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On Dispersive and Completely Unstable Random Dynamical Systems

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Abstract. *The desired goal of this work is to introduce and study the concept of* ***dispersivity*** *of RDS's when the sate space is locally compact separable metric space and characterize dispersive in terms of the prolongational limit Set and prolongation . The concept of completely unstable RDS is also studied in this work.*

Keywords: Random dynamical systems, , dispersive, non-wandering point, wandering point, completely unstable.

Introduction. The problem of dispersive and (deterministic) dynamical systems has a long history see [3,4] and the reference with in. The completely unstable is studied in detail in many literatures for example in [10] and the references within and in [6,7]

Our paper is divided into three sections: in section 1 we stated some general facts about RDS. In section 2 we study the prolongational limit set, prolongation of a random set, and dispersive RDS. In Section 3 the completely unstable RDS is introduced and studied.

Through this work (unless otherwise stated) we assume that $\mathbb{T}$ be a topological group and P be a replete semi group of the time space $\mathbb{T}$ and K be any a non-empty compact subset of P.

1. Primaries.

Definition 1.1[2,5]: Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A **metric dynamical system** (MDS) is the 5-tuple $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ such that

(i) $\theta: \mathbb{T} \times \Omega \to \Omega$ is $(\mathcal{H}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$ −measurable,

(ii) $\theta(e, \omega) = \omega$,

(iii) $\theta(t+s, \omega) = \theta(t, \theta(s, \omega))$ and

(iv) $\mathbb{P}(\theta_t F) = \mathbb{P}(F)$, for every $F \in \mathcal{F}$, $t \in \mathbb{T}$.

Definition 1.2[2,5]: Let $(X, \mathcal{B}(X))$ be the measurable space and $\varphi: \mathbb{T} \times \Omega \times X \to X$ be a measurable function with the following properties:

The function $\varphi(e, \omega): X \to X$ is the identity function , and

$$\varphi(t+s, \omega) = \varphi(t, \theta(s)\omega) \circ \varphi(s, \omega) \text{ for all } s, t \in \mathbb{T}, \omega \in \Omega.$$

Then $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called measurable dynamical system and is denoted by (θ, φ). If the function $\varphi(\cdot, \omega, \cdot): \mathbb{T} \times X \to X$, $(t, x) \mapsto \varphi(t, \omega, x)$, is continuous for each $\omega \in \Omega$, then (θ, φ) is called continuous or just random dynamical system (briefly RDS).

Theorem 1.3[2,5]: Let (θ, φ) be a measurable RDS. Then $\varphi(t, \omega): X \to X$ is a bimeasurable bijection and

$$\varphi(t, \omega)^{-1} \coloneqq \varphi(-t, \theta(t)\omega) \text{ for all } (t, \omega) \in \mathbb{T} \times \Omega,$$

or, equivalently,

$$\varphi(-t, \omega) = \varphi(t, \theta(t)^{-1}\omega)^{-1} \text{ for all } (t, \omega) \in \mathbb{T} \times \Omega.$$

Moreover, the mapping $(t, \omega, x) \longmapsto \varphi(t, \omega)^{-1}x$ is measurable.

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Definition 1.4[3,5,10]:

(a) A **random set** is a multifunction $A: \Omega \to \mathcal{B}(X)$ (where $\mathcal{B}(X)$ be a Borel $\sigma -$ algebra on a metric space (X, d)), with the property that for each $x \in$ X the function $\omega \longmapsto d\big(x, A(\omega)\big)$ is measurable. The random set called closed (compact) if for all $\omega \in \Omega$, the set $A(\omega)$ is closed (compact) set in X.

(b) Let $M: \Omega \longmapsto \mathcal{B}(X)$ be a random set . Then the multifunction

$$\gamma_M^P(\omega) \coloneqq \bigcup_{\tau \in P-K} \psi(\tau, \theta_{-\tau}\omega) M(\theta_{-\tau}\omega)$$

is called the $P -$trajectory of M [1].

Definition 1.5 [2,5]: A ***random fixed point*** for (θ, φ) is a random variable $v \in X_{\mathcal{B}}^{\Omega}$ with the property that

$$\mathbb{P}\{\omega: \varphi(t,\omega)v(\omega) = v(\theta_t\omega), \text{for every } t \in \mathbb{T}\} = 1.$$

Lemma 1.6: If $y \in \gamma_x^\tau(\omega)$, then

$$\gamma_x^\tau(\omega) = \gamma_y^\tau(\theta_{-t}\omega), \text{ for some } t \in P - K.$$

Definition 1.7[1]: The random set M that satisfy the condition

$$\mathbb{P}\{\omega: \varphi(\tau,\omega)M(\omega) \subseteq M(\theta_\tau\omega), \text{for every } \tau \in P - K\} = 1,$$

or equivalently

$$\varphi(\tau,\omega)x \in M(\theta_\tau\omega), \text{ for all } x \in M(\omega), \text{ and } \tau \in P - K,$$

is called $P -$invariant.

2. Prolongational Limit Set, Prolongation of a Random Set and Dispersive RDS:

The concepts of prolongational limit set and prolongation are introduced in [8] by means of sequences . Here we will get some essential properties of such concepts. Also the concept of dispersive RDS is introduced and studied.

Definition 2.1[1]: Let M be a random set. **The *P* −omega limit** set is a set

$$\Gamma_M^P(\omega) = \big\{y \in X: \exists \text{ net } \{t_\lambda\} \in P, t_\lambda \longrightarrow \infty, \{x_\lambda\} \in M\big(\theta_{-t_\lambda}\omega\big) \ni \psi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \longrightarrow y\big\}$$

Definition 2.2: Let M be a random set. The ***P −prolongational limit set of*** M is a set

$$J_M^P(\omega) \coloneqq \big\{y \in X: \exists \text{ nets } \{x_\lambda\}, \{t_\lambda\}, t_\lambda \longrightarrow +\infty \ni d(x_\lambda, M(\theta_{-t_\lambda}\omega)) \longrightarrow 0, \varphi\big(t_\lambda, \theta_{-t_\lambda}\omega\big)x_\lambda \longrightarrow y\big\}$$

Definition 2.3: Let M be a random set. The ***P* −prolongation of a random set** M is set

$$D_M^P(\omega) \coloneqq \big\{y \in X: \exists \text{ nets } \{x_\lambda\}, \{t_\lambda\} \ni d(x_\lambda, M(\theta_{-t_\lambda}\omega)) \to 0, \varphi\big(t_\lambda, \theta_{-t_\lambda}\omega\big)x_\lambda \to y\big\}$$

Theorem 2.4: The set $\Gamma_M^P(\omega)$, $J_M^P(\omega)$ and $D_M^P(\omega)$ are closed and are $P -$ invariant if M is $P -$ invariant.

Proof: To show that $\Gamma_M^P(\omega)$ is closed and $P -$ invariant see [1]. It is clear that, $J_M^P(\omega)$ is closed set. To show It is forward invariant. Let $x \in J_M^P(\omega)$ and let $p \in P$. So there are two nets $\{x_\lambda\}$ and $\{p_\lambda\}$ in X and in P respectively such that

$$p_\lambda \longrightarrow +\infty \quad , \quad x_\lambda \longrightarrow M(\theta_{-p_\lambda}\omega) \text{ and } \varphi\big(p_\lambda, \theta_{-p_\lambda}\omega\big)x_\lambda \longrightarrow x\ .$$

From continuity of φ , we have $\varphi(p,\omega) \circ \varphi\big(p_\lambda, \theta_{-p_\lambda}\omega\big)x_\lambda \longrightarrow \varphi(p,\omega)x$. By the cocycle property we have

$$\varphi\big(p + p_\lambda, \theta_{-(p+p_\lambda)}\theta_t\omega\big)x_\lambda \longrightarrow \varphi(t,\omega)x.$$

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Also since $x_\lambda \longrightarrow M(\theta_{-p_\lambda}\omega)$, then $x_\lambda \longrightarrow M(\theta_{-(p+p_\lambda)}\omega)$, then $\varphi(t,\omega)x \in J_M^+(\theta_t\omega)$.

Thus $J_M^P(\omega) \subset J_M^P(\theta_p\omega)$ for every $p \in P$.

Let $x \in J_M^P(\theta_p\omega)$ and let $p \in P$. So there are two nets $\{x_\lambda\}$ and $\{p_\lambda\}$ in X and in P respectively such that $x_\lambda \longrightarrow M(\theta_{-p_\lambda}\theta_p\omega)$ and $\varphi(p_\lambda, \theta_{-p_\lambda}\theta_p\omega)x_\lambda \longrightarrow x$. So

$$x_\lambda \longrightarrow M(\theta_{-p_\lambda+p}\omega) \text{ and } \varphi(p,\omega)z_\lambda \longrightarrow x, \qquad (*)$$

where $z_\lambda = \varphi(p_\lambda - p, \theta_{-p_\lambda+p}\omega)x_\lambda$

Moreover Definition(**2.2**) implies that $b \in J_M^P(\omega)$. From (*) we obtain that $x = \varphi(p,\omega)b$. Therefore for all $p \in P$ and $\omega \in \Omega$, $J_M^P(\theta_p\omega) \subset \varphi(p,\omega)J_M^P(\omega)$. Thus is $J_M^P(\omega)$ invariant.

Theorem 2.5: For any random set M in RDS (θ,φ) we have

$$\mathrm{D}_M^P(\omega) = \gamma_M^P(\omega) \cup J_M^P(\omega).$$

Proof. $\gamma_M^P(\omega) \cup J_M^P(\omega) \subset \mathrm{D}_M^P(\omega)$.Now, let $y \in \mathrm{D}_M^P(\omega)$. By definitions there are two nets $\{t_\lambda\}$ in P and $\{x_\lambda\}$ in $M(\theta_{-t_\lambda}\omega)$, $d(x_\lambda, M(\theta_{-t_\lambda}\omega)) \longrightarrow 0$ and $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \longrightarrow y$. Suppose that that either $t_\lambda \longrightarrow t \in P$ or $t_\lambda \longrightarrow +\infty$. In the first situation $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \longrightarrow \varphi(t, \theta_{-t}\omega)x$. Since the limit is unique we have $\varphi(t,\theta_{-t}\omega)x = y \in \gamma_M^P(\omega)$. In the second case $y \in J_M^P(\omega)$ by definitions of $J_M^P(\omega)$. Thus $y \in \gamma_M^P(\omega) \cup J_M^P(\omega)$. Hence $\mathrm{D}_M^P(\omega) = \gamma_M^P(\omega) \cup J_M^P(\omega)$.
■

Theorem 2.6: For every t in $\mathbb{T}$ and $\omega \in \Omega$. Then $y \in J_x^P(\omega)$ if and only if $x \in J_y^{P^{-1}}(\omega)$, where $y = \varphi(t,\theta_{-t}\omega)x$.

Proof. Let $y \in J_x^P(\omega)$. So there are two nets $\{t_\lambda\}$ in P and $\{x_\lambda\}$ in X such that $x_\lambda \longrightarrow x$, $t_\lambda \longrightarrow +\infty$ and $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \longrightarrow y$. Set $\tau_\lambda \coloneqq -t_\lambda$ and $y_\lambda \coloneqq \varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda$. Then $\{\tau_\lambda\}$ is a net in P^{-1} ,$\tau_\lambda \longrightarrow -\infty$ and $\{y_\lambda\}$ is a net in X , $y_\lambda \longrightarrow y$. Now

$$\begin{aligned} d\big(\varphi(\tau_\lambda,\theta_{-\tau_\lambda}\omega)y_\lambda, x\big) &= d\big(\varphi(\tau_\lambda,\theta_{-\tau_\lambda}\omega)\circ\varphi(t_\lambda,\theta_{-t_\lambda}\omega)x_\lambda, x\big) \\ &= d\big(\varphi(\tau_\lambda,\theta_{-\tau_\lambda}\omega)\circ\varphi(t_\lambda,\theta_{-t_\lambda}\omega)x_\lambda, \varphi(\tau_\lambda,\theta_{-\tau_\lambda}\omega)y\big), \\ &= d\big(\varphi(t_\lambda,\theta_{-t_\lambda}\omega)x_\lambda, y\big) \longrightarrow 0 \end{aligned}$$

Then we have $\varphi(\tau_\lambda, \theta_{-\tau_\lambda}\omega)y_\lambda \longrightarrow x$ .Thus $x \in J_y^{P^{-1}}(\omega)$.

Similarly we can prove the converse. ■

Definition 2.7: Let $A, B \subset X$ be two random sets. We say that A is ***P*** **−recursive** with respect to B if for any compact subset K in $\mathbb{T}$ there is a $t \in P - K$ and an $x \in B(\omega)$ such that

$$\mathbb{P}\{\omega: \varphi(t,\theta_{-t}\omega)x \in A(\omega)\} = 1,$$

or equivalently,

$$\mathbb{P}\{\omega: \varphi(-t,\omega)A(\omega) \cap B(\omega) \neq \emptyset\} = 1.$$

A set A is called self P −recursive if

$$\mathbb{P}\{\omega: \varphi(-t,\omega)A(\omega) \cap A(\omega) \neq \emptyset\} = 1$$

Definition 2.8: We say that a point x in X is ***P −non-wandering*** if for every random neighborhood U of x we have
$$\mathbb{P}\{\omega: \varphi(-t,\omega)U(\omega) \cap U(\omega) \neq \emptyset\} = 1.$$

Theorem 2.9: The following are equivalent for any $x \in X$:
(a) x is P −non-wandering,
(b) $x \in J_x^P(\omega)$,
(c) every neighborhood of x is self P^{-1} − recursive,
(d) $x \in J_x^{P^{-1}}(\omega)$.
Proof. Suppose (a). Consider a null net $\{r_\lambda\}$ of random variables, $0 < r_\lambda, r_\lambda \longrightarrow 0$, and a sequence $\{t_\lambda\}$ in $\mathbb{R}$ with $t_\lambda \longrightarrow +\infty$. Since each $S(x, r_\lambda(\omega))$ is self positively recursive, we have an $x_n \in S(x, r_n(\omega))$ and a $\tau_\lambda > t_\lambda$ with $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \in S(x, r_\lambda(\omega))$. Since $r_\lambda \longrightarrow 0$ we have $x_\lambda \longrightarrow x$ and $\varphi(\tau_\lambda, \theta_{-\tau_\lambda}\omega)x_\lambda \longrightarrow x$ and since $\tau_\lambda \longrightarrow +\infty$ we conclude $x \in J_x^+(\omega)$. Thus (b) holds. Now suppose (a). Then there exists a net $\{x_\lambda\}$ in X and a net $\{t_\lambda\}$ in $\mathbb{T}$ with $x_\lambda \longrightarrow x$ and $t_\lambda \longrightarrow +\infty$ such that $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \longrightarrow x$. Now for any random neighborhood $U(\omega)$ of x the net x_λ is eventually in $U(\omega)$ for every ω and $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda$ eventually in $U(\omega)$ for every ω. So the point x is non-wandering .

(c) if and only if (d). Is proved in the same way.

(b) if and only if (d): by Theorem (2.6). ■

Theorem 2.10: Let $A \subset X$. every limit point x of A is P −non-wandering whenever either $x \in \Gamma_x^P(\omega)$ or $x \in \Gamma_x^{P^{-1}}(\omega)$.

Proof: Let $\{x_\lambda\}$ be a net in A with $x_\lambda \longrightarrow x$. To show that $x \in J_x^P(\omega)$. In fact for each , either $x_\lambda \in \Gamma_{x_\lambda}^P(\omega)$ or $x_\lambda \in \Gamma_{x_\lambda}^{P^{-1}}(\omega)$. Assume $x_\lambda \in \Gamma_{x_\lambda}^P(\omega)$ for all λ. Then

$$d(x_\lambda, \varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda) \longrightarrow 0.$$

Then clearly

$$d(x, \varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda) < d(x, x_\lambda) + d(x_\lambda, \varphi\big(t_\lambda, \theta_{-t_\lambda}\omega\big)x_\lambda) \longrightarrow 0.$$

This shows that $\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x_\lambda \longrightarrow x$ and so $x \in J_x^P(\omega)$. In the second situation analogous attentions prove that $x \in J_x^{P^{-1}}(\omega)$. So from Definition 2.7 we get our result.

Definition 2.11: A random variable $x \in X^\Omega$ is called ***P* −wandering** of (θ, φ) if

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \cap B(\omega) = \emptyset, \text{ for } t \in P - K,$$

for some random ball B centered at x and some compact subset K of P.

Definition 2.12: We say that the RDS (θ, φ) is ***P* −dispersive** if for any $x, y \in X^\Omega$ there exist random balls U, V centered at x, y respectively and a compact subset K of P such that

$$\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega) \cap V(\omega) = \emptyset, \text{ for } t \in P - K.$$

Theorem 2.13: The necessarily and sufficiently conditions for the RDS (θ, φ) to be P −dispersive is , $J_x^P(\omega) = \emptyset$ for every $x \in X$.

Proof. Suppose that (θ, φ) be dispersive RDS. Assume if possible that $x \in X$ with $J_x^P(\omega) \neq \emptyset$, then there exist $y \in J_x^P(\omega)$. So, there are nets $\{x_\lambda\}$ in X, and $\{p_\lambda\}$ in P such that $x_\lambda \longrightarrow x$, $p_\lambda \longrightarrow +\infty$, and $\varphi(p_\lambda, \theta_{-p_\lambda}\omega)x_\lambda \longrightarrow y$ for all ω. This shows that for any neighborhoods U_x, U_y of x and y respectively $\varphi(p_\lambda, \theta_{-p_\lambda}\omega)U_x \cap U_y \neq \emptyset$ as the element $\varphi\big(p_\lambda, \theta_{-p_\lambda}\omega\big)x_\lambda = y_\lambda$ is contained in this intersection. Since $p_\lambda \longrightarrow +\infty$, this contradicts the definition of a P −dispersive flow as U_y is positively recursive relative to U_x. Hence $J_x^P(\omega) = \emptyset$, for each $x \in X$. Conversely, for each $x \in X$ assume that $J_x^P(\omega) = \emptyset$. We assertion that for $x, y \in X$ there are neighborhoods U_x of x and U_y of y and a compact neighborhood K subset of P such that $\varphi(t,\theta_{-t}\omega)U_x \cap U_y = \emptyset$ for all $t \in P - K$. For if not, then there will be nets $\{x_\lambda\}, \{p_\lambda\}$, $x_\lambda \longrightarrow x$, $y_\lambda = \varphi(p_\lambda, \theta_{-p_\lambda}\omega)x_\lambda$, $x_\lambda \longrightarrow y$, and $p_\lambda \longrightarrow +\infty$ for, so that $y \in J_x^P(\omega)$. This is absurd as $J_x^P(\omega) = \emptyset$.

Theorem 2.14: The RDS (θ,φ) is P −dispersive if and only if $D_x^P(\omega) = \gamma_x^P(\omega)$, for each $x \in X$ and there are no random fixed points or random periodic trajectories , i.e. $\gamma_x^P(\omega) = J_x^P(\omega)$ for every $x \in X$.

Proof: Suppose that (θ,φ) is $P -$ dispersive, so $J_x^P(\omega) = \emptyset$ for each $x \in X$. Consequently

$$D_x^P(\omega) = \gamma_x^P(\omega) \cup J_x^P(\omega) = \gamma_x^P(\omega)$$

for each x, hence we have no random fixed points or random periodic trajectories. For, whenever x is a random fixed point or $\gamma_x^P(\omega)$ is periodic, so $\gamma_x^P(\omega) = \Gamma_x^P(\omega) \subset J_x^P(\omega)$.

Conversely, if $D_x^P(\omega) = \gamma_x^{\mathbb{T}}(\omega) = \gamma_x^{P^{-1}P}(\omega)$ (note that, if P is replete semigroup, then $\mathbb{T} = P^{-1}P$) so we have no random fixed points or random periodic trajectories, then $J_x^P(\omega) = \emptyset$. For indeed

$$D_x^P(\omega) = \gamma_x^{P^{-1}P}(\omega) \cup J_x^P(\omega) = \gamma_x^{P^{-1}P}(\omega),$$

implies that $J_x^P(\omega) \subset \gamma_x^{P^{-1}P}(\omega)$, $J_x^P(\omega)$ being closed and invariant, we conclude that if $J_x^P(\omega)$ is not empty, then $\gamma_x^{P^{-1}P}(\omega) \subset J_x^P(\omega) \subset \gamma_x^P(\omega)$, i.e. $\gamma_x^{P^{-1}P}(\omega) = \gamma_x^P(\omega)$. This shows that if $\tau \in P^{-1}$ is arbitrary, then there is a $\tau' \in P$ such that

$$\varphi(\tau,\theta_{-\tau}\omega)x = \varphi(\tau',\theta_{-\tau'}\omega)x.$$

Then

$$x = \varphi(\tau' - \tau, \omega')x, \theta_{-\tau'}\omega \coloneqq \omega'.$$

Since $\tau' - \tau \in P$, the last equality shows that $\gamma_x^{\mathbb{T}}(\omega)$ is closed with period $\tau' - \tau$. Since we assumed that there are no random fixed points or periodic random trajectories, we have arrived at a contradiction. Thus $J_x^P(\omega) = \emptyset$ for each $x \in X$, and the dynamical system is dispersive. ■

3. Completely Unstable RDS:

The completely unstable systems whose properties are just the opposite of those for stable systems. Here the concept of completely unstable RDS is introduced and some essential properties that are related with our principle aim are proved.

Definition 3.1: An RDS (θ,φ) is called:

(a) $\boldsymbol{P} -$ **Lagrange stable** if $\gamma_x^P(\omega)$ lies in a compact set of the space X.

(b) $\boldsymbol{P} -$ **Lagrange unstable** if it is not $P -$ Lagrange stable .

Definition 3.2: An RDS (θ,φ) is said to be P −**completely unstable** if every $x \in X^\Omega$ is P −wandering.

Proposition 3.3: If (θ,φ) is P −completely unstable, then it is $P -$ Lagrange unstable .

Proof: If (θ,φ) $P -$ Lagrange stable at some x_0, then the set $\Gamma_{x_0}^P(\omega)$ is nonempty. Each point $y \in \Gamma_{x_0}^P(\omega)$ is P −nonwandering. In fact, for any tempered random variable $\varepsilon > 0$ consider $B(y,\varepsilon)$ and suppose that $\varphi(t_0,\omega)x_0 \in B(y,\varepsilon)$. According to the definition of an P −omega-limit point, for any compact subset K of P there can be found a $t \in P - K$ such that

$$\varphi(t + t_0, \theta_{-(t+t_0)}\omega)x_0 \in B(y,\varepsilon(\omega)),$$

but

$\varphi(t+t_0,\theta_{-(t+t_0)}\omega)x_0 \in \varphi(t,\theta_{-t}\omega)B(y,\varepsilon(\theta_{-t}\omega))$.

Consequently,

$B(y,\varepsilon(\omega)) \cap \varphi(t,\theta_{-t}\omega)B(y,\varepsilon(\theta_{-t}\omega) \neq \emptyset$,

i.e. y is P −non-wandering point. Thus (θ,φ) is not P −completely unstable. ■

Definition 3.4: An RDS (θ,φ) (with $\mathbb{T}=\mathbb{R}$) is said to have an **improper saddle point** if for some sequence $\{x_n\}$ in X and a divergent increasing sequences $\{t_n\}$, $\{\tau_n\}$ in $\mathbb{R}^+$ with $t_n > \tau_n > 0$ such that

$x_n \longrightarrow x$, $\varphi(t_n,\theta_{-t_n}\omega)x_n \longrightarrow y$ and $\{\varphi(\tau_n,\theta_{-\tau_n}\omega)x_n\}$

contains no convergent subsequence.

Lemma 3.5: If (θ,φ) is $P-$ Lagrange unstable without improper saddle point and $x_n \longrightarrow x$, $y_n \coloneqq \varphi(t_n,\theta_{-t_n}\omega)x_n \longrightarrow y$, then $\{t_n\}$ is bounded.

Proof. We let $A_n = \{\varphi(t,\theta_{-t}\omega)x_n : 0 < t < t_n\}$ and $A = \bigcup_{n=1}^{\infty} A_n$. To prove that A is compact in X. Suppose if possible that there exists a sequence $\{y_k\} \subset A$ without convergent subsequence. By compactness of A_n , then $\{y_{k_1}, y_{k_2}, \dots, y_{k_n}\} \subset A_n$. Thus, there are two unbounded, increasing sequences of $\{n_k\}$ and $\{l_k\}$ with

$y_{n_k} \in A_{l_k}$, i. e. , $y_{n_k} = \varphi\left(\tau_{l_k},\theta_{-\tau_{l_k}}\omega\right)x_{l_k}$, $0 < \tau_{l_k} < t_{l_k}$.

But then we have

$x_{l_k} \longrightarrow x$, $\varphi\left(t_{l_k},\theta_{-t_{l_k}}\omega\right)x_{l_k} \longrightarrow y$,

and $\left\{\varphi\left(\tau_{l_k},\theta_{-\tau_{l_k}}\omega\right)x_{l_k}\right\}$ has no convergent sequence, which is contradict the assumption that (θ,φ) admits an improper saddle point. Hence A is compact, so $\bar{A}$ is compact.

Suppose that $\{t_n\}$ is unbounded. Suppose that $t_n \longrightarrow +\infty$. Let $t \geq 0$; choose N with $t_n > t$ for $n > N$. For $n > N$ implies $\varphi(t,\theta_{-t}\omega)x_n \in A$, and because $\varphi(t,\theta_{-t}\omega)x_n \longrightarrow \varphi(t,\theta_{-t}\omega)x$ we obtain

$\varphi(t,\theta_{-t}\omega)x \in \bar{A}$ for any $t > 0$,

i.e. (θ,φ) is $P-$ Lagrange stable, which is a contradiction.

Corollary 3.6: According to the hypothesis of Lemma 3.5 the following relations hold

$t_n \longrightarrow t_0$ and $y = \varphi(t_0,\theta_{-t_0}\omega)x$.

Proof. Suppose that that the sequence $\{t_n\}$ divergent; since its bounded, there are two subsequences $\{t_{n_k}\}$ and $\{t_{n_l}\}$ with

$\lim_{k\to\infty} t_{n_k} = t'$, $\lim_{l\to\infty} t_{n_l} = t''$, $t' \neq t''$.

Then we would have

$$\lim_{k\to\infty} \varphi\left(t_{n_k},\theta_{-t_{n_k}}\omega\right)x_{n_k} = \lim_{k\to\infty} y_{n_k} = \varphi(t',\theta_{-t'}\omega)x = y ;$$

$$\lim_{l\to\infty}\varphi\left(t_{n_l},\theta_{-t_{n_l}}\omega\right)x_{n_l}=\lim_{k\to\infty}y_{n_l}=\varphi(t'',\theta_{-t''}\omega)x=y\ ,$$

i.e.

$$\varphi(t',\theta_{-t'}\omega)x=\varphi(t'',\theta_{-t''}\omega)x,$$

which is incredible. So

$t_n\longrightarrow t_0$ and $y=\varphi\left(t_0,\theta_{-t_0}\omega\right)x.$

Theorem 3.7: If (θ,φ) is $P-$ Lagrange unstable RDS which has no an improper saddle point, then it is $P-$ completely unstable.

Proof: Suppose if possible that ; (θ,φ) is $P-$ Lagrange unstable RDS without an improper saddle point and admits some $P-$nonwandering point x. Consider the sequence $\{T_n\}$ in $\mathbb{R}^+$ and a sequence of tempered random variables $\varepsilon_n:\Omega\longrightarrow\mathbb{R}$ such that

$$T_1<T_2<\cdots<T_n<\cdots,T_n\longrightarrow+\infty,$$

and

$$\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_n<\cdots,\varepsilon_n\longrightarrow 0.$$

So by definition of the $P-$nonwandering point , there are sequences $\{x_n\}$ and $\{t_n\}$ such that

$$t_n>T_n\ ,\ d(x_n,x)\ <\ \varepsilon_n(\omega)\ ,\ d\ (x,\varphi\left(t_n,\theta_{-t_n}\omega\right)x_n)<\ \varepsilon_n(\omega).$$

From this we obtain

$$x_n\longrightarrow x,\varphi\left(t_n,\theta_{-t_n}\omega\right)x_n\longrightarrow\ x,\ t_n\longrightarrow+\infty,$$

which contradicts lemma 3.5.

We close this work with an important result that relate the region of attraction and $J_x^P(\omega)$.

Definition 3.8: A closed random set $M\subset X$.

(a) The **region of attraction** of M is the set

$$A_M(\omega)=\{y\in X\colon d(\varphi(t,\theta_{-t}\omega)y,M(\omega))\longrightarrow 0\ as\ t\longrightarrow+\infty\}$$

(b) If $A_M(\omega)$ is a neighborhood of M ,then we say that the set M is an ***attractor***,

(c) If every neighborhood U of M contains a invariant neighborhood V of M, then we say that M is P ***−stable*** .

(d) If M is $P-$stable and is an attractor, we say that M is P ***−asymptotically stable*** .

Theorem 3.9: If a random set M is closed, $P-$invariant and $P-$asymptotically stable, then for each $x\in A_M(\omega)$, $J_x^P(\omega)\subset M$, and for each $x\in A_M(\omega)-M$,$J_x^{P^{-1}}(\omega)\cap A_M(\omega)=\emptyset$.

Proof. Let, if possible, $x^*\in A_M(\omega)$ and $y\ \in\ J_{x^*}^P(\omega)$, $y\notin\ M$. Set $d(y,M)=\alpha\ (>\ 0)$. Since M is uniformly stable, there is $\delta>\ 0$ such that $\gamma_{S(M,\delta)}^P(\omega)\subset S(M,\alpha/2)$. Since $x^*\in A_M(\omega)$, there is a $T\ >\ 0$ such that $\varphi(T,\theta_{-T}\omega)x^*\in S\ (M,\delta)$. Since $S\ (M,\delta)$ is open, there is an $\eta>0$ such that
$S(\varphi(T,\theta_{-T}\omega)x^*,\eta)\subset\ S(M,\delta)$. For each $x\ \in N$, with $\varphi(T,\theta_{-T}\omega)x\in S\ (\varphi(T,\theta_{-T}\omega)x^*,\eta)$ the set $N\equiv$
$\varphi(-T,\omega)\ S(\varphi(T,\theta_{-T}\omega)x^*,\eta)$ is a neighborhood of x^* and so $\gamma_x^{P-K}(\omega)\subset S(M,\alpha/2)$.

Now since $y \in J^{P}_{x^*}(\omega)$, there are $\{x_n\}$ in X and $\{t_n\}$ in P, $t_n \longrightarrow +\infty$, with $x_n \longrightarrow x^*$, $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y\cdot$ Suppose that, $t_n \in N$, and $t_n > T$. But then $\varphi(t_n, \theta_{-t_n}\omega)x_n \in S\,(M, \alpha/2)$. Thus if $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$, we must have $d(y, M) \leq \alpha/2$. This is a contradiction, as $d(y, M) = \alpha$. Thus $J^{P}_{x^*}(\omega) \subset M$. The second statement resulted since $y \in J^{P^{-1}}_{x}(\omega)$, implies that $x \in J^{P}_{y}(\omega)$. Now let $x \in A_M(\omega) - M$, and assume that $y \in J^{P^{-1}}_{x}(\omega) \cap A_M(\omega)$. Then we have $y \in A_M(\omega)$, $x \in J^{P}_{x}(\omega)$,$x \notin M$, which has already been ruled out.

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