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$\boldsymbol{\beta}$-multiplications Intuitionistic of BZ-algebra

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***Abstract**: The concepts of multiplications intuitionistic of fuzzy subalgebras and fuzzy BZ-ideals on BZ-algebras are introduced. Also, the relationships between multiplications intuitionistic fuzzy subalgebras, extensions intuitionistic and multiplications intuitionistic fuzzy BZ-ideals are investigated. The homomorphism of fuzzy subalgebra and fuzzy BZ-ideal on BZ-algebra are given. Also, the Cartesian product of multiplications intuitionistic of BZ-algebras are studied and investigate some related properties.*

Keywords— fuzzy BZ-ideal, intuitionistic fuzzy BZ-ideal, multiplication intuitionistic fuzzy BZ-ideal, image (pre-image) of BZ-ideal, Cartesian product.

1. INTRODUCTION

After the introduction of fuzzy sets by Zadeh [26], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [2-3] is one among them. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both degrees of membership and of nonmembership. Both degrees belong to the interval [α; 1], and their sum should not exceed 1. BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki [12,13]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. In 1991, Xi [25] applied the concept of fuzzy sets to BCK-algebras. In 1993, Jun [14] and Ahmad [1] applied it to BCI-algebras. After that Jun, Meng, Liu and several researchers investigated further properties of fuzzy subalgebras and ideals in BCK=BCI-algebras (see [15-18]). In [27], Zhan and Tan discussed characterization of fuzzy H-ideals and doubt fuzzy H-ideals in BCK-algebras. Recently, Satyanarayana et al. [21-22] introduced intuitionistic fuzzy H-ideals in BCK-algebras. The concept of fuzzy translations in fuzzy subalgebras and ideals in BCK/BCI-algebras has been discussed respectively, they investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. Motivated by this, in [19], the authors have studied fuzzy translations of fuzzy H-ideals in BCK/BCI-algebras. They also extend this study from fuzzy multiplication to intuitionistic fuzzy multiplication in BCK=BCI-algebras.

BZ-ideals and fuzzy BZ-ideals in BZ-algebras was defined by A.T. Hameed and S.H. Ghabue. They were introduced the notion of fuzzy BZ-ideal, intuitionistic fuzzy subalgebra and intuitionistic fuzzy BZ-ideal on BZ-algebras and a lot of properties are investigated of its.

In this paper, the concepts of multiplications intuitionistic of fuzzy subalgebras and fuzzy BZ-ideals on BZ-algebras are introduced. The notion of extensions of intuitionistic fuzzy and multiplications intuitionistic of fuzzy subalgebras and fuzzy BZ-ideals with several related properties are investigated. Also, the relationships between multiplications intuitionistic fuzzy subalgebras, extensions intuitionistic and multiplications intuitionistic fuzzy BZ-ideals are investigated. The image (pre-image) of fuzzy subalgebra and fuzzy BZ-ideal on BZ-algebra, and investigate some of these properties. Moreover, we introduce the notion of Cartesian product of multiplications intuitionistic of fuzzy BZ-ideal on BZ-algebras, and investigate some related properties.

2. 2. PRELIMINARIES

In this section, some definitions are recalled which are used in the later sections. The BZ-algebra is a very important branch of a modern algebra, which is defined.

Definition 2.1. ([18,19]). Let $(X;*,\alpha)$ be an algebra with operation $(+)$ and constant (α). X is called a BZ-algebra if it satisfies the following identities: for any $x, y, z \in X$,
(BZ-1) $((x * z) * (y * z)) * (x * y) = \alpha$;
(BZ-2) $x * \alpha = x$;
(BZ-3) $x * y = \alpha \; and \; y * x = \alpha$ implies that $x = y$.

Remark 2.2. ([18,19]). On BZ -algebra $(X,*,\alpha)$, we defined a binary relation $\leq$ on X by putting $x \leq y$ if and only if $x * y = \alpha$.

Proposition 2.3. ([1-3,18,19]). Let $(X;*,\alpha)$ be a BZ-algebra, then $(X,\leq)$ is a partially ordered set. It is easy to show that the following properties are true for a BZ -algebra. For any $x, y, z \in X$:
(P-1) $x * ((x * y) * y) = \alpha$;
(P-2) $x * x = \alpha$;
(P-3) $\partial \leq y \; implies \; y * z \leq x * z$;
(P-7) $x \leq y \; implies \; z * x \leq z * y$.

Definition 2.4. ([19]). A subset S of a BZ-algebra X is called **subalgebra of X** if $x * y \in S$ whenever $x, y \in S$.

Definition 2.5. ([1-3]). A non-empty subset I of a BZ-algebra $(X,*,\alpha)$ is called ***BZ*-ideal of X** if it satisfies the following conditions: for any $x, y, z \in X$

(I-1) $\alpha \in I$

(I-2) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$.

Proposition 2.6. ([18,19]). Every BZ-ideal of BZ-algebra $(X,*,\alpha)$ is a subalgebra of X.

Proposition 2.7. ([18,19]). Let $\{I_i \mid i\in\Lambda\}$ be a family of ideals of BZ-algebra $(X,*,\alpha)$. The intersection of any set of BZ-ideals of X is also an BZ-ideal of X.

Definition 2.8. ([23]). Let $(X,*,\alpha)$ be a nonempty set, a fuzzy subset μ of X is a mapping $\mu: X \to [\alpha, 1]$.

Definition 2.9. ([23]). Let μ be a fuzzy subset of a set . For t ∈ [α, 1], the set

$\mu_t = U(\mu, t) = \{x \in X \mid \mu(x) \geq t\}$, is called **upper level cut (level subset) of μ** and the set $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called **lower level cut of μ**.

Definition 2.10. ([9]). Let $f: (X;*,\alpha) \to (Y;*', \alpha')$ be a mapping nonempty sets X and Y respectively.

If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by:

$$f(\mu)(y) = \begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ \alpha & otherwise \end{cases}$$

is said to be the image of μ under f.

Similarly if β is a fuzzy subset of Y , then the fuzzy subset

$\mu = (\beta \circ f)$ of X (i.e the fuzzy subset defined by $\mu(\partial) = \beta(f(\partial))$ for all $x \in X$) is called the pre-image of β under f .

Definition 2.11. ([9]). A fuzzy subset μ of a set X **has sup property** if for any subset T of X, there exist $t_\alpha \in T$ such that $\mu(t_\alpha) = sup\{\mu(t) | t \in T\}$.

Definition 2.12. ([1,18]). Let $(X,*,\alpha)$ be an BZ-algebra, a fuzzy subset μ of X is called a **fuzzy subalgebra of X** if for all $x, y \in X$, $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$.

Proposition 2.13. ([1,19]). Let μ be a fuzzy subset of BZ-algebra $(X,*,\alpha)$. If μ is a fuzzy subalgebra of , then for any t ∈ [α,1], μ_t is a subalgebra of X .

Definition 2.14. [5]. Let $(X; *, \alpha)$ be an BZ-algebra. A fuzzy subset μ of X is called **a fuzzy BZ-ideal of X** if it satisfies the following conditions: for all $x, y \in X$,

(1) $\mu(\alpha) \geq \mu(x)$.

(2) $\mu(x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$.

Proposition 2.15. [5]. Every fuzzy BZ-ideal of BZ-algebra is fuzzy subalgebra.

Proposition 2.16. ([5]). Let $f: (X; *, \alpha) \to (Y; *`, \alpha`)$ be a homomorphism between BZ-algebras X and Y respectively.

1- For every fuzzy subalgebra β of Y, $f^{-1}(\beta)$ is a fuzzy subalgebra of X.

2- For every fuzzy subalgebra μ of X , $f(\mu)$ is a fuzzy subalgebra of Y.

3- For every fuzzy BZ-ideal β of Y, $f^{-1}(\beta)$ is a fuzzy BZ-ideal of X.

4- For every fuzzy BZ-ideal μ of X with sup property, $f(\mu)$ is a fuzzy BZ-ideal of Y, where f is onto.

Definition 2.17.([3,6]). An intuitionistic fuzzy subset A in a nonempty set X is an object having the form $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ where the functions $\mu_A: X \to [\alpha, 1]$ and $\nu_A: X \to [\alpha, 1]$ denote the degree of membership and the degree of non-membership respectively, and $\alpha \leq \mu_A(\partial) + \nu_A(\partial) \leq 1$ for all $\partial \in X$.

Remark 2.18.([3,6]). If an intuitionistic fuzzy subset A in a nonempty set X , then $\mu_A(\partial) + \nu_A(\partial) = 1$, i.e., when $\nu_A(\partial) = 1 - \mu_A(\partial) = \mu_A^c(\partial)$ for all that $\partial \in X$. Now μ_A is named fuzzy subset while $\nu_A = \mu_A^c$ is the complement of μ_A.

Definition 2.19.([3,6]). Let $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$be an intuitionistic fuzzy subset of BZ-algebra $(X;*,\alpha)$. A is said to be an **intuitionistic fuzzy subalgebra** of X if: for all ∂, y ∈X,

(IFS_1) $\mu_A(\partial * y) \geq \min\{\mu_A(\partial), \mu_A(y)\}$.

(IFS_2) $\nu_A(\partial * y) \leq \max\{\nu_A(\partial), \nu_A(y)\}$.

That mean μ_A is a fuzzy subalgebra and ν_A is a doubt fuzzy subalgebra.

Proposition 2.20.([3,6]). Every intuitionistic fuzzy subalgebra $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ of BZ-algebra X satisfies the inequalities

$\mu_A(\alpha) \geq \mu_A(\partial)$ and $\nu_A(\alpha) \leq \nu_A(\partial)$, for all $\partial \in X$.

Definition 2.21.([3,6]). For any $t \in [\alpha, 1]$ and a fuzzy subset μ in a nonempty set X, the set $U(\mu, t) = \{\partial \in X \mid \mu(\partial) \geq t\}$ is called **an upper t-level cut of μ**, and the set $L(\mu, t) = \{\partial \in X \mid \mu(\partial) \leq t\}$ is called **a lower t-level cut of μ.**

Theorem 2.22.([3,6]). An intuitionistic fuzzy subset $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ is an intuitionistic fuzzy subalgebra of BZ-algebra $(X;*,\alpha)$ if and only if, for all that, $t \in [\alpha, 1]$, the set $U(\mu_A, t)$ and $L(\nu_A, s)$ are subalgebras of X.

Definition 2.23.([3,6]). Let $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ be an intuitionistic fuzzy subset of BZ-algebra $(X;*,\alpha)$. A is said to be **an intuitionistic fuzzy BZ-ideal of X** if : for all that ∂, y, z $\in$X,

(IFI$_1$) $\mu_A(\alpha) \geq \mu_A(\partial)$ and $\nu_A(\alpha) \leq \nu_A(\partial)$.

(IFI$_2$) $\mu_A(\partial * z) \geq \min\{\mu_A((\partial * y) * z), \mu_A(y)\}$ and
$\nu_A(\partial * z) \leq \max\{\nu_A((\partial * y) * z), \nu_A(y)\}$.

That mean μ_A is a fuzzy BZ-ideal and ν_A is a doubt fuzzy BZ-ideal.

Theorem 2.24. ([3, 6]). An intuitionistic fuzzy subset $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ is an intuitionistic fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$ if and only if, for all t $\in$[α,1] , the set $U(\mu_A, t)$ and $L(\nu_A, s)$ are an BZ-ideal of X.

Proposition 2.25.[3,6]. Let $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ be an intuitionistic fuzzy BZ-ideal of BZ-algebra X, then A is an intuitionistic fuzzy subalgebra of X.

Theorem 2.26.([3,6]). An intuitionistic fuzzy subset $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ an intuitionistic fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$ if and only if, the fuzzy sets μ_A is a fuzzy BZ-ideal of X and ν_A is a doubt fuzzy BZ-ideal of X.

3. β-MULTIPLICATION INTUITIONISTIC OF FUZZY SUBALGEBRA.

We study the notion of multiplication intuitionistic of fuzzy subalgebra of BZ-algebra and we give some properties of it.

Definition 3.1. ([9]). Let μ be a fuzzy subset of a set X and β∈ (α, 1). A multiplication of μ, denoted by μ_β^M is defined to be a mapping μ_β^M: X→[α,1] define by $\mu_\beta^M(x) = \beta . \mu(x)$, for all $\partial \in$ X.

Definition 3.2. ([9]). Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$be an intuitionistic fuzzy subset of BZ-algebra $(X;*,\alpha)$ and let $\beta \in (\alpha, 1)$ an object having the form $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ is called **a β-multiplication intuitionistic of fuzzy subset of A** if $(\mu_A)_\beta^M(\partial) = \beta . \mu_A(\partial)$ and $(\nu_A)_\beta^M(\partial) = \beta . \nu_A(\partial)$, for all $\partial \in$ X.

Definition 3.3. Let $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ be a β-multiplication intuitionistic of fuzzy subset of A and $\beta \in (\alpha, 1)$, A_β^M is said to be **fuzzy subalgebra of X** if : for all ∂, y ∈X,

(IFS$_1$) $\beta . \mu_A(\partial * y) \geq \min\{\beta . \mu_A(\partial), \beta . \mu_A(y)\}$,

(IFS$_2$) $\beta . \nu_A(\partial * y) \leq \max\{\beta . \nu_A(\partial), \beta . \nu_A(y)\}$.

That mean μ_A is a fuzzy subalgebra of X and ν_A is a doubt fuzzy subalgebra of X.

Theorem 3.4. If $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ is an intuitionistic fuzzy subalgebra of BZ-algebra $(X;*,\alpha)$, then the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is a fuzzy subalgebra of X, for all $\beta \in (\alpha, 1)$.

Proof: Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an intuitionistic fuzzy subalgebra of X and $\beta \in (\alpha, 1)$, then for all ∂, y∈ X.

$$(\mu_A)_\beta^M(\partial * y) = \beta . \mu_A(\partial * y) \geq \beta . \min\{\mu_A(\partial), \mu_A(y)\}$$
$$= \min\{\beta . \mu_A(\partial), \beta . \mu_A(y)\}$$
$$= \min\{(\mu_A)_\beta^M(\partial), (\mu_A)_\beta^M(y)\}$$

and
$$(\nu_A)_\beta^M(\partial * y) = \beta . \nu_A(\partial * y)$$
$$\leq \beta . \max\{\nu_A(\partial), \nu_A(y)\}$$
$$= \max\{\beta . \nu_A(\partial), \beta . \nu_A(y)\}$$
$$= \max\{(\nu_A)_\beta^M(\partial), (\nu_A)_\beta^M(y)\}.$$

Hence, the β-multiplication intuitionistic of A is a fuzzy subalgebra of X. ■

Proposition 3.5. Every β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is fuzzy subalgebra of BZ-algebra $(X;*,\alpha)$ satisfies the inequalities $\beta . \mu_A(\alpha) \geq \beta . \mu_A(\partial)$ and $\beta . \nu_A(\alpha) \leq \beta . \nu_A(\partial)$, for all $\partial \in$ X.

Proof.
$$\beta . \mu_A(\alpha) = \beta . \mu_A(\partial * \partial) \geq \beta . \min\{\mu_A(\partial), \mu_A(\partial)\}$$
$$= \min\{\beta . \mu_A(\partial), \beta . \mu_A(\partial)\}$$
$$= \beta . \mu_A(\partial)$$
and
$$\beta . \nu_A(\alpha) = \beta . \nu_A(\partial * \partial) \leq \beta . \max\{\nu_A(\partial), \nu_A(\partial)\}$$
$$= \max\{\beta . \nu_A(\partial), \beta . \nu_A(\partial)\}$$
$$= \beta . \nu_A(\partial).$$ ■

Theorem 3.6. Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an intuitionistic fuzzy subset of BZ-algebra $(X;*,\alpha)$ such that the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is a fuzzy subalgebra of X, for some $\beta \in (\alpha, 1)$, then $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ is an intuitionistic fuzzy subalgebra of X.

Proof: Assume that $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ is an β-multiplication intuitionistic of fuzzy subalgebra of X, for some $\beta \in (\alpha, 1)$. Let ∂, y∈ X,

$$\beta . \mu_A(\partial * y) = (\mu_A)_\beta^M(\partial * y) \geq \min\{(\mu_A)_\beta^M(\partial), (\mu_A)_\beta^M(y)\}$$
$$= \min\{\beta . \mu_A(\partial), \beta . \mu_A(y)\}$$
$$= \beta . \min\{\mu_A(\partial), \mu_A(y)\}$$
and
$$\beta . \nu_A(\partial) = (\nu_A)_\beta^M(\partial) \leq \max\{(\nu_A)_\beta^M(\partial), (\nu_A)_\beta^M(y)\}$$
$$= \max\{\beta . \nu_A(\partial), \beta . \nu_A(y)\}$$
$$= \beta . \max\{\nu_A(\partial), \nu_A(y)\}$$

which implies that $\mu_A(\partial * y) \geq \min\{\mu_A(\partial), \mu_A(y)\}$and $\nu_A(\partial) \leq \max\{\nu_A(\partial), \nu_A(\partial)\}$, for all ∂, y ∈ X.

Hence, $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ is intuitionistic fuzzy subalgebra of X. ■

Definition 3.7. [6]. For any $t \in [\alpha, 1]$, $\beta \in (\alpha, 1)$ and a fuzzy subset μ in a nonempty set X, the set $U_\beta(\mu, t) = \{\partial \in X \mid \beta . \mu(\partial) \geq t\}$ is **called β - multiplication of upper t-level cut of μ**, and the set $L_\beta(\mu, t) = \{\partial \in X \mid \beta . \mu(\partial) \leq t\}$ is **called β - multiplication of lower t-level cut of μ.**

Theorem 3.8. If a β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ of A is fuzzy subalgebra of BZ-algebra $(X;*,\alpha)$, then for any $t, s \in [\alpha, 1]$, the set $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are subalgebras of X.

Proof. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ be fuzzy subalgebra of X and $U_\beta(\mu_A, t) \neq \emptyset \neq L_\beta(\nu_A, s)$ and follow for every $\partial, y \in X$ such as $\partial \in U_\beta(\mu_A, t)$, $y \in U_\beta(\mu_A, t)$, then $\beta . \mu_A(\partial) \geq t$ and $\beta . \mu_A(y) \geq t$, so therefore $\beta . \mu_A(\partial * y) \geq \min\{\beta . \mu_A(\partial), \beta . \mu_A(y)\} \geq t$, so as so $(\partial * y) \in U_\beta(\mu_A, t)$. thus $U_\beta(\mu_A, t)$ this a subalgebra from X.

In a similar way, we can prove that $L_\beta(\nu_A, s)$ is a subalgebra of X. ■

Theorem 3.9. A β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ of A of BZ-algebra $(X;*,\alpha)$. If for all that, $t, s \in [\alpha, 1]$, the set $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are subalgebras of X, then A^M_β is fuzzy subalgebra of X.

Proof. Assume that for each $, t, s \in [\alpha, 1]$, the sets $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are subalgebras of X.

I found $\partial', y' \in X$ be such that $\beta . \mu_A(\partial' * y') < \min\{\beta . \mu_A(\partial'), \beta . \mu_A(y')\}$, then by taking

$t_\alpha = \frac{1}{2}\{\beta.(\mu_A(\partial' * y')) + \min\{\beta . \mu_A(\partial'), \beta . \mu_A(y')\}\}$, we get

$\beta . \mu_A(\partial' * y') < t_\alpha < \min\{\beta . \mu_A(\partial'), \beta . \mu_A(y')\}$ and hence

$(\partial' * y') \in U_\beta(\mu_A, t_\alpha)$, $\partial' \in U_\beta(\mu_A, t_\alpha)$, $y' \in U_\beta(\mu_A, t_\alpha)$,

i.e., $U_\beta(\mu_A, t_\alpha)$, is not a subalgebra of X, which make a contradiction.

Hence $U_\beta(\mu_A, t_\alpha)$ is a subalgebra of X.

Finally, assume $\beta . \nu_A(\partial' * y') > \max\{\beta . \nu_A(\partial'), \beta . \nu_A(y')\}$. then by taking

$s_\alpha = \frac{1}{2}\{(\beta . \nu_A(\partial' * y')) + \max\{\beta . \nu_A(\partial'), \beta . \nu_A(y')\}\}$, we get

$\max\{\beta . \nu_A(\partial'), \beta . \nu_A(y')\} > s_\alpha > \beta . (\nu_A(\partial' * y'))$ and hence

$(\partial' * y') \in L_\beta(\nu_A, s_\alpha)$, $\partial' \in L_\beta(\nu_A, s_\alpha)$, $y' \in L_\beta(\nu_A, s_\alpha)$,

i.e., $L_\beta(\nu_A, s_\alpha)$, is not a subalgebra of BZ-algebra X, which make a contradiction.

Therefore, $L_\beta(\nu_A, s_\alpha)$ is a subalgebra of X.

Hence $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ is fuzzy subalgebra of X. ■

Theorem 3.10. Let β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ of A. If there exists as sequence $\{\partial_n\}$ in X such that $\beta.\mu_A(\partial_n) = 1$ and $\beta.\nu_A(\partial) = \alpha$, then $\beta.\mu_A(\alpha) = 1$ and $\beta.\nu_A(\alpha) = \alpha$.

Proof. By Proposition (3.6), $\mu_A(\alpha) \geq \mu_A(\partial)$ for all $\partial \in X$, therefore, $\beta.\mu_A(\alpha) \geq \beta.\mu_A(\partial_n)$ for every positive integer n.

Consider, $1 \geq \beta.\mu_A(\partial_n) = 1$. Hence $\beta.\mu_A(\alpha) = 1$.

Again by Proposition (3.6), $\nu_A(\alpha) \leq \nu_A(\partial)$, thus $\partial \in X$, thus

$\beta.\nu_A(\alpha) \leq \beta.\nu_A(\partial_n)$, for every positive integer n.

Now, $\alpha \leq \beta.\nu_A(\alpha) \leq \beta.\nu_A(\partial_n) = \alpha$. Hence $\beta.\nu_A(\alpha) = \alpha$. ■

Proposition 3.11. If the β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ of A is fuzzy subalgebra, then for all $\partial \in X$,

$\beta.\mu_A(\alpha * \partial) \geq \beta.\mu_A(\partial)$ and $\beta.\nu_A(\alpha * \partial) \leq \beta.\nu_A(\partial)$.

Proof. For all $\partial \in X$,

$$\begin{aligned}\beta.\mu_A(\alpha * \partial) &\geq \min\{\beta.\mu_A(\alpha), \beta.\mu_A(\partial)\} \\ &= \min\{\beta.\mu_A(\partial * \partial), \beta.\mu_A(\partial)\} \\ &\geq \min\{\min\{\beta.\mu_A(\partial), \beta.\mu_A(\partial), \beta.\mu_A(\partial)\} \\ &= \beta.\mu_A(\partial)\end{aligned}$$

and

$$\begin{aligned}\beta.\nu_A(\alpha * \partial) &\leq \max\{\beta.\nu_A(\alpha), \beta.\nu_A(\partial)\} \\ &= \max\{\beta.\nu_A(\partial * \partial), \beta.\nu_A(\partial)\} \\ &\leq \max\{\max\{\beta.\nu_A(\partial), \beta.\nu_A(\partial), \beta.\nu_A(\partial)\} \\ &= \beta.\nu_A(\partial).\end{aligned}$$ ■

Definition 3.12. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ of A and $B^M_\beta = \{(\partial, (\mu_B)^M_\beta, (\nu_B)^M_\beta) \mid x \in X\}$ of B be two β-multiplication intuitionistic of fuzzy subsets of X, then **the intersection of A^M_β and B^M_β** are denoted by $A^M_\beta \cap B^M_\beta$ and is given by

$A^M_\beta \cap B^M_\beta = \{min\{(\mu_A)^M_\beta, (\mu_B)^M_\beta\}, \max\{(\nu_A)^M_\beta, (\nu_B)^M_\beta\}\}$.

Also, **the complement of A** denoted by $\bar{A}^M_\beta$ and is defined by

$\bar{A}^M_\beta = \{(\partial, (\nu_A)^M_\beta, (\mu_A)^M_\beta) \mid \partial \in X\}$.

The intersection of two β-multiplication intuitionistic of fuzzy subalgebras, which proved in the following theorem.

Theorem 3.13. Let A and B be two β-multiplication intuitionistic of fuzzy subalgebras of BZ-algebra $(X;*,\alpha)$, then $A \cap B$ is β-multiplication intuitionistic of fuzzy subalgebra of X.

Proof. Let $\partial, y \in A \cap B$, then $\partial, y \in A$ and B, then

$$\begin{aligned}(\nu_B)^M_\beta(\partial * y) &= \min\{(\mu_A)^M_\beta(\partial * y), (\mu_B)^M_\beta(\partial * y)\} \\ &\geq \min\{\min\{(\mu_A)^M_\beta(\partial), (\mu_A)^M_\beta(y)\}, \min\{(\mu_B)^M_\beta(\partial), (\mu_B)^M_\beta(y)\}\} \\ &= \min\{\min\{(\mu_A)^M_\beta(\partial), (\mu_B)^M_\beta(y)\}, \min\{(\mu_A)^M_\beta(\partial), (\mu_B)^M_\beta(y)\}\}\end{aligned}$$

$= \min\{(\mu_{A\cap B})^M_\beta(\partial), (\mu_{A\cap B})^M_\beta(y)\}$ and

$(\nu_{A\cap B})^M_\beta(\partial * y) = \max\{(\nu_A)^M_\beta(\partial * y), (\nu_B)^M_\beta(\partial * y)\}$

$\leq \max\{\max\{(\nu_A)^M_\beta(\partial), (\nu_A)^M_\beta(y)\}, \max\{(\nu_B)^M_\beta(\partial), (\nu_B)^M_\beta(y)\}\}$

$= \max\{\max\{(\nu_A)^M_\beta(\partial), (\nu_B)^M_\beta(y)\}, \max\{(\nu_A)^M_\beta(\partial), (\nu_B)^M_\beta(y)\}\}$

$= \max\{(\nu_{A\cap B})^M_\beta(\partial), (\nu_{A\cap B})^M_\beta(y)\}$

Hence $A \cap B$ is β-multiplication intuitionistic of fuzzy subalgebra of X. ■

Theorem 3.14. The β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ is β-multiplication intuitionistic of fuzzy subalgebra of X if and only if, the fuzzy sets $(\mu_A)^M_\beta$ and $(\bar{\nu}_A)^M_\beta$ are fuzzy subalgebras of X.

Proof. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ be β-multiplication intuitionistic of fuzzy subalgebra of X.

Cleary$(\mu_A)^M_\beta$ is a fuzzy subalgebra of X, for every ∂, y ∈ X, we have

$(\bar{\nu}_A)^M_\beta(\partial * y) = 1 - (\nu_A)^M_\beta(\partial * y)$

$\geq 1 - \max\{(\nu_A)^M_\beta(\partial), (\nu_A)^M_\beta(y)\}$

$= \min\{1 - (\nu_A)^M_\beta(\partial), 1 - (\nu_A)^M_\beta(y)\}$

$= \min\{(\bar{\nu}_A)^M_\beta(\partial), (\bar{\nu}_A)^M_\beta(y)\}$.

Hence $(\bar{\nu}_A)^M_\beta$ is a fuzzy subalgebra of X.

Conversely, assume that $(\mu_A)^M_\beta$ and $(\bar{\nu}_A)^M_\beta$ are two β-multiplication intuitionistic of fuzzy subalgebras of X.

For every ∂, y ∈ X,$(\bar{\nu}_A)^M_\beta(\partial * y) \geq \min\{(\bar{\nu}_A)^M_\beta(\partial), (\bar{\nu}_A)^M_\beta(y)\}$ and

$1 - (\nu_A)^M_\beta(\partial * y) = (\bar{\nu}_A)^M_\beta(\partial * y)$

$\geq \min\{(\bar{\nu}_A)^M_\beta(\partial), (\bar{\nu}_A)^M_\beta(y)$

$= \min\{1 - (\nu_A)^M_\beta(\partial), 1 - (\nu_A)^M_\beta(y)\}$

$= 1 - \max\{(\nu_A)^M_\beta(\partial), (\nu_A)^M_\beta(y)\}$.

That is, $(\bar{\nu}_A)^M_\beta(\partial * y) \leq \max\{(\bar{\nu}_A)^M_\beta(\partial), (\bar{\nu}_A)^M_\beta(y)\}$.

Hence $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ be β-multiplication intuitionistic of fuzzy subalgebra of X. ■

For any element ∂ and y of X, let us write $\Pi^n \partial * y$ for $\partial * (\cdots * (\partial * y)))$, where ∂ occurs n times.

Theorem 3.15. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ be β-multiplication intuitionistic of fuzzy subalgebra of BZ-algebra $(X;*,\alpha)$ and let n∈ N (the set of natural numbers) then for all $\partial \in X$.

(i) $(\mu_A)^M_\beta(\Pi^n x * \partial) \geq (\mu_A)^M_\beta(\partial)$, for any add number n,

(ii) $(\nu_A)^M_\beta(\Pi^n x * \partial) \leq (\nu_A)^M_\beta(\partial)$, for any add number n,

(iii) $(\mu_A)^M_\beta(\Pi^n x * \partial) = (\mu_A)^M_\beta(\partial)$, for any even number n,

(iv) $(\nu_A)^M_\beta(\Pi^n x * \partial) = (\nu_A)^M_\beta(\partial)$, for any even number n.

Proof. Let $\partial \in X$ and assum that n is odd, then n=2p-1, for some positive integer p.

We prove the theorem by induction. Now, $(\mu_A)^M_\beta(\partial * \partial) = (\mu_A)^M_\beta(\alpha) \geq (\mu_A)^M_\beta(\partial)$ and $(\nu_A)^M_\beta(\partial * \partial) = (\nu_A)^M_\beta(\alpha) \leq (\nu_A)^M_\beta(\partial)$. Suppose that $(\mu_A)^M_\beta(\Pi^{2p-1}x*\partial) \geq (\mu_A)^M_\beta(\partial)$ and $(\nu_A)^M_\beta(\Pi^{2p-1}x*\partial) \leq (\nu_A)^M_\beta(\partial)$, then by assumption,

$(\mu_A)^M_\beta(\Pi^{2(p+1)-1}\partial*\partial) = (\mu_A)^M_\beta(\Pi^{2p+1}x*\partial)$

$= (\mu_A)^M_\beta(\Pi^{2p-1}x*(\partial*(\partial*\partial)))$

$= (\mu_A)^M_\beta(\Pi^{2p-1}x*\partial)$

$\geq (\mu_A)^M_\beta(\partial)$ and

$(\nu_A)^M_\beta(\Pi^{2(p+1)-1}x*\partial) = (\nu_A)^M_\beta(\Pi^{2p+1}x*\partial)$

$= (\nu_A)^M_\beta(\Pi^{2p-1}x*(\partial*(\partial*\partial)))$

$= (\nu_A)^M_\beta(\Pi^{2p-1}x*\partial)$

$\geq (\nu_A)^M_\beta(\partial)$, which proves (*i*) and (*ii*).

Proofs are similar to the cases (iii) and (iv). ■

Remark 3.16. The set $I_{\mu_A} = \{\partial \in X \mid (\mu_A)^M_\beta(\partial) = (\mu_A)^M_\beta(\alpha)\}$ is subset of X and the set $I_{\nu_A} = \{\partial \in X \mid (\nu_A)^M_\beta(\partial) = (\nu_A)^M_\beta(\alpha)\}$ is subset of X.

Theorem 3.17. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ be β-multiplication intuitionistic of fuzzy subalgebra of X, then the sets I_{μ_A} and I_{ν_A} are subalgebras of X.

Proof.

Let ∂, y ∈ I_{μ_A}, then $(\mu_A)^M_\beta(\partial) = \beta.\mu_A(\partial) = \beta.\mu_A(\alpha) = (\mu_A)^M_\beta(\alpha) = \beta.\mu_A(y) = (\mu_A)^M_\beta(y)$ and so, $(\mu_A)^M_\beta(\partial * y) = \beta.\mu_A(\partial * y) \geq \min\{\beta.\mu_A(\partial), \beta.\mu_A(y)\} = \beta.\mu_A(\alpha)$, by Proposition (3.6), we know that $(\mu_A)^M_\beta(\partial * y) = (\mu_A)^M_\beta(\alpha)$ or equivalently $\partial * y \in I_{\mu_A}$.

Again, let ∂, y∈ I_{ν_A}, then $(\nu_A)^M_\beta(\partial) = \beta.\nu_A(\partial) = \beta.\nu_A(\alpha) = (\nu_A)^M_\beta(\alpha) = \beta.\nu_A(y) = (\nu_A)^M_\beta(y)$ and so,$(\nu_A)^M_\beta(\partial * y) = \beta.\nu_A(\partial * y) \leq \max\{\beta.\nu_A(\partial), \beta.\nu_A(y)\} = \beta.\nu_A(\alpha)$.

Again by Proposition (3.5), we know that $(\nu_A)^M_\beta(\partial * y) = (\nu_A)^M_\beta(\alpha)$ or equivalently $\partial * y \in I_{\nu_A}$.

Hence, the sets I_{μ_A} and I_{ν_A} are subalgebra of X. ■

4. β-MULTIPLICATION INTUITIONISTIC OF FUZZY BZ-IDEALS.

We study the notion of β-multiplication intuitionistic of fuzzy BZ-ideal of BZ-algebra and we give some properties of it.

Definition 4.1. Let $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ be β-multiplication intuitionistic of fuzzy subset of BZ-algebra $(X;*,\alpha)$. A_β^M is said to be ***β*-multiplication intuitionistic of fuzzy BZ-ideal of X** if for all ∂, y, z ∈X,

(IFBZ1) $(\mu_A)_\beta^M(\alpha) = \beta.\mu_A(\alpha) \geq (\mu_A)_\beta^M(\partial) = \beta.\mu_A(\partial)$ and

$$(\nu_A)_\beta^M(\alpha) = \beta.\nu_A(\alpha) \leq (\nu_A)_\beta^M(\partial) = \beta.\nu_A(\partial).$$

(IFBZ2) $(\mu_A)_\beta^M(\partial * z) = \beta.\mu_A(\partial * z) \geq \min\{\beta.\mu_A((\partial * y) * z), \beta.\mu_A(y)\}$ and

$$(\nu_A)_\beta^M(\partial * z) = \beta.\nu_A(\partial * z) \leq \max\{\beta.\nu_A((\partial * y) * z), \beta.\nu_A(y)\}.$$

That mean $(\mu_A)_\beta^M$ is a fuzzy BZ-ideal of X and $(\nu_A)_\beta^M$ is a doubt fuzzy BZ-ideal of X.

Theorem 4.2. If $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ is an intuitionistic fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$, then the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is an intuitionistic fuzzy BZ-ideal of X, for all $\beta \in (\alpha, 1)$.

Proof: Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an intuitionistic fuzzy BZ-ideal of X and $\beta \in (\alpha, 1)$, then $(\mu_A)_\beta^M(\alpha) = \beta.\mu_A(\alpha) \geq \beta.\mu_A(\partial) = (\mu_A)_\beta^M(\partial)$ and

$(\nu_A)_\beta^M(\alpha) = \beta.\nu_A(\alpha) \leq \beta.\nu_A(\partial) = (\nu_A)_\beta^M(\partial)$, for all $\partial \in X$.

$$\begin{aligned}(\mu_A)_\beta^M(\partial * z) = \beta.\mu_A(\partial * z) &\geq \beta.\min\{\mu_A((\partial * y) * z), \mu_A(y)\}\\ &= \min\{\beta.\mu_A((\partial * y) * z), \beta.\mu_A(y)\}\\ &= \min\{(\mu_A)_\beta^M((\partial * y) * z), (\mu_A)_\beta^M(y)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_A)_\beta^M(\partial * z) &= \beta.\nu_A(\partial * z)\\ &\leq \beta.\max\{\nu_A((\partial * y) * z), \nu_A(y)\}\\ &= \max\{\beta.\nu_A((\partial * y) * z), \beta.\nu_A(y)\}\\ &= \max\{(\nu_A)_\beta^M((\partial * y) * z), (\nu_A)_\beta^M(y)\},\end{aligned}$$

for all ∂, y, z ∈ X.

Hence, the β-multiplication intuitionistic A_β^M of A is an intuitionistic fuzzy BZ-ideal of X. ■

Theorem 4.3. Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an intuitionistic fuzzy subset of BZ-algebra $(X;*,\alpha)$ such that the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is an intuitionistic fuzzy BZ-ideal of X. for some $\beta \in (\alpha, 1)$, then A is an intuitionistic fuzzy BZ-ideal of X.

Proof: Assume that $A_\alpha^T = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ is an intuitionistic fuzzy BZ-ideal of X for some α ∈ [α, ξ]. Let ∂, y∈ X,

$\beta.\mu_A(\alpha) = (\mu_A)_\beta^M(\alpha) \geq (\mu_A)_\beta^M(\partial) = \beta.\mu_A(\partial)$,

$\beta.\nu_A(\alpha) = (\nu_A)_\beta^M(o) \leq (\nu_A)_\beta^M(\partial) = \beta.\nu_A(\partial)$.

which implies $\mu_A(\alpha) \geq \mu_A(\partial)$ and $\nu_A(\alpha) \leq \nu_A(\partial)$.

Now, we have

$$\begin{aligned}\beta.\mu_A(\partial * z) = (\mu_A)_\beta^M(\partial * z) &\geq \min\{(\mu_A)_\beta^M((\partial * y) * z), (\mu_A)_\beta^M(y)\}\\ &= \min\{\beta.\mu_A((\partial * y) * z), \beta.\mu_A(y)\}\\ &= \beta.\min\{\mu_A((\partial * y) * z), \mu_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\beta.\nu_A(\partial * z) = (\nu_A)_\beta^M(\partial * z) &\leq \max\{(\nu_A)_\beta^M((\partial * y) * z), (\nu_A)_\beta^M(y)\}\\ &= \max\{\beta.\nu_A((\partial * y) * z), \beta.\nu_A(y)\}\\ &= \beta.\max\{\nu_A((\partial * y) * z), \nu_A(y)\},\end{aligned}$$

which implies that $\mu_A(\partial * z) \geq \min\{\mu_A((\partial * y) * z), \mu_A(y)\}$

and $\nu_A(\partial * z) \leq \max\{\nu_A((\partial * y) * z), \nu_A(y)\}$.

for all ∂, y, z ∈ X.

Hence, $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ is an intuitionistic fuzzy BZ-ideal of X.■

Theorem 4.4 If the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is an intuitionistic fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$, for all $\beta \in (\alpha, 1)$, then A_β^M must be an intuitionistic fuzzy subalgebra of X.

Proof: Let the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A be intuitionistic fuzzy BZ-ideal of X .

We have for all ∂, y, z ∈ X,

$(\mu_A)_\beta^M(\partial * z) \geq \min\{(\mu_A)_\beta^M((\partial * y) * z), (\mu_A)_\beta^M(y)\}$ and

$(\nu_A)_\beta^M(\partial * z) \leq \max\{(\nu_A)_\beta^M((\partial * y) * z), (\nu_A)_\beta^M(y)\}$, then

$$\begin{aligned}(\mu_A)_\beta^M(\partial * z) &\geq \min\{(\mu_A)_\beta^M((\partial * y) * z), (\mu_A)_\beta^M(y)\}\\ &= \min\{(\mu_A)_\beta^M(\partial * \alpha), (\mu_A)_\beta^M(y)\}\\ &= \min\{(\mu_A)_\beta^M(\partial), (\mu_A)_\beta^M(y)\}\end{aligned}$$

and

$$\begin{aligned}(\nu_A)_\beta^M(\partial * z) &\leq \max\{(\nu_A)_\beta^M(\partial * (y * z)), (\nu_A)_\beta^M(y)\}\\ &= \max\{(\nu_A)_\beta^M(\partial * \alpha), (\nu_A)_\beta^M(y)\}\\ &= \max\{(\nu_A)_\beta^M(\partial), (\nu_A)_\beta^M(y)\end{aligned}$$

Hence, A_β^M is of intuitionistic fuzzy subalgebra of X. ■

Theorem 4.5. If $A = \{(\partial, \mu_A(\partial), \nu_A(\partial)) \mid \partial \in X\}$ is an intuitionistic fuzzy subset of BZ-algebra $(X;*,\alpha)$ such that the β-multiplication intuitionistic $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ of A is an intuitionistic fuzzy BZ-ideal of X, for $\beta \in (\alpha, 1)$, then the sets I_{μ_A} and I_{ν_A} are BZ-ideals of X.

Proof: Let $A_\beta^M = \{(\partial, (\mu_A)_\beta^M, (\nu_A)_\beta^M) \mid x \in X\}$ is an intuitionistic fuzzy BZ-ideal of X, so$(\mu_A)_\beta^M$ is fuzzy BZ-ideal of X and $(\nu_A)_\beta^M$ is doubt fuzzy BZ-ideal of X.

Clearly $\alpha \in I_{\mu_A}, I_{\nu_A}$, suppose ∂, y, z ∈ X such that $((\partial * y) * z) \in I_{\mu_A}$ and $y \in I_{\nu_A}$, hence $(\mu_A)_\beta^M((\partial * y) * z) = (\mu_A)_\beta^M(\alpha) = (\mu_A)_\beta^M(y)$ and

$(\mu_A)_\beta^M(\partial * z) \geq \min\{(\mu_A)_\beta^M((\partial * y) * z), (\mu_A)_\beta^M(y)\} = (\mu_A)_\beta^M(\alpha)$.

Since, $(\mu_A)^M_\beta$ is fuzzy BZ-ideal of X, then $(\mu_A)^M_\beta(\partial * z) = (\mu_A)^M_\beta(\alpha)$.

Hence, $\beta.\mu_A(\partial * z) = \beta.\mu_A(\alpha)$ or $\mu_A(\partial * z) = \mu_A(\alpha)$ and $(\partial * z) \in I_{\mu_A}$ then I_{μ_A} is BZ-ideal of X.

Also, suppose that u, v, w, ∈ X such that $((u * v) * w) \in I_{\nu_A}$ and $v \in I_{\nu_A}$. Hence

$(\nu_A)^M_\beta((u * v) * w) = (\nu_A)^M_\beta(\alpha) = (\nu_A)^M_\beta(v)$ and

$$(N_A)^M_\beta(u * w) \le \max\{(\nu_A)^M_\beta((u * v) * w), (\nu_A)^M_\beta(v)\} = (\nu_A)^M_\beta(\alpha).$$

Since, $(\nu_A)^M_\beta$ is fuzzy BZ-ideal of X, then $(\nu_A)^M_\beta(u * w) = (\nu_A)^M_\beta(\alpha)$.

Hence $\beta.\nu_A(u * w) = \beta.\nu_A(\alpha), \nu_A(u * v) = \nu_A(\alpha)$ and $(u * v) \in I_{\nu_A}$, then I_{ν_A} is BZ-ideal of X. ■

Proposition 4.6. Let the β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ of A be intuitionistic fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$ for $\beta \in (\alpha, 1)$.

If $\partial \le y$ then $(\mu_A)^M_\beta(\partial) \ge (\mu_A)^M_\beta(y)$ and $(\nu_A)^M_\beta(x) \le (\nu_A)^M_\beta(y)$, that is, $(\mu_A)^M_\beta$ is order-reversing and $(\nu_A)^M_\beta$ is order-preserving.

Proof: Suppose that ∂, y ∈ X and $\partial \le y$, then $\partial * y = \alpha$ and

$$(\mu_A)^M_\beta(\partial) = (\mu_A)^M_\beta(\partial * \alpha) \ge \min\{(\mu_A)^M_\beta((\partial * y) * \alpha), (\mu_A)^M_\beta(y)\}$$
$$= \min\{(\mu_A)^M_\beta(\partial * y), (\mu_A)^M_\beta(y)\}$$
$$= \min\{(\mu_A)^M_\beta(\alpha), (\mu_A)^M_\beta(y)\} = (\mu_A)^M_\beta(y)$$ and

$$(\nu_A)^M_\beta(\partial) = (\nu_A)^M_\beta(\partial * \alpha)$$
$$\le \max\{(\nu_A)^M_\beta((\partial * y) * \alpha), (\nu_A)^M_\beta(y)\}$$
$$= \max\{(\nu_A)^M_\beta(\partial * y), (\nu_A)^M_\beta(y)$$
$$= \max\{(\nu_A)^M_\beta(\alpha), (\nu_A)^M_\beta(y)\} = (\nu_A)^M_\beta(y). \quad ■$$

Theorem 4.7. The β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ of A is a fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$, then for all $t, s \in [\alpha, 1]$, the set $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are BZ-ideals of X.

Proof. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ be fuzzy BZ-ideal of X and $U_\beta(\mu_A, t) \ne \emptyset \ne L_\beta(\nu_A, s)$.

Since $\beta.\mu_A(\alpha) \ge t$ and $\beta.\nu_A(\alpha) \le s$, let $\partial, y, z \in X$ be such that $((\partial * y) * z) \in U_\beta(\mu_A, t)$, $(y) \in U_\beta(\mu_A, t)$, then

$\beta.\mu_A((\partial * y) * z) \ge t$ and $\beta.\mu_A(y) \ge t$, it follows that $\beta.\mu_A(\partial * z) \ge \min\{\beta.\mu_A((\partial * y) * z), \beta.\mu_A(y)\} \ge t$, so that $(\partial * z) \in U_\beta(\mu_A, t)$.

Hence $U_\beta(\mu_A, t)$ is an BZ-ideal of X.

In a similar way, we can prove that $L_\beta(\nu_A, s)$ is BZ-ideal of X. ■

Theorem 4.8. The β-multiplication intuitionistic $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ of A is a fuzzy subset of BZ-algebra $(X;*,\alpha)$ such that for all $t, s \in [\alpha, 1]$, the set $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are BZ-ideals of X, then A^M_β is a fuzzy BZ-ideal of X.

Proof. Assume that for each , t , s ∈ $[\alpha, 1]$, the sets $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are BZ-ideal of X. For any ∈ X , let $\beta.\mu_A(\partial) = t$ and $\beta.\nu_A(\partial) = s$, then $\partial \in U_\beta(\mu_A, t) \cap L_\beta(\nu_A, s)$ and so $U_\beta(\mu_A, t) \ne \emptyset \ne L_\beta(\nu_A, s)$.

Since $U_\beta(\mu_A, t)$ and $L_\beta(\nu_A, s)$ are BZ-ideals of X, therefore $\alpha \in U_\beta(\mu_A, t) \cap L_\beta(\nu_A, s)$.

Hence $\beta.\mu_A(\alpha) \ge t = \mu_A(\partial)$ and $\beta.\nu_A(\alpha) \le s = \nu_A(\partial)$, for all $\partial \in X$.

If there exist $\partial', y', z' \in X$ be such that $\beta.\mu_A(\partial' * z') < \min\{\beta.\mu_A((\partial' *'^{y'}) * z'), \beta.\mu_A(y')\}$, then by taking

$t_\alpha = \frac{1}{2}\{\beta.\mu_A(\partial' * z') + \min\{\beta.\mu_A((\partial' *'^{y'}) * z'), \beta.\mu_A(y')\}\}$, we get $\beta.\mu_A(\partial' * z') < t_\alpha < \min\{\beta.\mu_A(\partial' *'(y' * z'), \beta.\mu_A(y')\}$ and hence

$(\partial' * z') \in U_\beta(\mu_A, t_\alpha)$, $((\partial' *' y') * z') \in U_\beta(\mu_A, t_\alpha)$, $(y') \in U_\beta(\mu_A, t_\alpha)$.

i.e., $U_\beta(\mu_A, t_\alpha)$, is not an BZ-ideal of X, Leading to contradiction.

Finally assume that there exist $\partial', y', z' \in X$ such that $\beta.\nu_A(\partial' * z') > \max\{\beta.\nu_A((\partial' *' y') * z'), \beta.\mu_A(y')\}$, then by taking

$s_\alpha = \frac{1}{2}\{\beta.\nu_A(\partial' * z') + \max\{\beta.\nu_A((\partial' *' y') * z'), \beta.\mu_A(y')\}\}$, we get $\max\{\beta.\nu_A((\partial' *' y') * z'), \beta.\nu_A(y')\} > s_\alpha > \nu_A(\partial' * z')$ and hence $(\partial' * z') \in U_\beta(\nu_A, s_\alpha)$, $((\partial' *'^{y'}) * z') \in L_\beta(\nu_A, s_\alpha)$, $(y') \in L_\beta(\nu_A, s_\alpha)$.

i.e., $L_\beta(\nu_A, s_\alpha)$, is not an BZ-ideal of BZ-algebra X, Leading to contradiction.

Hence $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ is fuzzy BZ-ideal of X.■

Theorem 4.9. Let A and B be two β-multiplication intuitionistic of fuzzy BZ-ideals of BZ-algebra $(X;*,\alpha)$. Then $A \cap B$ is β-multiplication intuitionistic of fuzzy BZ-ideal of X.

Proof. Let ∂, y∈ $A \cap B$, then ∂, y∈ A and B. Now,

$$(\mu_{A\cap B})^M_\beta(\alpha) = (\mu_{A\cap B})^M_\beta(\partial * \partial)$$
$$\ge \min\{(\mu_{A\cap B})^M_\beta(\partial), (\mu_{A\cap B})^M_\beta(\partial)\}$$
$$= (\mu_{A\cap B})^M_\beta(\partial)$$ and

$$(V_{A\cap B})^M_\beta(\alpha) = (\nu_{A\cap B})^M_\beta(\partial * \partial)$$
$$\le \max\{(\nu_{A\cap B})^M_\beta(\partial), (\nu_{A\cap B})^M_\beta(\partial)\}$$
$$= (\nu_{A\cap B})^M_\beta(\partial).$$

Also,

$(\mu_{A\cap B})^M_\beta(\partial * z) = \min\{(\mu_A)^M_\beta(\partial * z), (\mu_B)^M_\beta(\partial * z)\}$
$\ge \min\{\min\{(\mu_A)^M_\beta((\partial * y) * z), (\mu_A)^M_\beta(y)\}, \min\{(\mu_B)^M_\beta((\partial * y) * z), (\mu_B)^M_\beta(y)\}\}$
$= \min\{\min\{(\mu_A)^M_\beta((\partial * y) * z), (\mu_B)^M_\beta((\partial * y) * z)\}, \min\{(\mu_A)^M_\beta(y), (\mu_B)^M_\beta(y)\}\}$
$= \min\{(\mu_{A\cap B})^M_\beta((\partial * y) * z), (\mu_{A\cap B})^M_\beta(y)\}$ and

$(\nu_{A\cap B})^M_\beta(\partial * y) = \max\{(\nu_A)^M_\beta(\partial * z), (\nu_B)^M_\beta(\partial * z)\}$

$\leq \max\{\max\{(\nu_A)^M_\beta((\partial * y) * z), (\nu_A)^M_\beta(y)\}, \max\{(\nu_B)^M_\beta((\partial * y) * z), (\nu_B)^M_\beta(y)\}\}$
$=\max\{\max\{(\nu_A)^M_\beta((\partial * y) * z), (\nu_B)^M_\beta((\partial * y) * z)\}, \max\{(\nu_A)^M_\beta(y), (\nu_B)^M_\beta(y)\}\}$
$= \max\{(\nu_{A\cap B})^M_\beta((\partial * y) * z), (\nu_{A\cap B})^M_\beta(y)\}$

Hence, $A \cap B$ is β-multiplication intuitionistic of fuzzy BZ-ideal of X. ■

Theorem 4.10. The β-multiplication intuitionistic of fuzzy $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ of A is fuzzy BZ-ideal of BZ-algebra $(X;*,\alpha)$ if and only if, the fuzzy sets $(\mu_A)^M_\beta$ and $(\bar{\nu}_A)^M_\beta$ are fuzzy BZ-ideals of X.

Proof. Let $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ be β-multiplication intuitionistic of fuzzy BZ-ideal of X.

Cleary, μ_A is a fuzzy BZ-ideal of X, for every ∂, y ∈X, we have

$(\bar{\nu}_A)^M_\beta(\alpha)=1-(\nu_A)^M_\beta(\alpha) \geq 1-(\nu_A)^M_\beta(\partial)=(\bar{\nu}_A)^M_\beta(\partial)$ and

$(\bar{\nu}_A)^M_\beta(\partial * z)=1-(\nu_A)^M_\beta(\partial * z) \geq 1-\max\{(\nu_A)^M_\beta((\partial * y) * z), (\nu_A)^M_\beta(y)\}$

$=\min\{1-(\nu_A)^M_\beta((\partial * y) * z), 1-(\nu_A)^M_\beta(y)\}=\min\{(\bar{\nu}_A)^M_\beta((\partial * y) * z), (\bar{\nu}_A)^M_\beta(y)\}$.

Hence, $(\bar{\nu}_A)^M_\beta$ is a fuzzy BZ-ideal of X.

Conversely, assume that $(\mu_A)^M_\beta$ and $(\bar{\nu}_A)^M_\beta$ are two β-multiplication intuitionistic fuzzy BZ-ideals of X, for every ∂, y∈X, we get

$(\mu_A)^M_\beta(\alpha) \geq (\mu_A)^M_\beta(\partial)$ and $(\bar{\nu}_A)^M_\beta(\alpha) \geq (\bar{\nu}_A)^M_\beta(\partial)$.

This implies, $1-(\nu_A)^M_\beta(\alpha) \geq 1-(\nu_A)^M_\beta(\partial)$, that is, $(\nu_A)^M_\beta(\alpha) \leq (\nu_A)^M_\beta(\partial)$.

Also, $(\nu_A)^M_\beta(\partial * z) \geq \min\{(\nu_A)^M_\beta((\partial * y) * z), (\nu_A)^M_\beta(y)\}$ and

$1-(\nu_A)^M_\beta(\partial * z) = (\bar{\nu}_A)^M_\beta(\partial * z)$
$\geq \min\{(\bar{\nu}_A)^M_\beta((\partial * y) * z), (\bar{\nu}_A)^M_\beta(y)$
$= \min\{1-(\nu_A)^M_\beta((\partial * y) * z), 1-(\nu_A)^M_\beta(y)\}$
$=1-\max\{(\nu_A)^M_\beta((\partial * y) * z), (\nu_A)^M_\beta(y)\}$, that is,

$(\nu_A)^M_\beta(\partial * z) \leq \max\{(\nu_A)^M_\beta((\partial * y) * z), (\nu_A)^M_\beta(y)\}$.

Hence $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) | x \in X\}$ is fuzzy BZ-ideal of X. ■

5- HOMOMORPHISM OF β-MULTIPLICATION INTUITIONISTIC OF BZ-ALGEBRA.

Definition 5.1.[7]. Let $(X;*,\alpha)$ and $(Y;*',\alpha')$ be two nonempty sets. A mapping $f: X \to Y$ is said to be **a homomorphism** if $f(\partial * y) = f(\partial) *' f(y)$, for all ∂, y ∈ X.

Note that if $f: X \to Y$ is a homomorphism of sets, then $f(\alpha) = \alpha'$.

Definition 5.2. Let $f : (X;*,\alpha) \to (Y;*',\alpha')$ be a homomorphism of BZ-algebras for any $A^M_\beta = \{(y, (\mu_A)^M_\beta(y), (\nu_A)^M_\beta(y)) | y \in Y\}$ in Y and $\beta \in (\alpha, 1)$ we define new $(A^M_\beta)^f = \{(\partial, ((\mu_A)^M_\beta)^f, ((\nu_A)^M_\beta)^f) | \partial \in X\}$ in X by

$((\mu_A)^M_\beta)^f = (\mu_A)^M_\beta(f(\partial))$ and $((\nu_A)^M_\beta)^f(\partial) = (\nu_A)^M_\beta(f(\partial))$, for all $\partial \in X$.

Theorem 5.3. Let $f : (X;*,\alpha) \to (Y;*',\alpha')$ be a homomorphism of BZ-algebra X into BZ-algebra Y. If $A^M_\beta = \{(\partial, (\mu_A)^M_\beta(\partial), (\nu_A)^M_\beta(\partial)) | \partial \in X\}$ is β-multiplication intuitionistic fuzzy subalgebra in X, then the image of A^M_β is β-multiplication intuitionistic fuzzy subalgebra in Y.

Proof. We show that the image of A^M_β is β-multiplication intuitionistic of fuzzy subalgebra in Y, since A^M_β is β-multiplication intuitionistic of fuzzy subalgebra in X and for any a, b ∈ X there exist ∂, y ∈ Y such that $f(a) = \partial$, $f(b) = y$, then

$(\mu_A)^M_\beta(\partial *' y) = (\mu_A)^M_\beta(f(a) *' f(b))$
$\geq \min\{(\mu_A)^M_\beta(f(a)), (\mu_A)^M_\beta(f(b))\}$
$= \min\{(\mu_A)^M_\beta(\partial), (\mu_A)^M_\beta(y)\}$, and

$(\nu_A)^M_\beta(\partial *' y) = (\nu_A)^M_\beta(f(a) *' f(b))$
$\leq \max\{(\nu_A)^M_\beta(f(a)), (\nu_A)^M_\beta(f(b))\}$
$= \max\{(\nu_A)^M_\beta(\partial), (\nu_A)^M_\beta(y)\}$.

Hence the image of A^M_β is fuzzy subalgebra in Y. ■

Theorem 5.4. Let $f : (X;*,\alpha) \to (Y;*',\alpha')$ be a homomorphism of BZ-algebra X into BZ-algebra Y. If $A^M_\beta = \{(\partial, (\mu_A)^M_\beta(\partial), (\nu_A)^M_\beta(\partial)) | \partial \in X\}$ is β-multiplication intuitionistic of fuzzy subalgebra in Y, then the pre-image of A^M_β is β-multiplication intuitionistic of fuzzy subalgebra in X.

Proof. We show that $(A^M_\beta)^f$ is β-multiplication intuitionistic of fuzzy subalgebra in X, since A^M_β is β-multiplication intuitionistic of fuzzy BZ-ideal in Y and let ∂, y ∈ X,

$((\mu_A)^M_\beta)^f(\partial * y) = (\mu_A)^M_\beta(f(\partial * y))$
$= (\mu_A)^M_\beta(f(\partial) *' f(y))$
$\geq \min\{(\mu_A)^M_\beta(f(\partial)), (\mu_A)^M_\beta(f(y))\}$
$= \min\{((\mu_A)^M_\beta)^f(\partial), ((\mu_A)^M_\beta)^f(y)\}$, and

$((\nu_A)^M_\beta)^f(\partial * y) = (\nu_A)^M_\beta(f(\partial * y))$
$= (\nu_A)^M_\beta(f(\partial) *' f(y))$
$\leq \{(\nu_A)^M_\beta(f(\partial)), (\nu_A)^M_\beta(f(y))\}$
$= \{((\nu_A)^M_\beta)^f(\partial), ((\nu_A)^M_\beta)^f(y)\}$.

Hence $(A^M_\beta)^f$ is fuzzy subalgebra in X. ■

Theorem 5.5. Let $f : (X;*,\alpha) \to (Y;*',\alpha')$ be a homomorphism of BZ-algebras. If

$B^M_\beta = \{(\partial, (\mu_B)^M_\beta, (\nu_B)^M_\beta) \mid x \in X\}$ is β-multiplication intuitionistic of fuzzy BZ-ideal of Y, then the preimage $f^{-1}(B^M_\beta) = (f^{-1}((\mu_B)^M_\beta), f^{-1}((\nu_B)^M_\beta))$ of B^M_β under f in X is β-multiplication intuitionistic of fuzzy BZ-ideal of X.

Proof. For all $\partial \in$ X,

$f^{-1}((\mu_B)^M_\beta)(\partial) = (\mu_B)^M_\beta(f(\partial))$
$\leq \left((\mu_B)^M_\beta(\alpha) = (\mu_B)^M_\beta(f(\alpha)\right)$
$= f^{-1}((\mu_B)^M_\beta)(\alpha)$ and

$f^{-1}((\nu_B)^M_\beta)(\partial) = (\nu_B)^M_\beta(f(\partial))$
$\geq (\nu_B)^M_\beta(\alpha) = (\nu_B)^M_\beta(f(\alpha))$
$= f^{-1}((\nu_B)^M_\beta)(\alpha)$.

Let ∂, y, z $\in$ X , then

$(f^{-1}((\mu_B)^M_\beta)(\partial * z) = (\mu_B)^M_\beta(f(\partial * z))$
$\geq \min\{(\mu_B)^M_\beta\left(\left(f(\partial) *' f(y)\right) *' f(z)\right), (\mu_B)^M_\beta(f(y))\}$
$\geq \min\{(\mu_B)^M_\beta(f((\partial * y) * z), \left((\mu_B)^M_\beta(f(y))\right\}$
$= \min\{(f^{-1}((\mu_B)^M_\beta)((\partial * y) * z), (f^{-1}((\mu_B)^M_\beta)(y)\}$ and

$(f^{-1}((\nu_B)^M_\beta)(\partial * z) = ((\nu_B)^M_\beta(f(\partial * z))$
$\leq \max\{(\nu_B)^M_\beta\left(\left(f(\partial) *' f(y)\right) *' f(z)\right), (\nu_B)^M_\beta(f(y))\}$
$\geq \max\{(\nu_B)^M_\beta(f((\partial * y) * z), (\nu_B)^M_\beta(f(y))\}$
$= \max\{f^{-1}((\nu_B)^M_\beta)((\partial * y) * z), f^{-1}((\nu_B)^M_\beta)(y)\}$.

Hence, $f^{-1}(B)^M_\beta = (f^{-1}((\mu_B)^M_\beta), f^{-1}((\nu_B)^M_\beta))$ is β-multiplication intuitionistic of fuzzy BZ-ideal of X. ■

Theorem 5.6. Let $f : (X;*,\alpha) \to (Y;*',\alpha')$ be an epimorphism of BZ-algebras. If $A^M_\beta = \{(\partial, (\mu_A)^M_\beta, (\nu_A)^M_\beta) \mid x \in X\}$ is β-multiplication intuitionistic of fuzzy BZ-ideal of X, then $f(A^M_\beta)=\left(f((\mu_A)^M_\beta), f((\nu_A)^M_\beta)\right)$ of A is a fuzzy BZ-ideal of Y.

Proof. For any a$\in$ X, there exists y $\in$ Y such that $f(a) = y$, then

$f((\mu_A)^M_\beta)(y) = f((\mu_A)^M_\beta)(f(a))$
$= f^{-1}\left(f((\mu_A)^M_\beta\right)(a)$
$= (\mu_A)^M_\beta(a) \leq (\mu_A)^M_\beta(\alpha)$
$=f^{-1}\left(f((\mu_A)^M_\beta\right)(\alpha)$
$= f((\mu_A)^M_\beta)(f(\alpha)) = f((\mu_A)^M_\beta)(\alpha')$ and

$f((\nu_A)^M_\beta)(y) = f((\nu_A)^M_\beta)(f(a))$
$= f^{-1}\left(f((\nu_A)^M_\beta\right)(a)$
$= (\nu_A)^M_\beta(a) \geq (\nu_A)^M_\beta(\alpha)$
$= f^{-1}\left(f((\nu_A)^M_\beta\right)(\alpha)$
$= f((\nu_A)^M_\beta)(f(\alpha)) = f((\nu_A)^M_\beta)(\alpha')$.

Let ∂, y, z $\in$ Y, then f (a)=∂ and f (b)=y and f (c) =z for some a, b, c $\in$ X, thus

$f((\mu_A)^M_\beta)(\partial * z) = f((\mu_A)^M_\beta)(f(a) *' f(c))$
$= f^{-1}\left(f((\mu_A)^M_\beta\right)(a * c)$
$= (\mu_A)^M_\beta(a * c)$
$\geq \{ (\mu_A)^M_\beta((a * b) * c), (\mu_A)^M_\beta(b)\}$
$= \{ f^{-1}\left(f((\mu_A)^M_\beta)\right)((a * b) * c), f^{-1}\left(f((\mu_A)^M_\beta\right)(b)\}$
$= \{ \left(f((\mu_A)^M_\beta\right)\left(\left(f(a) *' f(b)\right) *' f(c)\right), \left(f((\mu_A)^M_\beta\right)(f(b))\}$
$= \{f((\mu_A)^M_\beta)((\partial *' y) *' z), f((\mu_A)^M_\beta(y)\}$ and

$f((\nu_A)^M_\beta)(\partial * z) = f((\nu_A)^M_\beta)(f(a) *' f(c))$
$= f^{-1}\left(f((\nu_A)^M_\beta)\right)(a * c) = (\nu_A)^M_\beta(a * c)$
$\leq \{ (\nu_A)^M_\beta((a * b) * c), (\nu_A)^M_\beta(b)\}$
$= \{ f^{-1}\left(f((\nu_A)^M_\beta\right)((a * b) * c), f^{-1}\left(f((\nu_A)^M_\beta\right)(b)\}$
$= \{ f((\nu_A)^M_\beta\left(\left(f(a) *' f(b)\right) *' f(c)\right), f((\nu_A)^M_\beta(f(b))\}$
$= \{ f((\nu_A)^M_\beta((\partial *' y) *' z), f((\nu_A)^M_\beta(y)\}$.

Hence $f(A^M_\beta)=\left(f((\mu_A)^M_\beta), f((\nu_A)^M_\beta)\right)$ is a fuzzy BZ-ideal of Y. ■

6- CARTESIAN PRODUCT OF β-MULTIPLICATION INTUITIONISTIC OF FUZZY BZ-IDEALS.

In this section, we will discuss, investigate a new notion called Cartesian product of α-translation of intuitionistic fuzzy BZ-ideals and we study several basic properties which related to α-translation of intuitionistic fuzzy BZ-ideals.

Definition 6.1.[1] . Let δ and λ be are two fuzzy subsets in the set X. the Cartesian product $\delta \times \lambda : X \times X \to [\alpha, 1]$ is defined by,
$\delta \times \lambda(\partial, y) = \{\delta(\partial), \lambda(y)\}$,for all ∂, y $\in$X.

Definition 6.2. Let $A^M_\beta = \{(\partial, (\delta_A)^M_\beta(\partial), (\lambda_A)^M_\beta(\partial)) \mid \partial \in X\}$ and $B^M_\beta = \{(\partial, (\delta_B)^M_\beta(\partial), (\lambda_B)^M_\beta(\partial)) \mid \partial \in X\}$ are two β-multiplication intuitionistic of fuzzy subsets of X, and $\beta \in (\alpha, 1)$, **the Cartesian product** $A^M_\beta \times B^M_\beta = \left(X \times X, (\delta_A \times \delta_B)^M_\beta, (\lambda_A \times \lambda_B)^M_\beta\right)$ such that $(\beta_A \times \beta_B)^M_\beta : X \times X \to [\alpha, 1]$ is defined by $(\delta_A \times \delta_B)^M_\beta(\partial, y) = \{ (\delta_A)^M_\beta(\partial), (\delta_B)^M_\beta(y)\} = \{\beta.\delta_A(\partial), \beta.\delta_B(\partial)\}$ and $(\lambda_A \times \lambda_B)^M_\beta : X \times X \to [\alpha, 1]$ is defined by
$(\lambda_A \times \lambda_B)^M_\beta(\partial, y) = \{ (\lambda_A)^M_\beta(\partial), (\lambda_B)^M_\beta(y)\} = \{\beta.\lambda_A(\partial), \beta.\lambda_B(\partial)\}$ for all ∂, y $\in$X.

Remark 6.3. Let $(X;*,\alpha)$ and $(Y;*',\alpha')$ be BZ-algebras, we define " ⋄ " on X × Y by: For every $(\partial, y), (u, v)X \times$

$Y, (\partial, y) \diamond (u, v) = (\partial * u, y *' v)$ then clearly $(X \times Y, \diamond, (\alpha, \alpha'))$ is a BZ-algebra.

Proposition 6.4.[3]. Let $A_\beta^M = \{(\partial, (\delta_A)_\beta^M(\partial), (\lambda_A)_\beta^M(\partial)) \mid \partial \in X\}$ and $B_\beta^M = \{(\partial, (\delta_B)_\beta^M(\partial), (\lambda_B)_\beta^M(\partial)) \mid \partial \in X\}$ are β-multiplication intuitionistic of fuzzy subalgebras of X, then $A_\beta^M \times B_\beta^M$ is fuzzy subalgebra of $X \times X$.

Proof. Let $(\partial_1, \partial_2), (y_1, y_2) \in X \times X$, then

$(\delta_A \times \delta_B)_\beta^M((\partial_1, \partial_2) * (y_1, y_2))$
$= (\delta_A \times \delta_B)_\beta^M(\partial_1 * y_1, \partial_2 * y_2)$
$=\{(\delta_A)_\beta^M(\partial_1 * y_1), (\beta_B)_\beta^M(\partial_2 * y_2)\}$
$\geq \{\min\{(\delta_A \times \delta_A)_\beta^M(\partial_1), (\delta_A \times \delta_A)_\beta^M(y_1)\}, min\{(\delta_B \times \delta_B)_\beta^M(\partial_2), (\delta_B \times \delta_B)_\beta^M(y_2)\}\}$
$= \{\min\{(\delta_A \times \delta_A)_\beta^M(\partial_1), (\delta_B \times \delta_B)_\beta^M(\partial_2)\}, min\{(\delta_A \times \delta_A)_\beta^M(y_1), (\delta_B \times \delta_B)_\beta^M(y_2)\}\}$
$= \{\min\{(\delta_A \times \delta_B)_\beta^M(\partial_1, \partial_2), (\delta_A \times \delta_B)_\beta^M(y_1, y_2)\}$ and

$(\lambda_A \times \lambda_B)_\beta^M((\partial_1, \partial_2) * (y_1, y_2))$
$= (\lambda_A \times \lambda_B)_\beta^M(\partial_1 * y_1, \partial_2 * y_2)$
$= \{(\lambda_A)_\beta^M(\partial_1 * y_1), (\lambda_B)_\beta^M(\partial_2 * y_2)\}$
$\leq \max\{(\lambda_A \times \lambda_A)_\beta^M(\partial_1)\}, (\lambda_A \times \lambda_A)_\beta^M(y_1)\}, \max\{(\lambda_B \times \lambda_B)_\beta^M(\partial_2), (\lambda_B \times \lambda_B)_\beta^M(y_2)\}\}$
$= \max\{(\lambda_A \times \lambda_B)_\beta^M(\partial_1, \partial_2), (\lambda_B \times \lambda_B)_\beta^M(y_1, y_2)\}$. ■

Proposition 6.5. [3]. Let $A_\beta^M = \{(\partial, (\delta_A)_\beta^M(\partial), (\lambda_A)_\beta^M(\partial)) \mid \partial \in X\}$ and $B_\beta^M = \{(\partial, (\delta_B)_\beta^M(\partial), (\lambda_B)_\beta^M(\partial)) \mid \partial \in X\}$ are β-multiplication intuitionistic of fuzzy BZ-ideals of X, then $A_\beta^M \times B_\beta^M$ is fuzzy BZ-ideal of $X \times X$.

Proof. For all $\partial \in X$,

$(\delta_A \times \delta_B)_\beta^M(\alpha, \alpha) = \{(\delta_A)_\beta^M(\alpha), (\beta_B)_\beta^M(\alpha)\}$
$\geq \{(\delta_A)_\beta^M(\partial), (\beta_B)_\beta^M(y)\}$
$= (\delta_A \times \delta_B)_\beta^M(\partial, y)$ and
$(\lambda_A \times \lambda_B)_\beta^M(\alpha, \alpha) = \{(\lambda_A)_\beta^M(\alpha), (\lambda_B)_\beta^M(\alpha)\}$
$\leq \{(\lambda_A)_\beta^M(\partial), (\lambda_B)_\beta^M(y)\}$
$= (\lambda_A \times \lambda_B)_\beta^M(\partial, y)$.

Now, let $(\partial_1, \partial_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$(\delta_A \times \delta_B)_\beta^M((\partial_1, \partial_2) * (z_1, z_2))$
$= (\delta_A \times \delta_B)_\beta^M(\partial_1 * z_1, \partial_2 * z_2)$
$=\{(\delta_A)_\beta^M(\partial_1 * z_1), (\beta_B)_\beta^M(\partial_2 * z_2)\}$
$\geq \{\min\{(\delta_A \times \delta_A)_\beta^M((\partial_1 * y_1) * z_1), (\delta_A \times \delta_A)_\beta^M(y_1)\}, min\{(\delta_B \times \delta_B)_\beta^M((\partial_2 * y_2) * z_2), (\delta_B \times \delta_B)_\beta^M(y_2)\}\}$
$= \{\min\{(\delta_A \times \delta_B)_\beta^M(((\partial_1 * y_1) * z_1), ((\partial_2 * y_2) * z_2))\}, \min\{(\delta_A \times \delta_B)_\beta^M(y_1, y_2)\}\}$
$=\min\{(\delta_A \times \delta_B)_\beta^M\left(((\partial_1, \partial_2) * (y_1, y_2)) * (z_1, z_2)\right), (\delta_A \times \delta_B)_\beta^M(y_1, y_2)\}$ and

$(\lambda_A \times \lambda_B)_\beta^M((\partial_1, \partial_2) * (z_1, z_2))$
$= (\lambda_A \times \lambda_B)_\beta^M(\partial_1 * z_1, \partial_2 * z_2)$
$= \{(\lambda_A)_\beta^M(\partial_1 * z_1), (\lambda_B)_\beta^M(\partial_2 * z_2)\}$
$\leq \max\{(\lambda_A \times \lambda_A)_\beta^M((\partial_1 * y_1) * z_1)\}, (\lambda_A \times \lambda_A)_\beta^M(y_1)\}, \max\{(\lambda_B \times \lambda_B)_\beta^M((\partial_2 * y_2) * z_2), (\lambda_B \times \lambda_B)_\beta^M(y_2)\}\}$
$= \max\{\{(\lambda_A \times \lambda_B)_\beta^M(((\partial_1 * y_1) * z_1), ((\partial_2 * y_2) * z_2))\}, \max\{(\lambda_B \times \lambda_B)_\beta^M(y_1, y_2)\}\}$
$=\min\{(\lambda_A \times \lambda_B)_\beta^M\left(((\partial_1, \partial_2) * (y_1, y_2)) * (z_1, z_2)\right), (\lambda_A \times \lambda_B)_\beta^M(y_1, y_2)\}$. ■

Definition 6.6. Let $A_\beta^M = \{(\partial, (\delta_A)_\beta^M(\partial), (\lambda_A)_\beta^M(\partial)) \mid \partial \in X\}$ and $B_\beta^M = \{(\partial, (\delta_B)_\beta^M(\partial), (\lambda_B)_\beta^M(\partial)) \mid \partial \in X\}$ are β-multiplication intuitionistic of fuzzy subsets of BZ-algebra X, for $s, t \in [\alpha, 1]$ and $\beta \in (\alpha, 1)$ the set $U_\beta(\delta_A \times \delta_B, t) = \{(\partial, y) \in X \times X \mid \beta.(\delta_A \times \delta_B(\partial, y)) \geq t\}$ is called ***β*-multiplication of upper level** of $U_\beta(\delta_A \times \delta_B, t)$ and $L_\beta(\lambda_A \times \lambda_B, s) = \{(\partial, y) \in X \times X \mid \beta.(\lambda_A \times \lambda_B(\partial, y)) \leq s\}$ is called ***β*-multiplication of lower level** of $L_\beta(\lambda_A \times \lambda_B, s)$.

Proposition 6.7. Let $A_\beta^M = \{(\partial, (\delta_A)_\beta^M(\partial), (\lambda_A)_\beta^M(\partial)) \mid \partial \in X\}$ and $B_\beta^M = \{(\partial, (\delta_B)_\beta^M(\partial), (\lambda_B)_\beta^M(\partial)) \mid \partial \in X\}$ are β-multiplication intuitionistic of fuzzy subalgebras of X, then the nonempty set $U_\beta(\delta_A \times \delta_B, s)$ and the nonempty set $L_\beta(\lambda_A \times \lambda_B, t)$ are subalgebras of X, for any $t, s \in [\alpha, 1]$.

Proof. Let A_β^M and B_β^M are β-multiplication intuitionistic of fuzzy subalgebras of X, therefore $(\partial_1, \partial_2), (y_1, y_2) \in X \times X$, $t, s \in [\alpha, 1]$, such that $(\partial_1, \partial_2), (y_1, y_2) \in U_\beta(\delta_A \times \delta_B, t)$ that mean $\{\beta.(\delta_A \times \delta_B)(\partial_1, \partial_2)\} \geq t$ and $\{\beta.(\delta_A \times \delta_B)(y_1, y_2)\} \geq t$, then

$\beta.(\delta_A \times \delta_B)((\partial_1, \partial_2) * (y_1, y_2)) = \beta.(\delta_A \times \delta_B)(\partial_1 * y_1, \partial_2 * y_2)$
$= \beta.\min\{\delta_A(\partial_1 * y_1), \delta_B(\partial_2 * y_2)\} \geq t$.

Therefore $(\partial_1 * y_1, \partial_2 * y_2) \in U_\beta(\delta_A \times \delta_B, t)$. Hence $U_\beta(\delta_A \times \delta_B, t)$ is subalgebra of $X \times X$.

In a similar way, we can prove that $L_\beta(\lambda_A \times \lambda_B, s)$ is subalgebra of $X \times X$. ■

Proposition 6.8.

Let $A_\beta^M = \{(\partial, (\delta_A)_\beta^M(\partial), (\lambda_A)_\beta^M(\partial)) \mid \partial \in X\}$ and $B_\beta^M = \{(\partial, (\delta_B)_\beta^M(\partial), (\lambda_B)_\beta^M(\partial)) \mid \partial \in X\}$ are β-multiplication intuitionistic of fuzzy BZ-ideals of X, then the nonempty sets $U_\beta(\delta_A \times \delta_B, t)$ and $L_\beta(\lambda_A \times \lambda_B, s)$ are BZ-ideals of X, for any $t, s \in [\alpha, 1]$.

Proof. Let A^M_β and B^M_β are β-multiplication intuitionistic of fuzzy subalgebras of X,

Let A^M_β and B^M_β are β-multiplication intuitionistic of fuzzy BZ-ideals of X, therefore for any $(\partial, y) \in X \times X$, for all $t, s \in [\alpha, 1]$

$\{\beta.(\delta_A \times \delta_B)(\alpha, \alpha)\} = \{\beta.\delta_A(\alpha), \beta.\delta_B(\alpha)\}$
$\geq \{\beta.\delta_A(\partial), \beta.\delta_B(y)\} = \beta.(\delta_A \times \delta_B(\partial, y)$ and

$\{\beta.(\lambda_A \times \lambda_B)(\alpha, \alpha)\} = \{\beta.\lambda_A(\alpha), \beta.\lambda_B(\alpha)\}\}$
$\leq \{\beta.\lambda_A(\partial), \beta.\lambda_B(y)\} = \beta.(\lambda_A \times \lambda_B)(\partial, y)$.

Let $(\partial_1, \partial_2), (y_1, y_2), (z_1, z_2) \in X \times X$, such that $(((\partial_1, \partial_2) * (y_1, y_2)) * (z_1, z_2)) \in U_\beta(\delta_A \times \delta_B, t)$ and $(y_1, y_2) \in U_\beta(\delta_A \times \delta_B, t)$, then

$\beta.(\delta_A \times \delta_B)\big((\partial_1, \partial_2) * (z_1, z_2)\big)$
$= \beta.(\delta_A \times \delta_B)(\partial_1 * z_1, \partial_2 * z_2)$
$\geq \beta.\min\{(\delta_A \times \delta_B)\big((\partial_1, \partial_2) * (y_1, y_2)\big) * (z_1, z_2)), (\delta_A \times \delta_B)(y_1, y_1)\}$
$= \beta.\min\{\min\{\delta_A((\partial_1 * y_1) * z_1), \delta_A(y_1)\}, \min\{\delta_B(\partial_2 * y_2) * z_2), \delta_B(y_2)\}\}$
$= \beta.\min\{\min\{t, t\}, \min\{t, t\}\} = \beta.t.$

Thus $(\partial_1, \partial_2) * (z_1, z_2) \in U_\beta(\delta_A \times \delta_B, t)$, then $U_\beta(\delta_A \times \delta_B, t)$ is subalgebra of X. And $(((\partial_1, \partial_2) * (y_1, y_2)) * (z_1, z_2)) \in L_\beta(\lambda_A \times \lambda_B, s)$ and $(y_1, y_2) \in L_\beta(\lambda_A \times \lambda_B, s)$, then

$\beta.(\lambda_A \times \lambda_B)\big((\partial_1, \partial_2) * (z_1, z_2)\big)$
$= \beta.(\lambda_A \times \lambda_B)(\partial_1 * z_1, \partial_2 * z_2)$
$\geq \beta.\min\{(\lambda_A \times \lambda_B)\big((\partial_1, \partial_2) * (y_1, y_2)\big) * (z_1, z_2)), (\lambda_A \times \lambda_B)(y_1, y_1)\}$
$= \beta.\min\{\min\{\lambda_A((\partial_1 * y_1) * z_1), \lambda_A(y_1)\}, \min\{\lambda_B(\partial_2 * y_2) * z_2), \lambda_B(y_2)\}\}$
$= \beta.\min\{\min\{s, s\}, \min\{s, s\}\} = \beta.s.$

Thus $(\partial_1, \partial_2) * (z_1, z_2) \in L_\beta(\lambda_A \times \lambda_B, s)$, then $L_\beta(\lambda_A \times \lambda_B, s)$ is subalgebra of X. ■

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