International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 6 Issue 9, September - 2022, Pages: 71-74

Continuous Functions in the $\boldsymbol{n}$-Normed Space

Noor Razak Jafar Majed

University of Babylon ,College of Education for pure Science ,Iraq
noorrazak1993@gmail.com

Abstract: *In this research we study the convergence between two well - known n-norms on L_∞ , the space of all continuous functions. The first is derived from Gähler's norm and the other is taken from Gunawan's norm. In fact , we explained that the convergence in each of these n- norms is equivalent to that in the usual norm on L_∞ .*

Keywords convergence spaces, dual space, 2 - normed space.

1. INTRODUCTION

Let $\mathcal{X}$ be a real vector space of dim$< n$, $n \geq 0$ and $\|.,\ldots,.\|$ be a real function on $\mathcal{X}^n$ satisfying the following properties :

1) $\|\varkappa_1,\ldots\ldots,\varkappa_n\| = 0 \leftrightarrow \varkappa_1,\ldots\ldots,\varkappa_n$ are linearly dependent ;

2) $\|\varkappa_1,\ldots\ldots,\varkappa_n\|$ is invariant under permutation ;

3) $\|\beta\varkappa_1,\varkappa_2,\ldots\ldots,\varkappa_n\| = |\beta|\|\varkappa_1,\varkappa_2,\ldots\ldots,\varkappa_n\| \;\; \forall\beta \in \mathcal{R}$;

4) $\|\varkappa+\varkappa^*,\varkappa_2,\ldots\ldots,\varkappa_n\| \leq \|\varkappa,\varkappa_2,\ldots\ldots,\varkappa_n\| + \|\varkappa^*,\varkappa_2,\ldots\ldots,\varkappa_n\|$, then the function $\|.,\ldots,.\|$

is called an n- norm on X , and $(\mathcal{X},\|.,\ldots,.\|)$ an n-normed space. In$(\mathcal{X},\|.,\ldots,.\|)$ **,** A sequence g_m is said to converge to $g \in \mathcal{X}$ when $\lim_{m\to\infty}\|g_m - g, g_2,\ldots\ldots,g_n\| = 0, \;\; \forall\, g_2,\ldots\ldots,g_n \in \mathcal{X}$.

In [4] ,Gunawan defined the following an n-norm on L_∞ **,**

$$\|f_1,\ldots,f_n\|_\infty = \sup_{\varkappa_1}.\sup_{\varkappa_2}\ldots\sup_{\varkappa_n}\left|\det(f_i(\varkappa_j))\right| .$$

In 1969 and 1970 [1,2,3] Gahler developed the theory of n- normed spaces. after that , the theory of n- normed spaces was studied by many researchers ,see [5,6,7,8] .We can define an n-norm on $\mathcal{X}$ depending to Gahler by the formula

$$\|\varkappa_1,\ldots,\varkappa_n\|^* = \sup_{\substack{g_i\in\mathcal{X}',\|g_i\|\leq 1\\ i=1,\ldots,n}}\begin{vmatrix} g_1(\varkappa_1) & \ldots. & g_1(\varkappa_n)\\ \vdots & \ddots & \vdots\\ g_n(\varkappa_1) & \cdots & g_n(\varkappa_n)\end{vmatrix}$$

Where $\mathcal{X}'$ which consists of bounded linear functional on $\mathcal{X}$ is dual of $\mathcal{X}$, and the dual of L_∞ is L_1.The above formula reduces to

$$\|\delta_1,\ldots\ldots,\delta_n\|_\infty^* = \sup_{\substack{\alpha_i\in L_1,\|\alpha_i\|_1\leq 1\\ i=1,\ldots,n}}.\sup_{\varkappa_{r1}}\ldots\sup_{\varkappa_{rn}}\begin{vmatrix} \delta_1(\varkappa_{r1}) & \ldots. & \delta_1(\varkappa_{rn})\\ \vdots & \ddots & \vdots\\ \delta_n(\varkappa_{r1}) & \cdots & \delta_n(\varkappa_{rn})\end{vmatrix}\begin{vmatrix} \alpha_1(\varkappa_{r1}) & \ldots. & \alpha_1(\varkappa_{rn})\\ \vdots & \ddots & \vdots\\ \alpha_n(\varkappa_{r1}) & \cdots & \alpha_n(\varkappa_{rn})\end{vmatrix},$$

Such that, $\|.\|_1$ denotes the usual norm on L_1 and $\delta_1,\ldots\ldots,\delta_n \in L_\infty$. So in the space L_∞ , we get two ideas of n- norms ,one referable to Gunawan[4] and the other is derived from Gähler formula [1]. We will first prove our results for $n = 2$, and then expand them to $n \geq 2$.

2. MATERIALS AND METHODS

On L_∞ , we give the flowing Gunawan's definition of 2- norm

$$\|\delta,\beta\|_\infty = \sup_{\varkappa_r}.\sup_{\varkappa_k}\left\{abs\begin{vmatrix}\delta(\varkappa_r) & \delta(\varkappa_k)\\ \beta(\varkappa_r) & \beta(\varkappa_k)\end{vmatrix}\right\}.$$

We have using the same consist as in [4]

$$\|\delta,\beta\|_\infty^* = \sup_{\alpha,\sigma\in L_1\|\sigma\|_1,\|h\|_1\le 1} . \sup_{\varkappa_r} . \sup_{\varkappa_k} \begin{vmatrix} \delta(\varkappa_r) & \delta(\varkappa_k) \\ \beta(\varkappa_r) & \beta(\varkappa_k) \end{vmatrix} . \begin{vmatrix} \alpha(\varkappa_r) & \alpha(\varkappa_k) \\ \sigma(\varkappa_r) & \sigma(\varkappa_k) \end{vmatrix} .$$

We prove the following facts using the last formula ,

Fact 2.1. $\forall\, \delta,\beta \in L_\infty$, then $\|\delta,\beta\|_\infty \le 2\|\delta\|_\infty . \|\beta\|_\infty$.

Proof:

Using triangle's and Minkowski's inequalities ,we get

$$\|\delta,\beta\|_\infty = \sup_{\varkappa_r} . \sup_{\varkappa_k} |\delta(\varkappa_r)\beta(\varkappa_k) - \delta(\varkappa_k)\beta(\varkappa_r)|$$

$$\le \sup_{\varkappa_r} . \sup_{\varkappa_k} [|\delta(\varkappa_r)||\beta(\varkappa_k)| + |\delta(\varkappa_k)||\beta(\varkappa_r|]$$

$$\le \left[\sup_{\varkappa_r} . \sup_{\varkappa_k} \; |\delta(\varkappa_r)||\beta(\varkappa_k)| + \sup_{\varkappa_r} . \sup_{\varkappa_k} \; |\delta(\varkappa_r)||\beta(\varkappa_k)|\right]$$

$$= 2\|\delta\|_\infty . \|\beta\|_\infty .$$

Fact 2.2. $\forall\, \delta,\beta \in L_\infty$, then $\|\delta,\beta\|_\infty^* \le 2\, \|\delta,\beta\|_\infty$.

Proof:

Through Holder's inequality ,we get

$$\sup_{\varkappa_r} . \sup_{\varkappa_k} \begin{vmatrix} \delta(\varkappa_r) & \delta(\varkappa_k) \\ \beta(\varkappa_r) & \beta(\varkappa_k) \end{vmatrix} . \begin{vmatrix} \alpha(\varkappa_r) & \alpha(\varkappa_k) \\ \sigma(\varkappa_r) & \sigma(\varkappa_k) \end{vmatrix}$$

$$\le \sup_{\varkappa_r} . \sup_{\varkappa_k} \left\{ abs \begin{vmatrix} \delta(\varkappa_r) & \delta(\varkappa_k) \\ \beta(\varkappa_r) & \beta(\varkappa_k) \end{vmatrix} \right\} \times \sup_{\varkappa_r} . \sup_{\varkappa_k} \left\{ abs \begin{vmatrix} \alpha(\varkappa_r) & \alpha(\varkappa_k) \\ \sigma(\varkappa_r) & \sigma(\varkappa_k) \end{vmatrix} \right\},$$

$$\sup_{\varkappa_r} . \sup_{\varkappa_k} \left\{ abs \begin{vmatrix} \alpha(\varkappa_r) & \alpha(\varkappa_k) \\ \sigma(\varkappa_r) & \sigma(\varkappa_k) \end{vmatrix} \right\}$$

$$\le \sup_{\varkappa_r} . \sup_{\varkappa_k} [|\alpha(\varkappa_r).\sigma(\varkappa_k)| + \alpha(\varkappa_k).\sigma(\varkappa_r)|]$$

$$\le \sup_{\varkappa_r} . \sup_{\varkappa_k} |\alpha(\varkappa_r).\sigma(\varkappa_k)| + \sup_{\varkappa_r} . \sup_{\varkappa_k} |\alpha(\varkappa_r).\sigma(\varkappa_k)|$$

$$= 2\|\alpha\|_1 . \|\sigma\|_1$$

But $\|\alpha\|_1, \|\sigma\|_1 \le 1 \quad \to \|\alpha,\sigma\|_1 \le 2$, and

$$\sup_{\alpha,\sigma\in L_1\|\sigma\|_1,\|h\|_1\le 1} . \sup_{\varkappa_r} . \sup_{\varkappa_k} \begin{vmatrix} \delta(\varkappa_r) & \delta(\varkappa_k) \\ \beta(\varkappa_r) & \beta(\varkappa_k) \end{vmatrix} . \begin{vmatrix} \alpha(\varkappa_r) & \alpha(\varkappa_k) \\ \sigma(\varkappa_r) & \sigma(\varkappa_k) \end{vmatrix}$$

$$\le \|\delta,\beta\|_\infty \; . \|\alpha,\sigma\|_1$$

$$\le 2\|\delta,\beta\|_\infty .$$

Thus $\|\delta,\beta\|_\infty^* \le 2\|\delta,\beta\|_\infty$.

Corollary 2.3. In $\|.,.\|_\infty$, if δ_m is convergent , then δ_m converges to δ in $\|.,.\|_\infty^*$.

Theorem2.4. In $\|.,.\|_\infty^*$, if δ_m is convergent, then δ_m converges to δ in $\|.\|_\infty$.

Proof:

In $\|.,.\|_\infty^*$, suppose $\delta_m \in L_\infty$ and converges to $\delta \in L_\infty$. Then, $\forall \varepsilon > 0$, there is $\tau \in \mathbb{N}$ where, $\forall m \geq \tau$, we get

$$\sup_{\varkappa_r} . \sup_{\varkappa_k} \begin{vmatrix} \delta_m(\varkappa_r) - \delta(\varkappa_r) & \delta_m(\varkappa_k) - \delta(\varkappa_k) \\ \beta(\varkappa_r) & \beta(\varkappa_k) \end{vmatrix} . \begin{vmatrix} \beta(\varkappa_r) & \beta(\varkappa_k) \\ \sigma(\varkappa_r) & \sigma(\varkappa_k) \end{vmatrix} < \varepsilon$$

$\forall \beta \in L_\infty$ and $\sigma, \alpha \in L_1$ with $\|\sigma\|_1, \|\alpha\|_1 \leq 1$. In fact ,if $\alpha = \{1,0,0, \dots.\}, \alpha = \alpha(\varkappa_r)$ with $\alpha(\varkappa_r) = \frac{sgn(\delta_m(\varkappa_r) - \delta(\varkappa_r))|\delta_m(\varkappa_r) - \delta(\varkappa_r)|}{\|\delta_m - \delta\|_\infty}$ and $\sigma = \{1,0,0, \dots.\}$, then we get

$\sum_{r=2}^{\infty} \frac{|\delta_m(\varkappa_r) - \delta(\varkappa_r)|}{\|\delta_m - \delta\|_\infty} < \epsilon$, $\ni \|\delta_m - \delta\|_\infty \neq 0$. If $\beta = \{0,1,0,0, \dots.\}, \alpha = \{\alpha(\varkappa_1), 0,0, \dots\}$ with $\alpha(\varkappa_1) = \frac{sgn(\delta_m(\varkappa_1) - \delta(\varkappa_1))|\delta_m(\varkappa_1) - \delta(\varkappa_1)|}{\|\delta_m - p\|_\infty}$ and $\sigma = \{0,1,0,0, \dots.\}$,then we get $\frac{|\delta_m(\varkappa_1) - \delta(\varkappa_1)|}{\|\delta_m - \delta\|_\infty} < \varepsilon$. Adding up ,we get

$\|\delta_m - \delta\|_\infty = \sum_{r=1}^{\infty} \frac{|\delta_m(\varkappa_r) - \delta(\varkappa_r)|}{\|\delta_m - \delta\|_\infty} < 2\varepsilon.$

Theorem2.5. In $\|.,.\|_\infty^*$, if δ_m is convergent , then δ_m is convergent in $\|.,.\|_\infty$.

Proof:

Through theorem 2.4**,** the convergence in $\|.,.\|_\infty^*$ leads to the convergence in $\|.\|_\infty$ **and through** Fact 2.1**.**, the convergence in $\|.\|_\infty$ leads to the convergence in $\|.,.\|_\infty$ **, thus** the convergence in $\|.,.\|_\infty^*$ leads to convergence in $\|.,.\|_\infty$.

Corollary 2.6. In $\|.,.\|_\infty^*$, δ_m is convergent iff δ_m is convergent in $\|.,.\|_\infty$.

Now , we can expanded all these results to n-normed spaces for any $n \geq 2$.

Fact 2.7. $\forall \delta_1, \dots, \delta_m \in L_\infty$, then $\|\delta_1, \dots, \delta_m\|_\infty^* \leq n! \|\delta_1, \dots, \delta_m\|_\infty$.

Corollary 2.8. In $\|., \dots, .\|_\infty$, if δ_m is convergent , then δ_m is also convergent in $\|., \dots, .\|_\infty^*$.

Theorem 2.9. In $\|., \dots, .\|_\infty^*$,if δ_m is convergent, then δ_m is also convergent in $\|.\|_\infty$.

Proof:

Suppose $\delta_{1m} \in L_\infty$ and converges to $\delta_1 = \{\delta_1(\varkappa_1), \delta_1(\varkappa_2), \dots\}$ in $\|., \dots, .\|_\infty^*$. Then , $\forall \varepsilon > 0$,there is $\tau \in \mathbb{N} \ni$ each $m \geq \tau$ we get

$$\sup_{\varkappa_{r_1}} . \sup_{\varkappa_{r_2}} \dots \sup_{\varkappa_{r_n}} \begin{vmatrix} \delta_{1m}\left(\varkappa_{r_1} - \delta_1(\varkappa_{r_1})\right) & \dots.. & \delta_{1m}\left(\varkappa_{r_n} - \delta_1(\varkappa_{r_n})\right) \\ \vdots & \ddots & \vdots \\ \delta_n(\varkappa_{r_1}) & \dots & \delta_n(\varkappa_{r_n}) \end{vmatrix} \begin{vmatrix} \alpha_1(\varkappa_{r_1}) & \dots & \alpha_1(\varkappa_{r_n}) \\ \vdots & \ddots & \vdots \\ \alpha_n(\varkappa_{r_1}) & \dots & \alpha_n(\varkappa_{r_n}) \end{vmatrix}$$

$$< \varepsilon$$

$\forall \delta_2, \dots, \delta_n \in L_\infty$ and $\alpha_1, \dots, \alpha_n \in L_1$ with $\|\alpha_1\|_1, \dots, \|\alpha_n\|_1 \leq 1$. If $\delta_k = \alpha_k = \{0, \dots, 0,1,0, \dots\}$ $\forall$ $k = 2, \dots., n$, $\ni$ 1 *is* $(n + 1 - k)-$ th term and $\alpha_1 = \{\alpha_1(\varkappa_1), \alpha_1(\varkappa_2), \dots\} \in L_1$ with $\alpha_1(\varkappa_r) \coloneqq \frac{sgn(\delta_{1m}(\varkappa_r) - \delta_1(\varkappa_r))|\delta_{1m}(\varkappa_r) - \delta_1(\varkappa_r)|}{\|\delta_{1m} - \delta_1\|_\infty}$, then we get $\sum_{r_1=n}^{\infty} \frac{|\delta_{1m}(\varkappa_{r_1}) - \delta_1(\varkappa_{r_1})|}{\|\delta_{1m} - \delta_1\|_\infty} < \varepsilon.$

if $\delta_k = \alpha_k = \{0, \dots, 0,1,0, \dots\}$ for every $k = 2, \dots, n$, $\ni$ 1 *is* $k-$ th term, and $\alpha_1 = \{\alpha_1(\varkappa_1), 0,0, \dots\}$ with $\alpha_1(\varkappa_1) = \frac{sgn(\delta_{1m}(\varkappa_1) - \delta_1(\varkappa_1))|\delta_{1m}(\varkappa_1) - \delta_1(\varkappa_1)|}{\|\delta_{1m} - \delta_1\|_\infty}$, we get

$$\frac{|\delta_{1m}(\varkappa_1) - \delta_1(\varkappa_1)|}{\|\delta_{1m} - \delta_1\|_\infty} < \varepsilon.$$

If we transform the situation of the entry 1 in δ_k and α_k for $k = 2, \dots\dots, n,$ and change the non-zero entry of α_1 based on that , we get

$$\frac{|\delta_{1m}(\varkappa_2) - \delta_1(\varkappa_2)|}{\|\delta_{1m} - \delta_1\|_\infty} < \varepsilon$$

And so on until

$$\frac{|\delta_{1m}(\varkappa_{n-1}) - \delta_1(\varkappa_{n-1})|}{\|\delta_{1m} - \delta_1\|_\infty} < \varepsilon.$$

Adding up ,we have

$$\|\delta_{1m} - \delta_1\|_\infty = \sum_{r_1=1}^{\infty} \frac{|\delta_{1m}(\varkappa_{r_1}) - \delta_1(\varkappa_{r_1})|}{\|\delta_{1m} - \delta_1\|_\infty} < n\varepsilon .$$

REFERENCES

[1] Gählher, S. , Untersuchungen über verallgemeinerte m-metrische Räume.I, Math. Nachr. 40 ,165-189, (1969).

[2] Gählher, S., Untersuchungen über verallgemeinerte m-metrische Räume.II ,Math. Nachr. 40, 229-264,(1969).

[3] Gählher, S., Untersuchungen über verallgemeinerte m-metrische Räume.III , Math. Nachr. 41 ,23-26, (1970).

[4] Gunawan, H. , The space of p-summable sequences and its natural n-norms, Bull. Austral. Math. Soc. 64, 137-147, (2001).

[5] Kavyasree, P.R. and Surender Reddy, B. , Some Results of Double Sequences in 2-Normed and n- Normed Spaces, Appl. Appl. Math. J. 4,97-105,(2018).

[6] Kim, S.S. and Cho, Y.J., Strict convexity in linear n-normed spaces, Demonstratio Math. 29, 739-744, (1996).

[7] Malceski, A. , Strong n-convex n-normed spaces , Mat. Bilten 21, 81-102, (1997).

[8] Mutaqin, A. and Gunawan, H., Equivalence of n- norms on the space of p- summable sequences , J. Indones. Math. Soc. 1, 1-7. (2010).