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$\boldsymbol{\delta}$ -Dot Cubic Ideals of BZ-algebra

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Abstract: In this paper, the notions of $\boldsymbol{\delta}$*-dot cubic ideals and* $\boldsymbol{\delta}$*-dot cubic subalgebra s in BZ-algebras are introduced and several properties are investigated. The image and inverse image of them in BZ-algebras are defined and studied.*

Keywords— BZ-algebras, δ-dot cubic ideal s, δ-dot cubic subalgebra s, of δ-dot cubic ideal s, homomorphism of BZ-algebra. product.

1. Introduction

K. Is´eki and S. Tanaka [22] studied ideals and congruences of BCK-algebras. S. M. Mostafa and et al. [26] were introduced a new algebraic structure which is called KUS-algebras and investigated some related properties. The concept of a fuzzy set, was introduced by L.A. Zadeh [30]. O.G. Xi [28] applied the concept of fuzzy set to BCK-algebras and gave some of its properties. Y. B. Jun and et al. [23] were introduced the notion of cubic ideals in BCK-algebras, and they discussed some related properties of it. In [21], A.T. Hameed and et al. introduced the notion of cubic KUS-ideals of KUS-algebra and they were studied the homomorphic of cubic KUS-ideals. In [1], A.T. Hameed and et al. introduced the notion of cubic AT-ideals of AT-algebra and they discussed some related properties of it. In this paper, we introduce the notion of cubic ideal s of BZ-algebra and we study the homomorphic image and inverse image of cubic ideal s of BZ-algebra.

2. Preliminaries

In this section, we give some basic definitions and preliminaries proprieties of ideal s and fuzzy ideal s in BZ-algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1([2-4]) Let X be a set with a binary operation $*$ and a constant 0. Then $(X;*,\ ℶ)$ is called **an BZ-algebra** if the following axioms satisfied: for all k, y, z ∈X ,

(BZ-1) ((k * z) * (y * z)) * (k * y) = ℶ;

(BZ-2) $k * ℶ = k$;

(BZ-3) $k * y = ℶ$ *and* $y * k = ℶ$ implies that $k = y$.

Example 2.2([2-4]) Let X = { ℶ, 1, 2, 3, 4} in which $(*)$ is defined by the following table:

$*$	ℶ	1	2	3	4
ℶ	ℶ	ℶ	ℶ	ℶ	ℶ
1	1	ℶ	1	ℶ	ℶ
2	2	2	ℶ	ℶ	ℶ
3	3	3	1	ℶ	ℶ
4	4	3	4	3	ℶ

Then $(X;*,\ ℶ)$ is an BZ-algebra.

Remark 2.3([2-4]) Define a binary relation ≤ on BZ-algebra $(X;*,ℶ)$ by letting $k \le y$ if and only if $k * y =$ 0.

Proposition 2.4([2-4]) In any BZ-algebra $(X;*,\ ℶ)$, the following properties hold: for all $k, y, z \in X$,

(P-1) $k * ((k * y) * y) = ℶ$;

(P-2) $k * k = ℶ$;

(P-3) $k * (y * z) = y * (k * z)$;

(P-4) $((k * y) * y) * y = k * y$;

(P-5) $(k * y) * ℶ = (k * ℶ) * (y * ℶ)$;

(P-6) $(k * y) * ((z * k) * (z * y)) = ℶ$;

Proposition 2.5([2-4]) Let $(X;*,\ ℶ)$ be an BZ-algebra. X is satisfies for all $k, y, z \in X$,

(P-7) $k \le y$ *implies* $y * z \le k * z$;

(P-8) $k \le y$ *implies* $z * k \le z * y$.

Definition 2.6([2-4]). Let $(X;*,ℶ)$ be an BZ-algebra and let S be a nonempty subset of X. S is called a **subalgebra of** X if $k * y \in S$ whenever $x \in S$ *and* $y \in S$.

Definition 2.7([2-4]). A nonempty subset I of an BZ-algebra $(X;*,ℶ)$ is called **an ideal of $\boldsymbol{X}$** if it satisfies the following conditions: for any $x, y, z \in X$,

(I_1) $ℶ \in I$,

(I_2) $(k * y) \in I$ *and* $k \in I$ *imply* $y \in I$.

Proposition 2.9 ([2-4]). Every ideal of BZ-algebra is a subalgebra.

Proposition 2.8 ([2-4]). Let $\{I_i \mid i \in \Lambda\}$ be a family of ideal s of BZ-algebra $(X;*,\ ℶ)$. The intersection of any set of ideal s of X is also an ideal.

Definition 2.9 ([13,14]). Let $(X;*,ℶ)$ and $(Y;*`,ℶ`)$ be nonempty sets. The mapping $f:(X;*,ℶ) \to (Y;*`,ℶ`)$ is called **a homomorphism** if it satisfies:

$f(k * y) = f(k) *` f(y)$, for all $k, y \in X$. The set $\{k \in X \mid f(k) = ℶ'\}$ is called **the kernel of $\boldsymbol{f}$** denoted by ker f.

Theorem 2.10 ([2-4]). Let $f:(X;*,ℶ) \to (Y;*`,ℶ`)$ be a homomorphism of an BZ-algebra X into an BZ-algebra Y, then:

A. $f(ℶ) = ℶ'$.

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B. f is injective if and only if ker f ={ ℶ}.

C. k ≤ y implies f (k) ≤ f (y).

Theorem 2.11 ([2-4]). Let $f: (X; *, ℶ) \rightarrow (Y; *`, ℶ`)$ be a homomorphism of an BZ-algebra X into an BZ-algebra Y, then:

(F_1) If S is an subalgebra of X, then f (S) is an subalgebra of Y.

(F_2) If I is an ideal of X, then f (I) is an ideal of Y , where f is onto.

(F_3) If H is an subalgebra of Y, then f^{-1} (H) is an subalgebra of X .

(F_4) If J is an ideal of Y, then f^{-1} (J) is an ideal of X .

(F_5) ker f is an ideal of X.

(F_6) Im(f) is a subalgebra of Y.

Definition 2.12([30]). Let $(X; *, ℶ)$ be a nonempty set, a fuzzy subset μ of X is a function $\mu: X \rightarrow [ℶ, 1]$.

Definition 2.13 ([29]). Let X be a nonempty set and μ be a fuzzy subset of $(X; *, ℶ)$, for $t \in [ℶ, 1]$, the set $L(\mu, t) = \mu_t = \{k \in X \mid \mu(k) \geq t\}$ is called **a level subset of μ** .

Definition 2.14([5]). Let $(X; *, ℶ)$ be an BZ-algebra, a fuzzy subset μ of X is called **a fuzzy subalgebra of X** if for all $k, y \in X$,

$\mu(k * y) \geq min\{\mu(k), \mu(y)\}$.

Definition 2.15([5]). Let $(X; *, ℶ)$ be an BZ-algebra, a fuzzy subset μ of X is called **a fuzzy ideal of X** if it satisfies the following conditions, for all $x, y \in X$,

(FBZ_1) $\mu(ℶ) \geq \mu(k)$,

(FBZ_2) $\mu(y) \geq min\{\mu(k * y), \mu(k)\}$.

Proposition 2.17([5]).

1- The intersection of any set of fuzzy ideal s of BZ-algebra is also fuzzy ideal.

2- The union of any set of fuzzy ideal s of BZ-algebra is also fuzzy ideal, where is chain.

Proposition 2.18([5]). Every fuzzy ideal of BZ-algebra is a fuzzy subalgebra.

Proposition 2.19([5]).

1- Let μ be a fuzzy subset of BZ-algebra $(X; *, ℶ)$. If μ is a fuzzy subalgebra of X if and only if for every t ∈ [ℶ,1], μ_t is a subalgebra of X .

2- Let μ be a fuzzy ideal of BZ-algebra $(X;*, ℶ)$, μ is a fuzzy ideal of X if and only if for every t ∈ [ℶ,1], μ_t is an ideal of X .

Lemma 2.20([5]). Let μ be a fuzzy ideal of BZ-algebra X and if $\leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in X$.

Definition 2.21 ([33]). Let $f: (X; *, ℶ) \rightarrow (Y; *`, ℶ`)$be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} sup\{\mu(k): x \in f^{-1}(y)\} & if\ f^{-1}(y) = \{k \in X, f(k) = y\} \neq \emptyset \\ ℶ & otherwise \end{cases}$$

is said to be **the image of μ under f.**

Similarly if β is a fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $k \in X$) is called **the pre-image of β under f** .

Definition 2.22 ([29]). A fuzzy subset μ of a set X has sup property if for any subset T of X , there exist t_0 ∈T such that $\mu(t_ℶ) = sup\{\mu(t) | t \in T\}$.

Proposition 2.23 ([5]). Let $f: (X; *, ℶ) \rightarrow (Y; *`, ℶ`)$ be a homomorphism between BZ-algebras X and Y respectively.

1- For every fuzzy subalgebra β of Y, $f^{-1}(\beta)$ is a fuzzy subalgebra of X.

2- For every fuzzy subalgebra μ of X , $f(\mu)$ is a fuzzy subalgebra of Y.

3- For every fuzzy ideal β of Y, $f^{-1}(\beta)$ is a fuzzy ideal of X.

4- For every fuzzy ideal μ of X with sup property, $f(\mu)$ is a fuzzy ideal of Y, where f is onto.

Now, we will recall the concept of interval-valued fuzzy subsets.

Remark 2.24[1,8]. An interval number is $\tilde{a} = [a^-, a^+]$, where ℶ ≤ a^-≤ a^+≤ 1. Let I be a closed unit interval, (i.e., I = [ℶ, 1]).

Let D[ℶ, 1] denote the family of all closed subintervals of I = [ℶ, 1] , that is, D[ℶ, 1] = { $\tilde{a} = [a^-, a^+] | a^- \leq a^+$, for $a^-, a^+ \in$ I} .

Now, we define what is known as refined minimum (briefly, rmin) of two element in D[ℶ,1].

Definition 2.25[1,7]. We also define the symbols (≽) , (≼) , (=) , rmin and rmax in case of two elements in D[ℶ, 1] . Consider two interval numbers (elements numbers) $\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+]$in D[ℶ, 1] : Then

(1) $\tilde{a} \succcurlyeq \tilde{b}$if and only if, $a^- \geq b^-$ and $a^+ \geq b^+$,

(2) $\tilde{a} \preccurlyeq \tilde{b}$if and only if, $a^- \leq b^-$ and $a^+ \leq b^+$,

(3) $\tilde{a} = \tilde{b}$if and only if, $a^- = b^-$ and $a^+ = b^+$,

(4) rmin {$\tilde{a}$, $\tilde{b}$}= [min {a^-,b^-}, min {a^+,b^+}],

(5) rmax {$\tilde{a}$, $\tilde{b}$}= [max {a^-,b^-}, max {a^+,b^+}],

Remark 2.26 [1,7]. It is obvious that (D[ℶ, 1] , ≼ , ∨ , ∧) is a complete lattice with $\tilde{ℶ}$ = [ℶ, ℶ] as its least element and $\tilde{1}$ = [1, 1] a sits greatest element. Let $\tilde{a}_i \in$ D[ℶ, 1] where i∈Λ .We define $rinf_{i \in \Lambda} \tilde{a}$= [r $inf_{i \in \Lambda} a^-$, $rinf_{i \in \Lambda} a^+$], $rsup_{i \in \Lambda} \tilde{a}$= [$rsup_{i \in \Lambda} a^-$, $rsup_{i \in \Lambda} a^+$].

Definition 2.27[1,7]. An **interval-valued fuzzy subset $\tilde{\mu}_A$ on X** is defined as

$\tilde{\mu}_A$ ={< k, [μ_A^-(k) ,μ_A^+(k)]>| k∈ X} .Where μ_A^-(k) ≤μ_A^+(k), for all k∈ X. Then the ordinary fuzzy subsetsμ_A^-: $X \rightarrow$ [ℶ, 1] and μ_A^+: $X \rightarrow$ [ℶ, 1] are called a **lower fuzzy subset and an upper fuzzy subset** of $\tilde{\mu}_A$respectively.

Let $\tilde{\mu}_A$ (k) = [μ_A^-(k), μ_A^+(k)], $\tilde{\mu}_A: X \rightarrow$ D[ℶ, 1], then A = {< k, $\tilde{\mu}_A$ (k) >| k∈ X } .

Definition 2.28([1,7]). Let (X ;∗, ℶ) be a nonempty set. A cubic set Ω in a structure Ω = {< k, $\tilde{\mu}_\Omega$ (k),λ_Ω (k)>| k∈ X }, which is briefly denoted by Ω =<$\tilde{\mu}_\Omega$, λ_Ω>, where $\tilde{\mu}_\Omega : X \rightarrow$ D[ℶ, 1], $\tilde{\mu}_\Omega$ is an interval-valued fuzzy subset of X and $\lambda_\Omega: X \rightarrow$ [ℶ, 1], λ_Ω is a fuzzy subset of X.

Definition 2.29([1,7]). For a family Ω_i = {⟨ k, $\tilde{\mu}_{\Omega i}$(k)⟩| k ∈ X} on fuzzy subsets of X, where i∈Λ

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and Λ is index set, we define the join ($\vee$)and meet ($\wedge$) operations as follows:

$\vee_{i\in\Lambda}\Omega_i = (\vee_{i\in\Lambda}\tilde{\mu}_{\Omega i})(k) = \sup\{\tilde{\mu}_{\Omega i}(k) | i \in \Lambda\}$,

$\wedge_{i\in\Lambda}\Omega_i = (\wedge_{i\in\Lambda}\tilde{\mu}_{\Omega i})(k) = \inf\{\tilde{\mu}_{\Omega i}(k) | i \in \Lambda\}$,

3. δ-dot Cubic Subalgebras of BZ-algebra

In this section, we will introduce a new notion called cubic subalgebra s of BZ-algebra and study several properties of it.

Definition 3.1[19]. Let $(X;*, ב)$ be an BZ-algebra. A cubic set

$\Omega =< \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ of X is called **cubic subalgebra of** $\boldsymbol{X}$ if, for all $x, y \in X$:

$\tilde{\mu}_\Omega(k*y) \succcurlyeq rmin\{\tilde{\mu}_\Omega(k), \tilde{\mu}_\Omega(y)\}, and\ \lambda_\Omega(k*y) \leq max\{\lambda_\Omega(k), \lambda_\Omega(y)\}$.

Definition 3.2. Let $(X;*, ב)$ be an BZ-algebra. A cubic set $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ of X is called **δ-dot cubic subalgebra of** $\boldsymbol{X}$ if
$\delta \in (ב, 1]$, for all $x \in X$, $\tilde{\mu}^\delta_\Omega = \tilde{\mu}_\Omega(x) \cdot \delta$ and $\lambda^\delta_\Omega = \lambda_\Omega(k) \cdot \delta$.

Example 3.3. Let $X = \{ב, 1, 2, 3\}$ in which the operation as in example $*$ be define by the following table:

*	ב	1	2	3
ב	ב	ב	ב	ב
1	1	ב	ב	ב
2	2	2	ב	ב
3	3	3	3	ב

Then $(X;*,ב)$ is an BZ-algebra. Define a cubic set $\Omega =< \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ of X is fuzzy subset $\mu: X \to [ב, 1]$ by:

$$\tilde{\mu}_\Omega(k) = \begin{cases} [0.3, \ 0.9] & if\ k = \{ב, 1\} \\ [0.1, \ 0.6] & otherwise \end{cases} \quad \text{and}$$

$$\lambda_\Omega = \begin{cases} 0.1 & if x = \{ב, 1\} \\ 0.6 & otherwise \end{cases}.$$

Define a cubic set $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ of X and $\delta = 0.5$ as follows:

$$\tilde{\mu}^\delta_\Omega(k) = \begin{cases} [0.15, \ 0.45] & if x = \{ב, 1\} \\ [0.05, \ 0.3] & otherwise \end{cases} \quad \text{and} \quad \lambda^\delta_\Omega(x) = \begin{cases} 0.05 & if x = \{ב, 1\} \\ 0.3 & otherwise \end{cases}.$$

The δ-dot cubic set $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ is a δ-dot cubic subalgebra of X.

Remark 3.4. Let $(X;*, ב)$ be an BZ-algebra, then it is clearly that

$\Omega^1 =< \tilde{\mu}^1_\Omega(k), \lambda^1_\Omega(k) > = \Omega =< \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$

Proposition 3.5. Let $(X;*, ב)$ be an BZ-algebra and $\Omega =< \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic subalgebra of X such that $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ is δ-dot cubic subalgebra of X, where $\delta \in (ב, 1]$, then for all $x, y \in X$, $\tilde{\mu}_\Omega(k*y) \cdot \delta \succcurlyeq rmin\{\tilde{\mu}_\Omega(k), \tilde{\mu}_\Omega(y)\} \cdot \delta$, and $\lambda_\Omega(k*y) \cdot \delta \leq max\{\lambda_\Omega(k), \lambda_\Omega(y)\} \cdot \delta$.

Proof. For all $k, y \in X$, we have

$$\begin{aligned} \tilde{\mu}^\delta_\Omega(x*y) = \tilde{\mu}_\Omega(x*y) \cdot \delta &\succcurlyeq rmin\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\} \cdot \delta, \\ &= rmin\{\tilde{\mu}_\Omega(x) \cdot \delta, \ \tilde{\mu}_\Omega(y) \cdot \delta\} \\ &= rmin\{\tilde{\mu}^\delta_\Omega(x), \ \tilde{\mu}^\delta_\Omega(y)\} \end{aligned}$$

and

$$\begin{aligned} \lambda^\delta_\Omega(x*y) = \lambda_\Omega(x*y) \cdot \delta &\leq max\{\lambda_\Omega(x), \lambda_\Omega(y)\} \cdot \delta \\ &= max\{\lambda_\Omega(x) \cdot \delta, \lambda_\Omega(y) \cdot \delta\} \\ &= max\{\lambda^\delta_\Omega(x), \lambda^\delta_\Omega(y)\}. \ \triangle \end{aligned}$$

It is clear that δ-dot cubic subalgebra of an BZ-algebra $(X;*, ב)$ is a generalization of a cubic subalgebra of X and a cubic subalgebra of X is special case, when $\delta = 1$.

Proposition 3.6. Let $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ be a δ-dot cubic subalgebra of BZ-algebra $(X;*, ב)$, then $\tilde{\mu}^\delta_\Omega(ב) \succcurlyeq \tilde{\mu}^\delta_\Omega(x)$ and $\lambda^\delta_\Omega(ב) \leq \lambda^\delta_\Omega(k)$, for all

$x \in X$.

Proof. For all $x \in X$, we have

$$\begin{aligned} \tilde{\mu}^\delta_\Omega(ב) &= \tilde{\mu}_\Omega(ב * x) \cdot \delta \\ &\succcurlyeq rmin\{\tilde{\mu}^\delta_\Omega((ב * x) * ב), \tilde{\mu}^\delta_\Omega(x)\} \cdot \delta \\ &= rmin\{[\mu^-_A((ב * x) * ב), \mu^-_A(x)], [\mu^+_A((ב * x) * ב), \mu^+_A(x)]\} \cdot \delta \\ &= rmin\{[\mu^-_A(ב), \mu^-_A(x)], [\mu^+_A(ב), \mu^+_A(x)]\} \cdot \delta \\ &= [\mu^-_A(x), \mu^+_A(x)] \cdot \delta \\ &= \tilde{\mu}_\Omega(x) \cdot \delta \\ &= \tilde{\mu}^\delta_\Omega(x). \end{aligned}$$

Similarly, we can show that

$\lambda^\delta_\Omega(ב) \leq max\{[\lambda^\delta_\Omega(ב), \lambda^\delta_\Omega(x)]\} = \lambda^\delta_\Omega(x)$. $\triangle$

Proposition 3.7. If a δ-dot cubic set $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ of BZ-algebra $(X;*, ב)$ is a δ-dot cubic subalgebra, then $\Omega^\delta(k*y) = \Omega^\delta(k*((y*ב)*ב))$, for all $k, y \in X$.

Proof.

Let X be an BZ-algebra and $x, y \in X$, then we know that $y = (y*ב)*\ 0$. Hence, $\tilde{\mu}^\delta_\Omega(k*y) = \tilde{\mu}^\delta_\Omega(k*((y*ב)*ב))$ and $\lambda^\delta_\Omega(k*y) = \lambda^\delta_\Omega(k*((y*ב)*ב))$. Therefore $\Omega^\delta(x*y) = \Omega^\delta(x*((y*ב)*ב))$. $\triangle$

Proposition 3.8.

Let $(X;*, ב)$ be an BZ-algebra and $\Omega =< \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic subset of X such that $\Omega^\delta =< \tilde{\mu}^\delta_\Omega(k), \lambda^\delta_\Omega(k) >$ is δ-dot cubic subalgebra of , for some $\delta \in [ב, 1]$, then $\Omega =< \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic subalgebra of X.

Proof.

Assume that $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ is a δ-dot cubic subalgebra of X for some $\delta \in (ב,1]$. Let $x, y \in X$, then

$\tilde{\mu}_{\Omega}(x * y) \cdot \delta = \tilde{\mu}^{\delta}_{\Omega}(x * y)$
$\succcurlyeq min\{\tilde{\mu}^{\delta}_{\Omega}(x), \tilde{\mu}^{\delta}_{\Omega}(y)\}$
$= min\{\tilde{\mu}_{\Omega}(x) \cdot \delta, \tilde{\mu}_{\Omega}(y) \cdot \delta\}$
$= min\{\tilde{\mu}_{\Omega}(x), \tilde{\mu}_{\Omega}(y)\} \cdot \delta$.

$\tilde{\mu}_{\Omega}(x * y) \geq min\{\tilde{\mu}_{\Omega}(x), \tilde{\mu}_{\Omega}(y)\}$ and so

$\lambda_{\Omega}(x * y) \cdot \delta = \lambda^{\delta}_{\Omega}(x * y)$
$\leq max\{\lambda^{\delta}_{\Omega}(x), \lambda^{\delta}_{\Omega}(y)\}$
$= max\{\lambda_{\Omega}(x) \cdot \delta, \lambda_{\Omega}(y) \cdot \delta\}$
$= max\{\lambda_{\Omega}(x), \mu(y)\} \cdot \delta$.

$\lambda_{\Omega}(x * y) \leq max\{\lambda_{\Omega}(x), \lambda_{\Omega}(y)\}$

Hence $\Omega =< \tilde{\mu}_{\Omega}(k), \lambda_{\Omega}(k) >$ is a cubic subalgebra of X. □

Proposition 3.9.

Let $(X; *, ב)$ be an BZ-algebra and $\Omega =< \tilde{\mu}_{\Omega}(k), \lambda_{\Omega}(k) >$ is a cubic subset of X such that $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ is δ-dot cubic subalgebra of X, for some $\delta \in [ב, 1]$, then then the cubic Ω of X is a fuzzy S-extension of the δ-dot cubic Ω^{δ} of X.

Proof:

Since $\tilde{\mu}_{\Omega}(x) \succcurlyeq \mu(x). \delta = \tilde{\mu}^{\delta}_{\Omega}(x)$, and $\lambda_{\Omega}(x) \geq \lambda_{\Omega}(x). \delta = \lambda^{\delta}_{\Omega}(x)$ then $\Omega(k)$ is a fuzzy S-extension of $\Omega^{\delta}(k)$, for all $k \in X$ and since Ω is a fuzzy subalgebra of X, then Ω^{δ} of μ is a δ-dot cubic subalgebra, by Proposition (3.8). □

Definition 3.10[19].

For a fuzzy subset μ of an BZ-algebra $(X; *, ב)$, $\delta \in (ב,1]$, $\tilde{t} \in D[ב, 1]$ and $s \in [ב, 1]$, with $t \leq \delta$, let $\widetilde{U}(\Omega; \tilde{t}, s) = \{k \in X \mid \tilde{\mu}_{\Omega}(k) \succcurlyeq \tilde{t}, \lambda_{\Omega}(k) \leq s\}$.

Proposition 3.11.

Let $(X; *, ב)$ be an BZ-algebra. A δ-dot cubic subset $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ of X If Ω^{δ} is a δ-dot cubic subalgebra of X, then for all $\delta \in (ב,1]$, $\tilde{t} \in D[ב, 1]$ and $s \in [ב, 1]$, with $t \leq \delta$, then the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is a subalgebra of X.

Proof.

Assume that $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ is a δ-dot cubic subalgebra of X and let $\tilde{t} \in D[ב, 1]$ and $s \in [ב, 1]$, be such that $\widetilde{U}(\Omega; \tilde{t}, s) \neq \emptyset$.

Let $k, y \in X$ such that $, y \in \widetilde{U}(\Omega; \tilde{t}, s)$, then $\tilde{\mu}^{\delta}_{\Omega}(k) \succcurlyeq \tilde{t}$, $\tilde{\mu}^{\delta}_{\Omega}(y) \succcurlyeq \tilde{t}$ and $\lambda^{\delta}_{\Omega}(k) \leq s$, $\lambda^{\delta}_{\Omega}(y) \leq s$. Since Ω^{δ} is a δ-dot cubic subalgebra of X, we get

$\tilde{\mu}^{\delta}_{\Omega}(k * y) \succcurlyeq rmin\{\tilde{\mu}^{\delta}_{\Omega}(k), \tilde{\mu}^{\delta}_{\Omega}(y)\} \succcurlyeq \tilde{t}$ and $\lambda^{\delta}_{\Omega}(k * y) \leq max\{\lambda^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(y)\} \leq s$.

Hence the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is a subalgebra of X. ⌂

Proposition 3.12. Let $(X; *, ב)$ be an BZ-algebra. A δ-dot cubic subset $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ of . If the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is a subalgebra of X, for all $\delta \in (ב, 1]$, $\tilde{t} \in D[ב, 1]$ and $s \in [ב, 1]$, with $t \leq \delta$, then Ω^{δ} is a δ-dot cubic subalgebra of X.

Proof.

Suppose that $\widetilde{U}(\Omega; \tilde{t}, s)$ is a subalgebra of X and let $x, y \in X$ be such that $\tilde{\mu}^{\delta}_{\Omega}(k * y) \prec rmin\{\tilde{\mu}^{\delta}_{\Omega}(k), \tilde{\mu}^{\delta}_{\Omega}(y)\}$ and $\lambda^{\delta}_{\Omega}(k * y) > max\{\lambda^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(y)\}$.

Consider $\tilde{\delta} = 1/2 \{\tilde{\mu}^{\delta}_{\Omega}(k * y) + rmin\{\tilde{\mu}^{\delta}_{\Omega}(k), \tilde{\mu}^{\delta}_{\Omega}(y)\}\}$ and

$$\delta = 1/2 \{\lambda^{\delta}_{\Omega}(k * y) + max\{\lambda^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(y)\}\}.$$

We have $\tilde{\delta} \in D[ב, 1]$ and $\delta \in (ב, 1]$, and $\tilde{\mu}^{\delta}_{\Omega}(k * y) \prec \tilde{\delta} \prec rmin\{\tilde{\mu}^{\delta}_{\Omega}(k), \tilde{\mu}^{\delta}_{\Omega}(y)\}$, and $\lambda^{\delta}_{\Omega}(k * y) > \delta > max\{\lambda^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(y)\}$.

It follows that $x, y \in \widetilde{U}(\Omega; \tilde{t}, s)$, and $(k * y) \notin \widetilde{U}(\Omega; \tilde{t}, s)$. This is a contradiction and therefore Ω^{δ} is a δ-dot cubic subalgebra of . ⌂

Theorem 3.13. Let $(X; *, ב)$ be an BZ-algebra. A δ-dot cubic subset

$\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ of X is a δ-dot cubic subalgebra of X if and only if, $\mu^{-\delta}_{\Omega}$, and $\mu^{+\delta}_{\Omega}$ are fuzzy subalgebras of X and $\lambda^{\delta}_{\Omega}$ are anti-fuzzy subalgebra of X.

Proof. Let $\mu^{-\delta}_{\Omega}$, $\mu^{+\delta}_{\Omega}$ and $\lambda^{\delta}_{\Omega}$ be fuzzy subalgebras of X and $x, y \in X$, then

$\mu^{-\delta}_{\Omega}(k * y) \geq min\{\mu^{-\delta}_{\Omega}(k), \mu^{-\delta}_{\Omega}(y)\}$, $\mu^{+\delta}_{\Omega}(k * y) \geq min\{\mu^{+\delta}_{\Omega}(k), \mu^{+\delta}_{\Omega}(y)\}$ and

$\lambda^{\delta}_{\Omega}(k * y) \leq max\{\lambda^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(y)\}$. Now,

$\tilde{\mu}^{\delta}_{\Omega}(k * y) = [\mu^{-\delta}_{\Omega}(k * y), \mu^{+\delta}_{\Omega}(k * y)]$
$\succcurlyeq [min\{\mu^{-\delta}_{\Omega}(k), \mu^{-\delta}_{\Omega}(y)\}, min\{\mu^{+\delta}_{\Omega}(k), \mu^{+\delta}_{\Omega}(y)\}]$
$= rmin\{[\mu^{-\delta}_{\Omega}(k), \mu^{+\delta}_{\Omega}(k)], [\mu^{-\delta}_{\Omega}(y), \mu^{+\delta}_{\Omega}(y)]\}$
$= rmin\{\tilde{\mu}^{\delta}_{\Omega}(k), \tilde{\mu}^{\delta}_{\Omega}(y)\}$,

therefore Ω is a δ-dot cubic subalgebra of X.

Conversely, assume that Ω^{δ} is a δ-dot cubic subalgebra of X, for any $k, y \in X$,

$[\mu^{-\delta}_{\Omega}(k * y), \mu^{+\delta}_{\Omega}(k * y)] = \tilde{\mu}^{\delta}_{\Omega}(k * y) \succcurlyeq rmin\{\tilde{\mu}^{\delta}_{\Omega}(k), \tilde{\mu}^{\delta}_{\Omega}(y)\}$
$= rmin\{[\mu^{-\delta}_{\Omega}(k), \mu^{+\delta}_{\Omega}(k)], [\mu^{-\delta}_{\Omega}(y), \mu^{+\delta}_{\Omega}(y)]\}$
$= [min\{\mu^{-\delta}_{\Omega}(k), \mu^{-\delta}_{\Omega}(k), min\{\mu^{+\delta}_{\Omega}(y), \mu^{+\delta}_{\Omega}(y)\}]$.

Thus

$\mu^{-\delta}_{\Omega}(k * y) \geq min\{\mu^{-\delta}_{\Omega}(k), \mu^{-\delta}_{\Omega}(k)\}, \mu^{+\delta}_{\Omega}(k * y)$
$\geq min\{\mu^{+\delta}_{\Omega}(k), \mu^{+\delta}_{\Omega}(k)\}$ and

$\lambda^{\delta}_{\Omega}(k * y) \leq max\{\lambda^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(y)\}$.

Therefore, $\mu^{-\delta}_{\Omega}$ and $\mu^{+\delta}_{\Omega}$ are fuzzy subalgebras of X and $\lambda^{\delta}_{\Omega}$ are anti-fuzzy subalgebra of X. □

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Proposition 3.14.

Let $(X;*, ב)$ be an BZ-algebra and $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic subalgebra of X and $\delta_1, \delta_2 \in (ב, 1]$. If $\delta_2 \geq \delta_1$, then the δ-dot cubic subalgebra Ω^{δ_2} is a fuzzy S-extension of the Ω^{δ_1} δ-dot cubic subalgebra of X.

Proof: For every $k \in X$ and $\delta_1, \delta_2 \in (ב, 1]$ and $\delta_2 \geq \delta_1$, we have

$\tilde{\mu}_\Omega^{\delta_2}(k) = \tilde{\mu}_\Omega(k).\delta_2 \succcurlyeq \tilde{\mu}_\Omega(k).\delta_1 = \tilde{\mu}_\Omega^{\delta_1}(k)$, and

$\lambda_\Omega^{\delta_2}(k) = \lambda_\Omega(k).\delta_2 \geq \lambda_\Omega(k).\delta_1 = \lambda_\Omega^{\delta_1}(k)$, then

$\tilde{\mu}_\Omega^{\delta_2}(k) \succcurlyeq \tilde{\mu}_\Omega^{\delta_1}(k)$, and $\lambda_\Omega^{\delta_2}(k) \geq \lambda_\Omega^{\delta_1}(k)$, therefore Ω^{δ_2} is a fuzzy S-extension of Ω^{δ_1}.

Since Ω is a cubic subalgebra of X, then Ω^δ is a δ-dot cubic subalgebra of μ, by Proposition (3.8). Hence Ω^{δ_2} of X is a fuzzy S-extension of the δ-dot cubic subalgebra Ω^{δ_1} of X. □

4. δ-dot Cubic Ideals of BZ-algebra

In this section, we shall define the notion of δ-dot cubic of ideals, and we study some of the relations, theorems, propositions and examples of δ-dot cubic of ideals of BZ-algebra.

Definition 4.1.

Let $(X;*, ב)$ be an BZ-algebra. A cubic set $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ of X is called **cubic ideal of *X*** if, for all $x, y \in X$:

(1) $\tilde{\mu}_\Omega(ב) \succcurlyeq \tilde{\mu}_\Omega(x)$ and $\lambda_\Omega(ב) \leq \lambda_\Omega(k)\}$,

(2) $\tilde{\mu}_\Omega(y) \succcurlyeq rmin\{\tilde{\mu}_\Omega(k*y), \tilde{\mu}_\Omega(k)\}$ and $\lambda_\Omega(y) \leq max\{\lambda_\Omega(k*y), \lambda_\Omega(k)\}$.

Definition 4.2.

Let $(X;*, ב)$ be an BZ-algebra. A cubic set $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ of X is called **δ-dot cubic ideal of *X*** if it satisfies the following conditions: for all $x, y \in X$:

(1) $\tilde{\mu}_\Omega^\delta(ב) \succcurlyeq \tilde{\mu}_\Omega^\delta(x)$ and $\lambda_\Omega^\delta(ב) \leq \lambda_\Omega^\delta(x)\}$,

(2) $\tilde{\mu}_\Omega^\delta(y) \succcurlyeq rmin\{\tilde{\mu}_\Omega^\delta(k*y), \tilde{\mu}_\Omega^\delta(k)\}$ and $\lambda_\Omega^\delta(y) \leq max\{\lambda_\Omega^\delta(k*y), \lambda_\Omega^\delta(k)\}$.

Example 4.3. Let $X = \{ב, 1, 2, 3\}$ in which the operation as in example $*$ be define by the following table:

*	ב	1	2	3
ב	ב	ב	ב	ב
1	1	ב	ב	ב
2	2	2	ב	ב
3	3	3	3	ב

Then $(X;*, ב)$ is an BZ-algebra. Define a cubic set $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ of X is fuzzy subset $\mu: X \rightarrow [ב, 1]$ by:

$$\tilde{\mu}_\Omega(k) = \begin{cases} [0.3, \ 0.9] & if x = \{ב, 2\} \\ [0.1, \ 0.6] & otherwise \end{cases}$$ and

$$\lambda_\Omega = \begin{cases} 0.1 & if x = \{ב, 2\} \\ 0.6 & otherwise \end{cases}.$$

Define a cubic set $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ of X and $\delta = 0.4$ as follows:

$$\tilde{\mu}_\Omega^\delta(k) = \begin{cases} [0.12, \ 0.32] & if x = \{ב, 2\} \\ [0.04, \ 0.24] & otherwise \end{cases}$$ and $$\lambda_\Omega^\delta = \begin{cases} 0.04 & if x = \{ב, 2\} \\ 0.24 & otherwise \end{cases}.$$

The δ-dot cubic set $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ is a δ-dot cubic ideal of X.

Theorem 4.4.

If $(X;*, ב)$ be an BZ-algebra and $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic ideal of X, then $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ is δ-dot cubic ideal of X, where $\delta \in (ב, 1]$.

Proof :

Assume that Ω is a δ-dot cubic ideal of X and let $\delta \in (ב, 1]$. Then for all $x, y \in$ X.

$\tilde{\mu}_\Omega(ב) = \tilde{\mu}_\Omega(ב).\delta \succcurlyeq \tilde{\mu}_\Omega(k).\delta = \tilde{\mu}_\Omega(k)$ and so $\tilde{\mu}_\Omega^\delta(ב) \succcurlyeq \tilde{\mu}_\Omega^\delta(k)$.

$\lambda_\Omega^\delta(ב) = \lambda_\Omega(ב).\delta \leq \lambda_\Omega(k).\delta = \lambda_\Omega^\delta(k)$ and so $\lambda_\Omega^\delta(ב) \leq \lambda_\Omega^\delta(k)$.

$$\begin{aligned} \tilde{\mu}_\Omega^\delta(y) &= \tilde{\mu}_\Omega(y).\delta \\ &\succcurlyeq min\{\tilde{\mu}_\Omega(k*y), \tilde{\mu}_\Omega(k)\}.\delta \\ &= min\{\tilde{\mu}_\Omega(k*y).\delta, \tilde{\mu}_\Omega(k).\delta\} \\ &= min\{\tilde{\mu}_\Omega^\delta(k*y), \tilde{\mu}_\Omega^\delta(k)\}. \end{aligned}$$ And

$$\begin{aligned} \lambda_\Omega^\delta(y) &= \lambda_\Omega(y).\delta \\ &\leq max\{\lambda_\Omega(k*y), \lambda_\Omega(k)\}.\delta \\ &= max\{\lambda_\Omega(k*y).\delta, \lambda_\Omega(k).\delta\} \\ &= max\{\lambda_\Omega^\delta(k*y), \lambda_\Omega^\delta(k)\}. \end{aligned}$$

Hence $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ is a δ-dot cubic ideal of X. □

Proposition 4.5.

Let $(X;*, ב)$ be an BZ-algebra and $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic subset of X such that $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ is δ-dot cubic ideal of , for some $\delta \in [ב, 1]$, then $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic ideal of X.

Proof.

Assume that $\Omega^\delta = < \tilde{\mu}_\Omega^\delta(k), \lambda_\Omega^\delta(k) >$ is a δ-dot cubic subalgebra of X for some $\delta \in (ב, 1]$. Let $k, y, z \in X$, then

$\tilde{\mu}_\Omega^\delta(ב) = \tilde{\mu}_\Omega(ב).\delta \succcurlyeq \tilde{\mu}_\Omega(x).\delta = \tilde{\mu}_\Omega^\delta(x)$ and so $\tilde{\mu}_\Omega^\delta(ב) \succcurlyeq \tilde{\mu}_\Omega^\delta(k)$.

$\lambda_\Omega^\delta(ב) = \lambda_\Omega(ב).\delta \leq \lambda_\Omega(k).\delta = \lambda_\Omega^\delta(k)$ and so $\lambda_\Omega^\delta(ב) \leq \lambda_\Omega^\delta(k)$.

$$\begin{aligned} \tilde{\mu}_\Omega(y) \cdot \delta &= \tilde{\mu}_\Omega^\delta(y) \\ &\succcurlyeq min\{\tilde{\mu}_\Omega^\delta(k*y), \tilde{\mu}_\Omega^\delta(k)\} \\ &= min\{\tilde{\mu}_\Omega(k*y) \cdot \delta, \tilde{\mu}_\Omega(k) \cdot \delta\} \\ &= min\{\tilde{\mu}_\Omega(k*y), \tilde{\mu}_\Omega(k)\} \cdot \delta. \end{aligned}$$

$\tilde{\mu}_\Omega(y) \geq min\{\tilde{\mu}_\Omega(k*y), \tilde{\mu}_\Omega(k)\}$ and so

$$\begin{aligned} \lambda_\Omega(y) \cdot \delta &= \lambda_\Omega^\delta(y) \\ &\leq max\{\lambda_\Omega^\delta(k*y), \lambda_\Omega^\delta(k)\} \end{aligned}$$

$= max\{\lambda_{\Omega}(k * y) \cdot \delta \ , \ \lambda_{\Omega}(k) \cdot \delta\}$

$= max\{\lambda_{\Omega}(k * y), \lambda_{\Omega}(k)\} \cdot \delta$.

$\lambda_{\Omega}(y) \leq max\{\lambda_{\Omega}(k * y), \lambda_{\Omega}(x)\}$

Hence $\Omega =< \tilde{\mu}_{\Omega}(k), \lambda_{\Omega}(k) >$ is a cubic ideal of X . □

Proposition 3.6.

Let (X ;∗, ב) be an BZ-algebra and $\Omega =< \tilde{\mu}_{\Omega}(k), \lambda_{\Omega}(k) >$ is a cubic subset of X such that $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ is δ-dot cubic ideal of X , for some δ ∈ [ב, 1], then then the cubic Ω of X is a fuzzy S-extension of the δ-dot cubic Ω^{δ} of X.

Proof:

Since $\tilde{\mu}_{\Omega}(k) \succcurlyeq \mu(k).\delta = \tilde{\mu}^{\delta}_{\Omega}(k)$, and $\lambda_{\Omega}(k) \geq \lambda_{\Omega}(k).\delta = \lambda^{\delta}_{\Omega}(k)$ then $\Omega(k)$ is a fuzzy S-extension of $\Omega^{\delta}(k)$, for all k∈ X and since Ω is a fuzzy ideal of X, then Ω^{δ} of μ is a δ-dot cubic ideal, by Proposition (4.4) . □

Proposition 4.6.

Let (X ;∗, ב) be an BZ-algebra. A δ-dot cubic subset $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ of . If Ω^{δ} is a δ-dot cubic AB- ideal of X, then for all δ ∈ (ב,1], $\tilde{t}$∈ D[ב, 1] and s ∈ [ב, 1], with t ≤ δ, then the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is an ideal of X.

Proof.

Assume that $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ is a δ-dot cubic ideal of X and let $\tilde{t}$∈ D[ב, 1] and s ∈ [ב, 1], be such that $\widetilde{U}(\Omega; \tilde{t}, s) \neq \emptyset$.

Let $x, y \in X$ such that $k * y \in \widetilde{U}(\Omega; \tilde{t}, s)$, then $\tilde{\mu}^{\delta}_{\Omega}(k * y) \succcurlyeq \tilde{t}$, $\tilde{\mu}^{\delta}_{\Omega}(k) \succcurlyeq \tilde{t}$ and $\lambda^{\delta}_{\Omega}(k * y) \leq s$, $\lambda^{\delta}_{\Omega}(k) \leq s$.

Since Ω^{δ} is a δ-dot cubic ideal of X, we get

$\tilde{\mu}^{\delta}_{\Omega}(y) \succcurlyeq rmin\{\tilde{\mu}^{\delta}_{\Omega}(k * y), \tilde{\mu}^{\delta}_{\Omega}(k)\} \succcurlyeq \tilde{t}$ and

$\lambda^{\delta}_{\Omega}(y) \leq max\{\lambda^{\delta}_{\Omega}(k * y), \lambda^{\delta}_{\Omega}(k)\} \leq s$.

Hence the set $\widetilde{U}(\Omega; \tilde{t}, s)$is an ideal of X. □

Proposition 4.7.

Let (X ;∗, ב) be an BZ-algebra. A δ-dot cubic subset $\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ of . If the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is an ideal of X, for all δ ∈ (ב, 1], $\tilde{t}$ ∈ D[ב, 1] and s ∈ [ב, 1], with t ≤ δ, then Ω^{δ} is a δ-dot cubic ideal of X.

Proof.

Suppose that $\widetilde{U}(\Omega; \tilde{t}, s)$is an ideal of X and let $x, y \in X$ be such that

$\tilde{\mu}^{\delta}_{\Omega}(y) \prec rmin\{\tilde{\mu}^{\delta}_{\Omega}(k * y), \tilde{\mu}^{\delta}_{\Omega}(y)\}$ and $\lambda^{\delta}_{\Omega}(y) > max\{\lambda^{\delta}_{\Omega}(k * y), \lambda^{\delta}_{\Omega}(k)\}$.

Consider $\tilde{\delta} = 1/2\ \{\tilde{\mu}^{\delta}_{\Omega}(y) + rmin\{\tilde{\mu}^{\delta}_{\Omega}(k * y), \tilde{\mu}^{\delta}_{\Omega}(k)\}\}$ and

$\delta = 1/2\ \{\lambda^{\delta}_{\Omega}(y) + max\{\lambda^{\delta}_{\Omega}(k * y), \lambda^{\delta}_{\Omega}(k)\}\}$.

We have $\tilde{\delta}$∈ D[ב, 1] and δ ∈ (ב, 1], and

$\tilde{\mu}^{\delta}_{\Omega}(y) \prec \tilde{\delta} \prec rmin\{\tilde{\mu}^{\delta}_{\Omega}(k * y), \tilde{\mu}^{\delta}_{\Omega}(k)\}$, and

$\lambda^{\delta}_{\Omega}(y) > \delta > max\{\lambda^{\delta}_{\Omega}(k * y), \lambda^{\delta}_{\Omega}(k)\}$.

It follows that $x * y, x \in \widetilde{U}(\Omega; \tilde{t}, s)$, and (y) $\notin \widetilde{U}(\Omega; \tilde{t}, s)$. This is a contradiction and therefore Ω^{δ} is a δ-dot cubic ideal of . □

Theorem 4.8. Let (X ;∗, ב) be an BZ-algebra. A δ-dot cubic subset

$\Omega^{\delta} =< \tilde{\mu}^{\delta}_{\Omega}(k), \lambda^{\delta}_{\Omega}(k) >$ of X is a δ-dot cubic ideal of X if and only if, $\mu^{-\delta}_{\ \Omega}$ and $\mu^{+\delta}_{\ \Omega}$ are fuzzy ideal s of X and $\lambda^{\delta}_{\Omega}$ are anti-fuzzy ideal of X.

Proof.

Let $\mu^{-\delta}_{\ \Omega}$,$\mu^{+\delta}_{\ \Omega}$ and $\lambda^{\delta}_{\Omega}$ be fuzzy ideal s of X and $x, y \in X$, then

$\mu^{-\delta}_{\ \Omega}(y) \geq min\{\mu^{-\delta}_{\ \Omega}(k * y), \mu^{-\delta}_{\ \Omega}(k)\}$,

$\mu^{+\delta}_{\ \Omega}(y) \geq min\{\mu^{+\delta}_{\ \Omega}(k * y), \mu^{+\delta}_{\ \Omega}(k)\}$ and

$\lambda^{\delta}_{\Omega}(y) \leq max\{\lambda^{\delta}_{\Omega}(k * y), \lambda^{\delta}_{\Omega}(k)\}$. Now,

$\tilde{\mu}^{\delta}_{\Omega}(y) = [\mu^{-\delta}_{\ \Omega}(y), \mu^{+\delta}_{\ \Omega}(y)]$

$\succcurlyeq [min\{\mu^{-\delta}_{\ \Omega}(k * y), \mu^{-\delta}_{\ \Omega}(k)\}, min\{\mu^{+\delta}_{\ \Omega}(k * y), \mu^{+\delta}_{\ \Omega}(k)\}]$

$= rmin\{[\mu^{-\delta}_{\ \Omega}(k * y), \mu^{+\delta}_{\ \Omega}(k * y)], [\mu^{-\delta}_{\ \Omega}(k), \mu^{+\delta}_{\ \Omega}(k)]\}$

$= rmin\{\tilde{\mu}^{\delta}_{\Omega}(k * y), \tilde{\mu}^{\delta}_{\Omega}(k)\}$, therefore Ω is a δ-dot cubic ideal of X.

Conversely, assume that Ω^{δ} is a δ-dot cubic ideal of X, for any $k, y \in X$,

$[\mu^{-\delta}_{\ \Omega}(y), \mu^{+\delta}_{\ \Omega}(y)] = \tilde{\mu}^{\delta}_{\Omega}(y) \succcurlyeq rmin\{\tilde{\mu}^{\delta}_{\Omega}(k * y), \tilde{\mu}^{\delta}_{\Omega}(k)\}$

$= rmin\{[\mu^{-\delta}_{\ \Omega}(k * y), \mu^{+\delta}_{\ \Omega}(k * y)], [\mu^{-\delta}_{\ \Omega}(k), \mu^{+\delta}_{\ \Omega}(k)]\}$

$= [min\{\mu^{-\delta}_{\ \Omega}(k * y), \mu^{-}_{\ \Omega}(k * y)\}, min\{\mu^{+\delta}_{\ \Omega}(k), \mu^{+\delta}_{\ \Omega}(k)\}]$.

Thus

$\mu^{-\delta}_{\ \Omega}(y) \geq min\{\mu^{-\delta}_{\ \Omega}(k * y), \mu^{-\delta}_{\ \Omega}(k)\}$,

$\mu^{+\delta}_{\ \Omega}(y) \geq min\{\mu^{+\delta}_{\ \Omega}(k * y), \mu^{+\delta}_{\ \Omega}(k)\}$ and

$\lambda^{\delta}_{\Omega}(y) \leq max\{\lambda^{\delta}_{\Omega}(k * y), \lambda^{\delta}_{\Omega}(k)\}$,

Therefore, $\mu^{-\delta}_{\ \Omega}$ and $\mu^{+\delta}_{\ \Omega}$ are fuzzy ideals of X and $\lambda^{\delta}_{\Omega}$ is anti-fuzzy ideal of X. □

Proposition 4.9.

Let (X ;∗, ב) be an BZ-algebra and $\Omega =< \tilde{\mu}_{\Omega}(k), \lambda_{\Omega}(k) >$ is a cubic ideal of X and $\delta_1, \delta_2 \in$ (ב, 1]. If $\delta_2 \geq \delta_1$, then the δ-dot cubic ideal $\Omega^{\delta 2}$ is a fuzzy S-extension of the $\Omega^{\delta 1}$ δ-dot cubic ideal of X.

Proof:

For every k∈ X and $\delta_1, \delta_2 \in$ (ב,1] and $\delta_2 \geq \delta_1$,we have

$\tilde{\mu}^{\delta 2}_{\Omega}(x) = \tilde{\mu}_{\Omega}(x).\delta_2 \succcurlyeq \tilde{\mu}_{\Omega}(x).\delta_1 = \tilde{\mu}^{\delta 1}_{\Omega}(x)$, and

$\lambda^{\delta 2}_{\Omega}(x) = \lambda_{\Omega}(x).\delta_2 \geq \lambda_{\Omega}(x).\delta_1 = \lambda^{\delta 1}_{\Omega}(x)$, then

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$\tilde{\mu}_\Omega^{\delta 2}(x) \succcurlyeq \tilde{\mu}_\Omega^{\delta 1}(x)$, and $\lambda_\Omega^{\delta 2}(x) \geq \lambda_\Omega^{\delta 1}(x)$, therefore $\Omega^{\delta 2}$ is a fuzzy S-extension of $\Omega^{\delta 1}$.

Since Ω is a cubic ideal of X, then Ω^{δ} is a δ-dot cubic ideal of μ, by Proposition (4.4).

Hence $\Omega^{\delta 2}$ of X is a fuzzy S-extension of the δ-dot cubic ideal $\Omega^{\delta 1}$ of X. □

Theorem 4.10.

Every δ-dot cubic ideal of BZ-algebra (X; ∗, ב) is a δ-dot cubic subalgebra of an BZ-algebra (X ;∗, ב) .

Proof: Let (X ;∗, ב) be an BZ-algebra and $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ is a cubic ideal of X and $\Omega^{\delta} = < \tilde{\mu}_\Omega^{\delta}(k), \lambda_\Omega^{\delta}(k) >$ is a δ-dot cubic subset of .

Since Ω^{δ} is an δ-dot cubic ideal of X, then by Proposition (4.6), for every δ ∈ (ב,1] , $\tilde{t}$∈ D[ב, 1] and s ∈ [ב,1] , $\tilde{U}(\Omega; \tilde{t}, s) = \{ k \in X \mid \tilde{\mu}_\Omega(k) \succcurlyeq \tilde{t}, \lambda_\Omega(k) \leq s\}$, is ideal of X .

By Proposition (2.9), for every δ ∈ (ב,1], $\tilde{t}$∈ D[ב, 1] and s ∈ [ב,1], $\tilde{U}(\Omega; \tilde{t}, s)$ is subgalgebra of X . □

Hence μ is a δ-dot cubic subalgebra of X by Proposition (3.12). □

Remark 4.11. The converse of proposition (4.10) is not true as the following example:

Example 4.12. Let X = { ב, 1, 2, 3, 4} in which (∗) is defined by the following table:

∗	ב	1	2	3	4
ב	ב	ב	ב	ב	ב
1	1	ב	ב	ב	ב
2	2	ב	ב	ב	ב
3	3	2	1	ב	ב
4	4	3	4	3	ב

Then (X;∗,ב) is an BZ-algebra. Define a cubic set $\Omega = < \tilde{\mu}_\Omega(k), \lambda_\Omega(k) >$ of X is fuzzy subset μ: X→ [ב,1] by:

$\tilde{\mu}_\Omega(k) = \begin{cases} [0.3, \ 0.9] & if x = \{ב, 1, 2\} \\ [0.1, \ 0.6] & otherwise \end{cases}$ and $\lambda_\Omega = \begin{cases} 0.1 & if x = \{ב, 1, 2\} \\ 0.6 & otherwise \end{cases}$.

Define a δ-dot cubic set $\Omega^{\delta} = < \tilde{\mu}_\Omega^{\delta}(k), \lambda_\Omega^{\delta}(k) >$ of X and δ = 0.4 as follows:

$\tilde{\mu}_\Omega^{\delta}(k) = \begin{cases} [0.12, \ 0.32] & if x = \{ב, 1, 2\} \\ [0.04, \ 0.24] & otherwise \end{cases}$ and $\lambda_\Omega^{\delta} = \begin{cases} 0.04 & if x = \{ב, 1, 2\} \\ 0.24 & otherwise \end{cases}$.

The δ-dot cubic set $\Omega^{\delta} = < \tilde{\mu}_\Omega^{\delta}(k), \tilde{\mu}_\Omega^{\delta}(k) >$ is not a δ-dot cubic subalgebra of X.

Note that λ_Ω is not an anti-fuzzy ideal of X since

$\lambda_\Omega(4 * 2) = \lambda_\Omega(4) = 0.24$
$> 0.04 = \max\{\lambda_\Omega((4 * 1) * 2), \lambda_\Omega(1)\}$
$= \max\{\lambda_\Omega(3 * 2), \lambda_\Omega(1)\} =$
$\max\{\lambda_\Omega(1), \lambda_\Omega(1)\} = \lambda_\Omega(1)$.

Hence Ω^{δ} is not δ-dot cubic ideal of X.

5. Homomorphism of δ-dot Cubic ideals (subalgebras) of BZ-algebra

In this section, we will present some results on images and preimages of

δ-dot cubic ideal s of BZ-algebras.

Definition 5.1[3].

Let : (X;∗,ב) → (Y;∗ ′,ב′) be a mapping from the set X to a set Y. If $\Omega^{\delta} = < \tilde{\mu}_\Omega^{\delta}(k), \lambda_\Omega^{\delta}(k) >$ is a δ-dot cubic subset of X, then the cubic subset β =<$\tilde{\mu}_\beta$, λ_β>of Y defined by:

$$f(\tilde{\mu}_\Omega^{\delta})(y) = \begin{cases} \underset{x \in f^{-1}(y)}{rsup} \ \tilde{\mu}_\Omega^{\delta}(k) if f^{-1}(y) = \{k \in X, f(k) = y\} \neq \varphi \\ ב \qquad otherwise \end{cases}$$

$$f(\lambda_\Omega^{\delta})(y) = \begin{cases} \underset{x \in f^{-1}(y)}{inf} \ \lambda_\Omega^{\delta}(k) if f^{-1}(y) = \{x \in X, f(k) = y\} \neq \varphi \\ 1 \qquad otherwise \end{cases}$$

is said to be **the image of Ω under f.**

Similarly if $\beta^{\delta} = < \tilde{\mu}_\beta^{\delta}(k), \lambda_\beta^{\delta}(k) >$ is a δ-dot cubic subset of Y , then the cubic subset $\Omega^{\delta} = (\beta^{\delta} \circ f)$ in X (i.e., the δ-dot cubic subset defined by $\tilde{\mu}_\Omega^{\delta}(k) = \tilde{\mu}_\beta^{\delta}(f(k))$, $\lambda_\Omega^{\delta}(k) = \lambda_\beta^{\delta}(f(k))$, for all k ∈X) is called **the preimage of β under f**).

Theorem 5.2. A homomorphic preimage of δ-dot cubic subalgebra is also

δ-dot cubic subalgebra.

Proof. Let f: (X;∗, ב) →(Y;∗ ', ב') be homomorphism from an BZ-algebra X into an BZ-algebra Y.

If $\beta^{\delta} = < \tilde{\mu}_\beta^{\delta}(k), \lambda_\beta^{\delta}(k) >$ is a cubic subalgebra of Y and $\Omega^{\delta} = < \tilde{\mu}_\Omega^{\delta}(k), \lambda_\Omega^{\delta}(k) >$the preimage of β^{δ} under f, then

$\tilde{\mu}_\Omega^{\delta}(k) = \tilde{\mu}_\beta^{\delta}(f(k))$, $\lambda_\Omega^{\delta}(k) = \lambda_\beta^{\delta}(f(k))$, for all k ∈$X$.

Let k ∈X , then

$(\tilde{\mu}_\Omega^{\delta})(ב) = \tilde{\mu}_\beta^{\delta}(f(ב)) \succcurlyeq \tilde{\mu}_\beta^{\delta}(f(k)) = \tilde{\mu}_\Omega^{\delta}(k)$, and
$(\lambda_\Omega^{\delta})(ב) = \lambda_\beta^{\delta}(f(ב)) \leq \lambda_\beta^{\delta}(f(k)) = \lambda_\Omega^{\delta}(k)$.

Now, let x , y ∈ X, then

$\tilde{\mu}_\Omega^{\delta}(k * y) = \tilde{\mu}_\beta^{\delta}(f(k * y)) = \tilde{\mu}_\beta^{\delta}(f(k) *' f(y))$

$\geqslant$ rmin $\{\tilde{\mu}_\beta^\delta(f(\ \ k), \tilde{\mu}_\beta^\delta(f(y))\}$

$=$ rmin $\{\tilde{\mu}_\Omega^\delta(\ \ k), \tilde{\mu}_\Omega^\delta(y)\}$, and

$\lambda_\Omega^\delta(\ \ k*y) = \lambda_\beta^\delta(f(\ \ k*y)) = \lambda_\beta^\delta(f(\ \ k)*' f(y))$

$\leq \max\{\lambda_\beta^\delta(f(\ \ k), \lambda_\beta^\delta(f(y))\}$

$= \max\{\lambda_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(y)\}$. □

Definition 5.3. Let $f: (X;*,ב) \to (Y;*',ב')$ be a mapping from a set X into a set Y. $\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$ is a δ-dot cubic subset of X **has sup and inf properties** if for any subset T of X, there exist t, s ∈ T such that

$$\tilde{\mu}_\Omega^\delta(t) = \underset{t0\in T}{rsup}\,\tilde{\mu}_\Omega^\delta(t0) \text{ and } \lambda_\Omega^\delta(s) = \underset{s0\in T}{inf}\,\lambda_\Omega^\delta(s0).$$

Theorem 5.4. Let $: (X;*,ב) \to (Y;*',ב')$ be an epimorphism from an BZ-algebra X into an BZ-algebra Y. For every δ-dot cubic subalgebra

$\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$ of X with **sup and inf properties**, then $f(\Omega^\delta)$ is a

δ-dot cubic subalgebra of Y.

Proof. By definition $\tilde{\mu}_\beta^\delta(y') = f(\tilde{\mu}_\Omega^\delta)(y') = \underset{x\in f^{-1}(y')}{rsup}\ \tilde{\mu}_\Omega^\delta(k)$ and

$\lambda_\beta^\delta(y') = f(\lambda_\Omega^\delta)(y') = \underset{x\in f^{-1}(y')}{inf}\ \lambda_\Omega^\delta(x)$ for all y' ∈ Y and

rsup(∅) = [ב, ב] and inf (∅) = ב . We have prove that

$\tilde{\mu}_\beta^\delta(\ \ k'*y') \geqslant$ rmin $\{\tilde{\mu}_\beta^\delta(\ \ k'), \tilde{\mu}_\beta^\delta(y')\}$, and

$\lambda_\beta^\delta(\ \ k'*y') \leq \max\{\lambda_\beta^\delta(\ \ k'), \lambda_\beta^\delta(y')\}$, for all k', y' ∈ Y .

$\tilde{\mu}_\beta^\delta(k'*y') = \underset{t\in f^{-1}(x'*y')}{rsup}\ \tilde{\mu}_\Omega^\delta(t) = \tilde{\mu}_\Omega^\delta(k_ב * y_ב)$

$\geqslant rmin\{\tilde{\mu}_\Omega^\delta(k_ב), \tilde{\mu}_\Omega^\delta(y_ב)\}$,

$\geqslant rmin\{\underset{t\in f^{-1}(x')}{rsup}\ \tilde{\mu}_\Omega^\delta(t)\ ,\ \underset{t\in f^{-1}(y')}{rsup}\ \tilde{\mu}_\Omega^\delta(t)\}$,

$=$ rmin $\{\tilde{\mu}_\beta^\delta(k'), \tilde{\mu}_\beta^\delta(y')\}$ and

$\lambda_\Omega^\delta(k'*y') = \underset{t\in f^{-1}(x'*y')}{inf}\ \lambda_\Omega^\delta(t)$

$\leq \max\{\lambda_\Omega^\delta(k_ב), \lambda_\Omega^\delta(y_ב)\}$

$= \max\{\underset{t\in f^{-1}(x')}{inf}\ \lambda_\Omega^\delta(t),\ \underset{t\in f^{-1}(y')}{inf}\ \lambda_\Omega^\delta(t)\}$

Hence, $\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$ is a δ-dot cubic subalgebra of . □

Theorem 5.5.

A homomorphic pre-image of δ-dot cubic ideal is also δ-dot cubic ideal.

Proof.

Let $f: (X;*,ב) \to (Y;*',ב')$ be homomorphism from an BZ-algebra X into an BZ-algebra Y.

If $\beta^\delta =< \tilde{\mu}_\beta^\delta(\ \ k), \lambda_\beta^\delta(\ \ k) >$ is a δ-dot cubic ideal of Y and $\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$the pre-image of β^δ under f, then $\tilde{\mu}_\Omega^\delta(\ \ k) = \tilde{\mu}_\beta^\delta(f(\ \ k))$, $\lambda_\Omega^\delta(\ \ k) = \lambda_\beta^\delta(f(\ \ k))$, for all $x \in X$. Let k ∈X , then

$(\tilde{\mu}_\Omega^\delta)(ב) = \tilde{\mu}_\beta^\delta(f(ב)) \geqslant \tilde{\mu}_\beta(f(\ \ k)) = \tilde{\mu}_\Omega^\delta(\ \ k)$, and $(\lambda_\Omega^\delta)(ב) = \lambda_\beta^\delta(f(ב)) \leq \lambda_\beta^\delta(f(\ \ k)) = \lambda_\Omega^\delta(\ \ k)$.

Now, let $x, y \in X$, then

$\tilde{\mu}_\Omega^\delta(y) = \tilde{\mu}_\beta^\delta(f(y)) \geqslant$ rmin $\{\tilde{\mu}_\beta^\delta(f(k*y), \tilde{\mu}_\beta^\delta(f(\ \ k))\}$

$=$ rmin $\{\tilde{\mu}_\Omega^\delta(\ \ k*(y*z)), \tilde{\mu}_\Omega^\delta(y)\}$, and

$\lambda_\Omega^\delta(y) = \lambda_\beta^\delta(f(y)) \leq \max\{\lambda_\beta^\delta(f(x*y), \lambda_\beta^\delta(f(\ \ k))\}$

$= \max\{\lambda_\Omega^\delta(x*y), \lambda_\Omega^\delta(\ \ k)\}$. □

Theorem 5.6.

Let $: (X;*,ב) \to (Y;*',ב')$ be an epimorphism from an BZ-algebra X into an BZ-algebra Y. For every δ-dot cubic ideal $\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$ of X with **sup and inf properties**, then $f(\Omega^\delta)$ is a δ-dot cubic ideal of Y.

Proof.

By definition $\tilde{\mu}_\beta^\delta(y') = f(\tilde{\mu}_\Omega^\delta)(y') = \underset{x\in f^{-1}(y')}{rsup}\ \tilde{\mu}_\Omega^\delta(x)$ and

$\lambda_\beta^\delta(y') = f(\lambda_\Omega^\delta)(y') = \underset{x\in f^{-1}(y')}{inf}\ \lambda_\Omega^\delta(x)$ for all y' ∈ Y and

rsup(∅) = [ב, ב] and inf (∅) = ב . We have prove that

$\tilde{\mu}_\beta^\delta(y' \geqslant$ rmin $\{\tilde{\mu}_\beta^\delta(\ \ k'*y'), \tilde{\mu}_\beta^\delta(\ \ k')\}$, and

$\lambda_\beta^\delta(\ \ k'*z') \leq \max\{\lambda_\beta^\delta(\ \ k'*y'), \lambda_\beta^\delta(\ \ k')\}$, for all k', y' ∈ Y .

Let $f: (X;*,ב) \to (Y;*',ב')$ be epimorphism of BZ-algebras,

$\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$is a δ-dot cubic ideal of X has sup and inf properties and $\beta^\delta =< \tilde{\mu}_\beta^\delta(\ \ k), \lambda_\beta^\delta(\ \ k) >$ the image of Ω under f.

Since $\Omega^\delta =< \tilde{\mu}_\Omega^\delta(\ \ k), \lambda_\Omega^\delta(\ \ k) >$ is a δ-dot cubic ideal of X, we have

$(\tilde{\mu}_\Omega^\delta)(ב) \geqslant \tilde{\mu}_\Omega^\delta(\ \ k)$ and $\lambda_\Omega^\delta(ב) \leq \lambda_\Omega^\delta(\ \ k)$, for all k ∈X .

Note that, ב $\in f^{-1}$ (ב') where ב, ב' are the zero of X and Y, respectively.

Thus

$\tilde{\mu}_\beta(ב') = \underset{t\in f^{-1}(ב')}{rsup}\ \tilde{\mu}_\Omega(t) = \tilde{\mu}_\Omega(ב)$

$\geqslant \tilde{\mu}_\Omega(\ \ k) = \underset{t\in f^{-1}(x')}{rsup}\ \tilde{\mu}_\Omega(t) = \tilde{\mu}_\beta(x')$, and

$\lambda_\beta(ב') = \underset{t\in f^{-1}(ב')}{inf}\ \lambda_\Omega^\delta(t) = \lambda_\Omega^\delta(ב)$

$\leq \lambda_\Omega^\delta(k) = \underset{t\in f^{-1}(x')}{inf}\ \lambda_\Omega^\delta(t) = \lambda_\beta^\delta(k')$, for all k ∈X ,

which implies that

$\tilde{\mu}_\beta^\delta(ב') \geqslant \tilde{\mu}_\beta^\delta(k')$, and $\lambda_\beta^\delta(ב') \leq \lambda_\beta^\delta(k')$, for all k' ∈Y .

For any k' , y' ∈ Y, let $x_0 \in f^{-1}(\ \ k')$ and $y_0 \in f^{-1}(y')$ be such that

$$\lambda_\beta^\delta(y') = f(\lambda_\Omega^\delta)(y') = \underset{x\in f^{-1}(y')}{inf}\ \lambda_\Omega^\delta(x)$$

$\tilde{\mu}_\Omega^\delta(x_ב * y_ב) = \underset{t\in f^{-1}(x'*y'))}{rsup}\ \tilde{\mu}_\Omega^\delta(t)$, and

$\tilde{\mu}_\Omega^\delta(y_ב) = \underset{t\in f^{-1}(y')}{rsup}\ \tilde{\mu}_\Omega^\delta(t)$. then

$\tilde{\mu}_\Omega^\delta(y_ב) = \tilde{\mu}_\beta^\delta(f(y_ב))$

$= \tilde{\mu}_{\beta}^{\delta}(y')$

$= \underset{(y_{\beth}) \in f^{-1}(y')}{rsup} \tilde{\mu}_{\Omega}^{\delta}(y_{\beth})$

$= \underset{t \in f^{-1}(y')}{rsup} \tilde{\mu}_{\Omega}^{\delta}(t)$. Also ,

$\lambda_{\Omega}^{\delta}(k_{\beth} * y_{\beth}) = \underset{t \in f^{-1}(k * y')}{inf} \lambda_{\Omega}^{\delta}(t), \lambda_{\Omega}^{\delta}(k_{\beth}) = \underset{t \in f^{-1}(k')}{inf} \lambda_{\Omega}^{\delta}(t)$

and

$\lambda_{\Omega}^{\delta}(y_{\beth}) = \lambda_{\beta}^{\delta}(f(y_{\beth}))$

$= \lambda_{\beta}^{\delta}\left(f(y')\right)$

$= \underset{(y_{\beth}) \in f^{-1}(y')}{inf} \lambda_{\Omega}^{\delta}(y_{\beth})$

$= \underset{t \in f^{-1}(y')}{inf} \lambda_{\Omega}^{\delta}(t)$. Then

$\tilde{\mu}_{\beta}^{\delta}(y') = \underset{t \in f^{-1}(y')}{rsup} \tilde{\mu}_{\Omega}^{\delta}(t) = \tilde{\mu}_{\Omega}^{\delta}(y_{\beth})$

$\geqslant$ rmin $\{\tilde{\mu}_{\Omega}^{\delta}(k_0 * y_0), \tilde{\mu}_{\Omega}^{\delta}(k_0)\}$,

$=$ rmin$\{ \underset{t \in f^{-1}(x' * y')}{rsup} \tilde{\mu}_{\Omega}^{\delta}(t), \underset{t \in f^{-1}(x')}{rsup} \tilde{\mu}_{\Omega}^{\delta}(t)\}$

$=$ rmin $\{ \tilde{\mu}_{\beta}^{\delta}(x' * y'), \tilde{\mu}_{\beta}^{\delta}(x') \}$and

$\lambda_{\Omega}^{\delta}(y') = \underset{t \in f^{-1}(y')}{inf} \lambda_{\Omega}^{\delta}(t)$

$\leq$ max $\{\lambda_{\Omega}^{\delta}(x_{\beth} * y_{\beth}), \lambda_{\Omega}^{\delta}(x_{\beth})\}$

$=$ max $\{ \underset{t \in f^{-1}(x' * y')}{inf} \lambda_{\Omega}(t), \underset{t \in f^{-1}(x')}{inf} \lambda_{\Omega}(t) \}$

Hence, $\beta^{\delta} = < \tilde{\mu}_{\beta}^{\delta}(\ k), \lambda_{\beta}^{\delta}(\ k) >$ is a δ-dot cubic ideal of . □

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