ is any nonempty topological space and $\mathcal{F}$ be any filter on $\mathcal{W}$, so $H = \mathcal{F} \cap Z_{Rc}(\mathcal{W})$ is a Z_{Rc} −filter on $\mathcal{W}$. On other hand, if H is a Z_{Rc} −filter on $\mathcal{W}$, then $\mathcal{F} = \{A \subseteq \mathcal{W} | D \subseteq A\}$ for some D $\in H$ is a filter on $\mathcal{W}$ and it is the smallest filter on $\mathcal{W}$ containing H and satisfies $\mathcal{F} \cap Z_{Rc}(\mathcal{W}) = H$.

Theorem 2.17. [4]
For an ideal j in $C_{Rc}(\mathcal{W})$, $Z_{Rc}[j] = \{Z_{Rc}(g) | g \in j\}$ is a Z_{Rc}- filter on $\mathcal{W}$.

Theorem 2.118. [4]
If H is a Z_{Rc} −filter on j, then $Z_{Rc}^{-1}[H] = \{g \in C_{Rc}(\mathcal{W}) | Z_{Rc}(g) \in H\}$ is an ideal in $C_{Rc}(\mathcal{W})$.

Remark 2.19.

1. For any two elements $g, L \in C_{Rc}(\mathcal{W})$and let (g, L) denotes to the ideal generated by g, L in $C_{Rc}(\mathcal{W})$. If $j = (g, L) \neq C_{Rc}(\mathcal{W})$, then $Z_{Rc}[j]$ is a Z_{Rc} −filter.

(2) Suppose $Z_{Rc}: C_R(\mathcal{W}) \rightarrow Z_{Rc}(\mathcal{W})$ is a surjective map, then for any subset $H \in Z_{Rc}(\mathcal{W})$, $Z_{Rc}[Z_{Rc}^{-1}[H]] = H$ and J $\subset$ $Z_{Rc}^{-1}[Z[\mathrm{j}]]$.

3. Ultra-filter in $C_{Rc}(\mathcal{W})$

Definition 3.1.
$Z_{Rc} -$ ultrafilter H on $\mathcal{W}$ is a maximal Z_{Rc} $-$filter on $\mathcal{W}$, in other words, it is not contained in any Z_{Rc} $-$filter on $\mathcal{W}$.

Proposition 3.2.
For a topological space $(\mathcal{W}, \tau)$, any $Z_{Rc} -$ filter on $\mathcal{W}$ is contained in Z_{Rc} $-$ultrafilter.

Proof:
Obviously, same proof as filter condition.

Proposition 3.3.
For any Z_{Rc} $-$ultrafilter H on $\mathcal{W}$, there is an ultrafilter $K \ni K \cap Z_{Rc}(\mathcal{W}) = H$.
Proof.
Suppose that H is Z_{Rc} $-$ultrafilter on $\mathcal{W}$.
Let S={A⊆ $\mathcal{W}$|H⊆A} for some H∈ $\mathcal{W}$ need not be an ultrafilter on $\mathcal{W}$. However, there is an ultrafilter $K \in \mathcal{W} \ni$ S⊆ K. Hence, $K \cap Z_{Rc}(\mathcal{W})$ is a Z_{Rc} $-$filter on $\mathcal{W}$ satisfying $H = S \cap Z_{Rc}(\mathcal{W}) \subset K \cap Z_{Rc}(\mathcal{W})$.
Therefore, $K \cap Z_{Rc}(\mathcal{W}) = H$ that because of H is Z_{Rc} $-$ultrafilter.

Theorem 3.4.
Whenever N is a maximal ideal in $C_{Rc}(\mathcal{W})$, then $Z_{Rc}[N]$ is a Z_{Rc} $-$ultrafilter.
Proof.
Z_{Rc} and Z_{Rc}^{-1} are both keep embed. We can see that V⊆ $Z_{Rc}^{-1}[Z_{Rc}[V]]$ for any ideal V.
If V= P is a maximal ideal, $Z_{Rc}^{-1}[Z_R[P]] = P$.
If $Z_{Rc}[P]$ is not $Z_{Rc} -$ filter, there exists a $Z_{Rc} -$ ultrafilter S $\ni Z_{Rc}[P] \subset S$.
Because of Z_{Rc}: $C_{Rc}(\mathcal{W}) \longrightarrow Z_{Rc}(\mathcal{W})$ is surjective, we have $Z_{Rc}^{-1}[Z_R[P]] \subset Z_{Rc}^{-1}[S]$.
But$Z_{Rc}^{-1}[Z_{Rc}[P]] = P$ is a maximal ideal, then there is no ideal in $C_{Rc}(\mathcal{W})$ properly contained P.
Therefore, this contradiction explains $Z_{Rc}[P]$ is $Z_{Rc} -$ ultrafilter.

Theorem 3.5.
Whenever H is $Z_{Rc} -$ ultrafilter, $Z_{Rc}^{-1}[H]$ is a maximal ideal of $C_{Rc}(\mathcal{W})$. Where the function Z is a bijection from the set of maximal ideal of $C_R(\mathcal{W})$ to the set of $Z_{Rc} -$ ultrafilter on $\mathcal{W}$.
Proof.
Let$H$ is $Z_{Rc} -$ ultrafilter and $Z_{Rc}^{-1}[H]$ is not maximal ideal in $C_{Rc}(\mathcal{W})$.
Hence, there exists a maximal ideal P in $C_{Rc}(\mathcal{W}) \ni Z_{Rc}^{-1}[H] \subset P$.
Obviously, $Z_{Rc}^{-1}[H] \subset Z_{Rc}^{-1}[Z_{Rc}[P]]$………..(*) [since $Z_{Rc}^{-1}[Z_{Rc}[P]] = P$].
Therefore, $Z_{Rc}Z_{Rc}^{-1}[H] \subset Z_{Rc}Z_{Rc}^{-1}[Z_{Rc}[P]]$ and that leads to $H \subset Z_{Rc}[P]$.
However, because of H is a $Z_{Rc} -$ ultrafilter, then $Z_{Rc}^{-1}[H] = Z_{Rc}^{-1}[Z_{Rc}[P]]$, but this a contradiction to (*).
So, that implies to $Z_{Rc}^{-1}[H]$ is a maximal ideal. Because of $Z_{Rc}^{-1}[Z_{Rc}[P]] = P$ for all maximal ideal in $C_{Rc}(\mathcal{W})$ and $Z_{Rc}\, Z_{Rc}^{-1}[H] = H$ for all $Z_{Rc} -$ ultrafilter, hence [from part 2 of 2.9.] the proof is completed.

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