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Nearly Endo Small T–ABSO Submodules and Related Concepts

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***Abstract**: This study compares the interactions between the Jacobson radical, the Endo small prime submodule, and the Endo small-T-ABSO submodule. To answer this relation etween these concepts, we offer two novel concepts: Nearly Endo small prime submodule and Nearly Endo small T-ABSO submodule. A variety of characteristics are found via the investigation to support the novel theories. The Nearly Endo small T-ABSO submodule structure is also examined, and it is shown that the Nearly Endo T-ABSO submodule and Nearly Endo small T-ABSO submodule are linked using straightforward algebraic techniques. It is also addressed whether the Nearly Endo tiny T-ABSO submodule is compatible with other submodule types. For the purpose of creating a new Nearly Endo small TABSO submodule, the study's findings are essential.*

Keywords: *Endo* prime submodules; *Endo* small prime submodules; *Nearly Endo T– ABSO submodules;* *Endo small T– ABSO submodules.*

1. INTRODUCTION

In this work, G has the identity and would be a commutative ring, while W is a unitary G-module. Research on the concept of a prime submodule of modules was done by Lu [1] in 1983. For the 2-Absorbing submodule, Yousefian and Soheilnia created the concept [2]. Dakheel originally introduced the Endo prime submodule [3] in 2010. Wisbauer suggested the idea of a small submodule in 1991 [4]. Harfash [5] proposed the ideas of a small prime submodule and small 2-absorbing submodule in his paper (2015). In 2015[5], Harfash widened these concepts to incorporate the Endo 2-absorbing (Endo small 2-absorbing) submodule. The terms Nearly Endo T-ABSO submodule and Nearly Endo prime submodule were introduced by Abd Ali and Hanoon [6].

The terms Nearly Endo small prime submodule and Nearly Endo small T-ABSO submodule, respectively, are used to generalize the notions of Endo small prime submodule and Endo small T-ABSO submodule in this article. There are two sections to this article. In the first section, we present some fundamental terms and characteristics that are necessary. Section 2 examines the numerous crucial properties, results, and outputs of the Nearly Endo smallT-ABSO submodule.

2. PRELIMINARIES

This section discusses the several fundamental concepts as well as any prerequisites they may have for the following section.

***Definition 2.1** [7]*

A submodule $P \leq W$ is referred to as ***minimal*** (respectively ***maximal***) submodule of W if $P\neq 0$, $\forall\, B \leq W$, $B \subsetneq P \Longrightarrow B=(0)$ [respectively $P \subsetneqq W$, $\forall\, B \leq W$, $P \subset B \Longrightarrow B=W$]

***Definition 2.2** [7]*

A $G-$module W is referred to as ***a cyclic*** if $m \in W$ such that $W = < m > = \{rm : r \in G\}$.

***Definition 2.3** [7]*

If a módulè W has a finite generating set, it is said to be finitely generated., say X, that is $W = \langle X \rangle$.

***Definition 2.4** [8]*

A submod P of a $G-$module W is referred to as a direct summand of W, for short $P \leq^{\oplus} W$ if, there exists a submódulè K of W such that $P + K = W$ and $P \cap K = 0$.

Definition 2..5 [1]

Let W as G-Módulè and $P \subset W$. P is referred to as a prime submódulè if $g \in G$, $s \in W$, *with* $gs \in P$ implies that $s \in P$ or $g \in (P :_G W)$.

Definition 2.6 [3]

Let W as G-Módulè and $P \subset W$. P is referred to as Endo Prime submódulè if L ∈ End(W), L(g)∈N, g ∈ W implies that g∈P or (W) ⊆ P.

Definition 2.7 [2]

Let W as G-Módulè and $P \subset W$. P is referred to as T − ABSO submódulè if whenever a,∈G, x∈W, with abx∈P implies that ax∈P or bx∈P or ab∈P or ab ∈ (P:$_G$ W).

Definition 2.8 [5]

Let W as G-Módulè and $P \subset W$. P is referred to as Endo T-ABSO submod if for each f, g ∈ End(W), m ∈ W With (f ∘ g)(m)∈P implies that f(m) ∈P or g(m) ∈P or (f ∘ g)(W) ⊆P.

***Definition* 2.9 [4]**

Let W as G-Módulè and $P \subset W$. P is referred to as a small submódulè of W and denoted by P≪W if P+L≠W for any submódulè $L \subset$ W .

***Proposition* 2. 10 [4]**

Let K, L be submódulès of a G-módulè W, and W' a G-módulès.Then
1) If K<<W and f: M⟶W' is an èiphómórphism, then f(K)<<W'.
2) (2) If K ⊂ L ⊂ W, then L ≪W if and only if K ≪W and L/K ≪W/K.
3) If K<<W and L<<W then K+L<<W.

Definition 2.11 [4]

A G-módulè W is refered to as a hóllów módulè if and only if every submódulè in W is small

Definition 2.12 [7]

Let W be a G-module. The Jacóbsón radical of W is denoted by J(W), and defined as the intersection of all maximal submódulè of W, and denoted by sum of all small submod of W. If W has no maximal submod, then we set J(W) = W.

***Theorem* 2. 13 [7]**

If ∅:W⟶W' is a G- hómómórphism, then ∅ (J W)⊆ J(W') , If ∅:W⟶W′ is a G-epimórphism and ker∅≪W,then ∅ (J (W)) = J(W′) , and J(W) .R⊆J(W),where G is a ring , if W is projective module then J(W).R=J(W).

Definition 2.14 [6]

Let W as G-module and $P \subset W$, P is refered to as Nearly Endo prime submódulè (in short N-E- prime submod) if for each $L \in End$W , $x \in$ W such that $L(x) \in P$ *implies that* $x \in P + J(W)$ or $L(W) \subseteq P + J(W)$.

***Definition* 2. 15** [6]

Let W as G-module and $P \subset W$, P is refered to as Nearly Endo T-ABSO submódulè (in short N-E- T-ABSO submod) if for each $L, h \in End$W , $x \in$ W such that $(L \circ h)(x) \in P$ implies that either $L(x) \in P + J(W)$ or $h(x) \in P + J(W)$ or $(L \circ h)(W) \subseteq P + J(W)$.

***Definition* 2. 16 [9]**

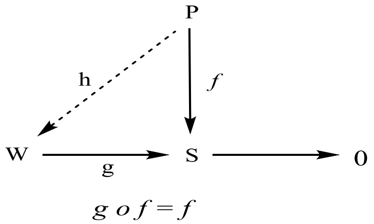
A G- module W is refered to as a scalar module if for each $f \in End(W)$, *there exists* $r \in G$ *such that* $f(x) = rx$, for $x \in W$

***Corollary* 2. 17** [9]

Every finitely generated multiplication G-modul W is scalar módulè.

***Definition* 2.18 [10]**

A G-module P is referred to as W-Projective modul if each pattern diagram :



with exactrow can be extended commutatively via homomorphism $h: P \to W$ that is $goh = f$.

Definition 2.19[4]

A submod P of a module W is refered to as fully invariant if, $f(P) \subseteq P$ for all $L \in End_G(W)$.

3. Nearly Endo Small T– ABSO Submodule

Definition 3.1

Let W as G-módul and $P \subset W$, P is refered to as Nearly Endo small prime submódulè (in short N-E-smal prime submod) if for each $L \in End W$, $< x > \ll$ W such that $L(x) \in P$ *implies that* $x \in P + J(W)$ or $L(W) \subseteq P + J(W)$.

Example 3.2 Consider Z_8 as Z-module
$P = (\bar{2})$ is N-E- small prime submodule since if $L(x) = 2x$, $\forall x \in Z_8$ where $< \bar{2} > \ll Z_8$
$L \in End(Z_8)$, $J(Z_8) = (\bar{2})$, $L(2) = 4 \in P = (\bar{2})$, implies that *either* $2 \in (\bar{2}) + J(Z_8) = (\bar{2})$ *or* $L(Z_8) \subseteq (\bar{2}) + J(Z_8) = (\bar{2})$

$$L(Z_8) = \left\{ \begin{array}{l} L(0) = 0 \\ L(1) = 2 \\ L(2) = 4 \\ L(3) = 6 \\ L(4) = 8 \\ L(5) = 10 \\ L(6) = 4 \\ L(7) = 6 \end{array} \right\} \subseteq P + J(Z_8) = (\bar{2})$$

***Definition* 3.3**

Let W as G-module and $P \subset W$, P is refered to as Nearly Endo small T-ABSO submódulè (in short N-E-small T-ABSO submod) if for each $L, h \in End W$, $< x > \ll W$ such that $(L \circ h)(x) \in P$ implies that either $L(x) \in P + J(W)$ or $h(x) \in P + J(W)$ or $(L \circ h)(W) \subseteq P + J(W)$.

***Example* 3.4**

Consider Z_{16} as Z-module, $P = (\bar{8})$ is N-E- small T-ABSO submod, since if $L, h \in Endo(Z_{16})$, $L(x) = 2x$, $h(x) = x$, $\forall x \in Z_{16}$ and $J(Z_{16}) = (\bar{2})$, $(L \circ h)(4) = L(4) = 8 \in P = (\bar{8})$ implies that either $L(4) = 8 \in P + J(Z_{16}) = (\bar{2})$ *or* $h(4) = 4 \in P + J(Z_{16}) = (\bar{2})$ *or* $(L \circ h)(Z_{16}) \subseteq P + J(Z_{16}) = (\bar{2})$

$$(f \circ g)(Z_{15}) = \begin{cases} L(h(0)) = 0, \ L(h(1)) = 2, \\ L(h(2)) = 4, \ L(h(3)) = 6, \\ L(h(4)) = 8, \ L(h(5)) = 10, \\ L(h(6)) = 12, \ L(h(7)) = 14, \\ L(h(8)) = 0, \ L(h(9)) = 2, \\ L(h(10)) = 4, \ L(h(11)) = 6, \\ L(h(12)) = 8, \ L(h(13)) = 10, \\ L(h(14)) = 12, \ L(h(15)) = 14 \end{cases}$$

So that $(f \circ g)(Z_{16}) \subseteq P + J(Z_{16}) = (\bar{2})$

Remarks and Examples 3.5

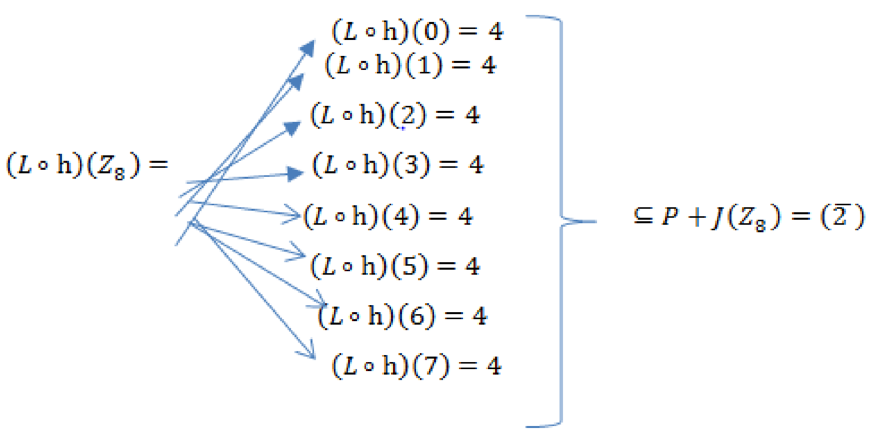
1) Every E-small prime submod of a G-module W is N- E-small prime submod of W. But the converse is not true in general, for example: Consider Z_8 as Z-module, $P = (\bar{4})$ is N-E- small prime submod of Z_8 since if $L \in Endo(Z_8)$, $f(x) = x + 2$, where $\forall x \in Z_8 J(Z_8) = (\bar{2})$, $L(2) = 4 \in P = (\bar{4})$, where $< \bar{2} > \ll Z_8$, then $2 \in P + J(Z_8) = (\bar{2})$, but P is not E-small prime

submod of W, since $L(2) = 4 \in P = (\bar{4})$, then $2 \notin P = (\bar{4})$ and $L(Z_8) \nsubseteq P = (\bar{4})$ $When$ $L(3) = 5 \notin P = (\bar{4})$.

2) Every E-small T-ABSO submod of a G-module W is N-E- small T-ABSO submod of W.
Proof:Let P be E-small T-ABSO submod of W and L, h $Endo(W)$, $x \in W$, $< x > \ll$ W such that $(L \circ \mathrm{h})(x) \in P$, but P is E-small T-ABSO submod of W, then $L(x) \in P$ or $h(x) \in P$ or $(L \circ \mathrm{h})(W) \subseteq P$, hence $L(x) \in P + J(W)$ or $h(x) \in P + J(W)$ or $(L \circ \mathrm{h})(W) \subseteq P + J(W)$.Thus, P is N-E-small T-ABSO submod of W.
But the converse incorrect in general, for example: Consider Z_{16} as Z-module, $P = (\bar{4})$ is N-E- small T-ABSO submod of Z_{16} since if $L, h \in Endo(Z_{16})$, $L(x) = x + 2$, $h(x) = 3x$, $\forall x \in Z_{16}$, $< x > \ll Z_{16}$ where $J(\mathrm{Z}_{16}) = (\bar{2})$, such that $(L \circ \mathrm{h})(2) = L(6) = 8 \in P = (\bar{8})$, then $L(2) = 4 \in P + J(\mathrm{Z}_{16}) = (\bar{2})$ and $h(2) = 6 \in P + J(\mathrm{Z}_{16}) = (\bar{2})$, but P is not E-small T-ABSO submod of W, since $(L \circ \mathrm{h})(2) = 8 \in P = (\bar{8})$, then $L(2) = 4 \notin P = (\bar{8})$, $h(2) = 6 \notin P = (\bar{8})$ and $(L \circ \mathrm{h})(Z_8) \nsubseteq P = (\bar{8})$ $When$ $(L \circ \mathrm{h})(1) = 5 \notin P = (\bar{8})$

3) Let P, S be two submods of a G-module W, and $P \subset S$. If P is N-E-small T-ABSO submod of W, then P is N-E-small T-ABSO submod of S with , $J(W) \subseteq J(S)$.
Proof: Let $(L \circ h)(x) \in P$, $\forall x \in S, < x > \ll$ S since $S < W, so$ $x \in W, L, h \in Endo(W)$, Since P is N-E-small T-ABSO submod of W, then either $L(x) \in P + J(W)$ or $h(x) \in P + J(W)$ or $(L \circ h)(W) \subseteq P + J(W)$, since $J(W) \subseteq J(S)$, hence $L(x) \in P + J(S)$ or $h(x) \in P + J(S)$ or $(L \circ h)(W) \subseteq P + J(S)$, but $S < W$, so that $(L \circ h)(S) \subseteq (L \circ h)(W)$, hence $(L \circ h)(S) \subseteq (L \circ h)(W) \subseteq P + J(S)$, $(L \circ h)(S) \subseteq P + J(S)$ Thus P is N-E-small T-ABSO submod of S.

4) Every N-E- prime submod is N-E-small T-ABSO submod, but the converse incorrect in general, for example :
Consider Z_8 as $\mathrm{Z}-$ module, $P = (\bar{4})$ is N-E- small T-ABSO submod of Z_8 since if $L, h \in Endo(Z_8)$, $L(x) = x + 1$, $h(x) = 3$, $\forall x \in Z_8$, $< x > \ll Z_8$ where $J(\mathrm{Z}_8) = (\bar{2})$, such that $(L \circ \mathrm{h})(2) = L(3) = 4 \in P = (\bar{4})$, then $(L \circ \mathrm{h})(\mathrm{Z}_8) \subseteq P + J(\mathrm{Z}_8)$



So that $(L \circ \mathrm{h})(Z_8) \subseteq P + J(\mathrm{Z}_8)$. But it is not N-E- prime submod of Z_8 since $L(3) = 4 \in P = (\bar{4})$, then $3 \notin P + J(\mathrm{Z}_8) = (\bar{2})$ and $L(\mathrm{Z}_8) \nsubseteq P + J(\mathrm{Z}_8) = (\bar{2})$ when $f(0) = 1 \notin P + J(\mathrm{Z}_8)$

5) Every E-T-ABSO(N-E-T-ABSO) submod. is N-E-small-T-ABSO submod. But the converse incorrect in general, for example:Consider Z_{24} as Z-module, $P = (\overline{12})$ is N-E- small T-ABSO submod of Z_{24} since if $L, h \in End(Z_{24})$, $L(x) = 2x, h(x) = 3x$, $\forall x \in Z_{24}$, $< x > \ll Z_{24}$ where $J(\mathrm{Z}_{24}) = (\bar{6})$, such that $(L \circ \mathrm{h})(6) = L(18) = 12 \in P = (\overline{12})$, Then $L(6) = 12 \in P + J(\mathrm{Z}_{24}) = (\bar{6})$ and $h(6) = 18 \in P + J(\mathrm{Z}_{24}) = (\bar{6})$, but P is not E-T-ABSO submod of W, since $(L \circ \mathrm{h})(2) = 12 \in P = (\overline{12})$, then $L(2) = 4 \notin P = (\overline{12})$, $h(2) = 6 \notin P = (\overline{12})$ and $(L \circ \mathrm{h})(Z_8) \nsubseteq P = (\overline{12})$
When $(L \circ \mathrm{h})(1) = 6 \notin P = (\overline{12})$

***Proposition* 3.6**
If W as G-módùè is hóllów. If P is N-E-small T-ABSO submod of W and J(W)⊆ P, then P is N-E -T-ABSO (E -T-ABSO) submod of W
Proof: Suppose P is N-E-small T-ABSO submod. Let h, g ∈End(W) ,x ∈ W such that (h∘g)(x) ∈P, but W is hollow module, then $< x > \ll$ W. So that either h(x)∈P +J(W) or g(x)∈P +J(W) or (h∘g)(W) ⊆ P+J(W) since P is N-E-small T-ABSO submod. (but J(W)⊆ P, hence either h(x)∈P or g(x)∈P or (h∘g)(W) ⊆ P). Thus, P is N-E -T-ABSO (E -T-ABSO) submod of W
***Proposition* 3.7**
If W G- módùè is hóllów and J(W) is N-E-small T-ABSO submod of W, then every submod of W is N-E-small T-ABSO submod.

Proof: Suppose P be a submod of W. Let h, g ∈End(W) ,$< x >\ll$ W such that (h∘g)(x) ∈P, but W is hollow module, then P is small submod in W ,hence (h∘g)(x)∈ P ⊆ J(W), so that either h(x)∈J(W) +J(W)= J(W)⊆ P +J(W) or g(x)∈J(W) +J(W)= J(W)⊆ P +J(W) or (h∘g)(W) ⊆J(W)+J(W) =J(W) ⊆ P+J(W). Thus, P is N-E-small T-ABSO submod of W.

***Proposition* 3.8**

Let P a submod of a scalar G-module W. Then P is small T-ABSO submod if and only if P is N-E-Small T-ABSO submod and $J(W) \subseteq P$.

Proof: (⟹) Let $(L \circ h)(x) \in P$ $where\ L, h \in \text{End}(W),\ \forall x \in W, < x >\ll W, since$ W is Scalar module , then there exist a,b $\in G$ such that $ax = L(x)$, $bx = h(x)$ for each $\forall x \in W , < x >\ll$ W, but P is small T-ABSO submod of W, then $(L \circ h)(x) = abx \subseteq P$, implice that either $ax = L(x) \in P$ or $bx = h(x) \in P\ or\ abW \subseteq P$, Since $J(W) \subseteq P$ hence $L(x) \in P + J(W)$ or $h(x) \in P + J(W)$ or $(L \circ h)(W) \subseteq P + J(W)$.Then P is N-E- smallT-ABSO submod of W.

(⟸) Let $L, h \in \text{Endo}(W)$, $\forall x \in W , < x >\ll$ W such that $L(x) = ax$, $h(x) = bx$ $(L \circ h)(M) = abx \subseteq P$, $since$ W is scalar module, but P is N-E-small T-ABSO submod of W, then either $ax = L(x) \in P + J(W)\ or\ bx = h(x) \in P + J(W)\ or\ (L \circ h)(W) = abW \subseteq P + J(W)$, hence $L(x) = ax \in P$ or $bx = h(x) \in P$ or $abW = (L \circ h)(W) \subseteq P$, so that $ab \in (P:_G W)$ since $J(W) \subseteq P$. Then P is small T-ABSO submod of W.

***Remark* 3.9**

If delete the condition of scalar module the converse of Proposition 3.8 is not true in general, for example: Consider Z_{12} as Z-module, $P = (\bar{3})$ is small T- ABSO submodule, $2 \cdot 1 \cdot (\bar{6}) = 0 \in P = (\bar{3})$ where $< \bar{6} >\ll Z_{12}$, implies that $1 \cdot (\bar{6}) = 6 \in P = (\bar{3})$ and $2 \cdot (\bar{6}) = 0 \in P = (\bar{3})$. But P is not N-E-small T-ABSO submod of Z_{12} since if $f, g \in End(Z_{12})$, $f(x) = x - 2$, $g(x) = x - 1$ such that $(f \circ g)(6) = f(g(6)) = 3 \in P = (\bar{3})$, where $J(Z_{12}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$, then $f(6) = 4 \notin P + J(Z_{12}) = (\bar{3})$, $g(6) = 5 \notin P + J(Z_{12}) = (\bar{3})$ and $(f \circ g)(Z_{12}) \nsubseteq P + J(Z_{12})$ When $(f \circ g)(4) = f(g(4)) = 1 \notin (\bar{3}) = P + J(Z_{12})$

Corollary 3.10

Let P a própèr submod of a finitely generated multiplication G-módulè W $J(W) \subseteq P$. then P is small T-ABSO submod if and only if P is N-E-small T-ABSO submod

Proof : By Proposition 3.8 and Corollary 2.14 we get the result .

Proposition 3.11

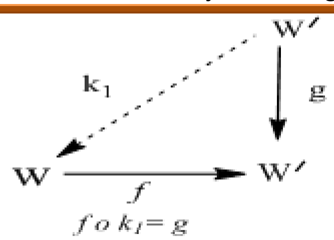
Let W_1, W_2, be two G-modules. If $P_1 \oplus P_2$ is N-E-small T-ABSO submod of $W_1 \oplus W_2$, then $P_1\ and\ P_2$ are N-E-small T-ABSO submods of $W_1\ and\ W_2$ rèspèctivièly.

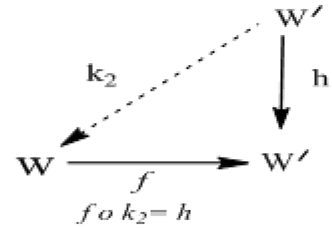
Proof: Suppose that $rax \in P_1$ and $ray \in P_2$, for all r,a∈G and $< x >\ll W_1$ and $< y >\ll W_2$, then $ra(x, y) \subseteq P_1 \oplus P_2$, such that $< x >\oplus< y >\ll W_1 \oplus W_2$ by Proposition 2.10. But $P_1 \oplus P_2$ is N-E-small T-ABSO submod of $W_1 \oplus W_2$, then either $r_s(x, y) \in P_1 \oplus P_2 + J(W_1 \oplus W_2)$ or $a(x, y) \subseteq P_1 \oplus P_2 + J(W_1 \oplus W_2)$ or $ra \in (P_1 \oplus P_2 + J(W_1 \oplus W_2): W_1 \oplus W_2) = (P_1 + J(W_1): W_1) \cap (P_2 + J(W_2): W_2)$, hence $rx \in P_1 + J(W_1)$ or $ax \in P_1 + J(W_1)$or $ra \in (P_1 + J(W_1): W_1)$ and $ry \in P_2 + J(W_2)$ or $ay \in P_2 + J(W_2)$ or $ra \in (P_2 + J(W_2): W_2)$. Thus, $P_1\ and\ P_2$ are N-E-small T-ABSO submods of $W_1\ and\ W_2$ respectively.

Theorem 3.12

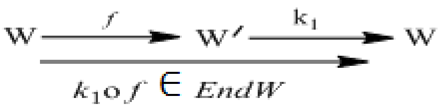
Let P be a fully invariant N-E-small T-ABSO submod of G-module W, let $f: W \to W'$ be èpimorphism such that $Ker\ f \subseteq P$. Then $f(P)$ is N-E- small T- ABSO submod of W' where W ' is W '-projective module.

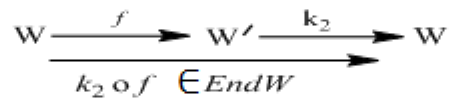
***Proof* :** Let $g, h \in EndW', < m' >\ll \in W'$ such that $(g \circ h)(m') \in f(P)$ since f is epimorphism, then $m' = f(m)$ for some $m \in P$ and since W' is projective Module, then there exist $K_1, K_2: W' \to W$, such that $f \circ K_1 = g\ and\ f \circ K_2 = h$





Now consider the following digrams





$(g\circ h)(m') = (f\circ K_2)\circ(m') \in f(P) = f[(f\circ K_1\circ\ K_2)(m')] \in\ f(P)$. Then$(K_1\circ f\circ\ K_2)(m') \in\ P+\ Kerf$,since$Kerf\ \subseteq P$, $and\ m'\ = f(m), so\ (K_1\circ f\circ\ K_2)(f(m))\ \in\ P$. That is $(K_1\circ f)(\ K_2\circ f\ )(m))\ \in\ P$, since P is N-E-smallT-ABSO submod of W, then either $(K_1\circ f)(m) \in\ P + J(W)$ ***or*** $(K_2\circ f)(m) \in\ P+\ J(W)$ ***or*** $(K_1\circ f)\circ(K_2\circ f)(W) \subseteq\ P + J(W)$. If $(K_1\circ f)(m) \in P + J(W)$, then $(K_1\big(f(m)\big) \in P + J(W), i.e\ K_1(m') \in P + J(W)$, so $f(K_1(m')) \in f(P) + f\big(J(W)\big) \subseteq f(P) + J(W')$ by Theorem 2.13, thus $g(m') \in\ f(P) + J(W')$

If $(K_2\circ f)(m) \in P + J(W), then (K_2\big(f(m)\big) \in P + J(W), i.e\ [\ K_2\ (m') \in P + J(W)]$,
$so\ f(\ K_2(m')) \in\ f(P) +\ f\big(J(W)\big)f(P) + J(W')$by Theorem 2.13, $\quad thus\ h(m') \in f(P)\ +\ (J(W'))$

If $(K_1\circ f)\circ(K_2\circ f)(W) \subseteq\ P + J(W)$, then $(K_1\circ f\circ K_2)\ f(W)) \subseteq\ P + J(W)$, Since $f(W) =\ W', (K_1\circ f\circ K_2)(W')\ \subseteq\ P + J(W), since\ (foK_2) = h, then\ (K_1\circ h)\ (W')\ \subseteq\ P + J(W)$. So $f[(K_1\circ h)\ (W')]\ \subseteq\ f(P) + f\ (J(W)), i.e\ ((\ f\circ K_1)\circ h)\ (W')\ \subseteq\ f(P) + f\ (J(W))f(P) + J(W')$ by Theorem 2.13. Since $f\circ K_1 = g$, then $(goh)(W') \subseteq f(P) + J(W')$
Thus $f(N) is\ \ N - E - small\ T - ABSO\ submod\ of\ W'$.

***Proposition* 3.13**

Let f: W → W′be a G-hómomórphism. If P is fully invariant N-E-T-small ABSO submod of fully invariant W', such that $f(W) \nsubseteq P$. Then $f^{-1}(P)$ is N-E-smallT-ABSO submod of W.

Proof : Since P is a proper submod of W', So $f^{-1}(P)$ is a proper submod of W (if not, then $f^{-1}(P) = W$, so $ff^{-1}(P) = f(W)$, hence $P = f(W)$ this contradication) Let $h, g \in \mathrm{End}(W), < m >\ll W$ such that $(h\circ \mathrm{g})(m) \in f^{-1}(P), then\ f(h\circ \mathrm{g})(m) \in P\ , so\ (f\circ \mathrm{h})g(m) \in P\ where\ <\ g(m) >\ll W\ by\ Proposition\ 2.17\ .\ Since$ P is N-E-smallT-ABSO submod of W', then either $f\big(g(m)\big) \in P + J(W')\ or\ h(g(m)) \in P + J(W')\ or\ f(h(W)) \subseteq P + J(W')$.

***Case 1*:** If $f(g(m)) \in P + J(W'), f^{-1}\ [f(g(m)) \in P + J(W')]$, then $g(m) \in f^{-1}(P) + J\big(f^{-1}(W')\big), so\ g(m) \in f^{-1}(P) + J(W)$

Case 2: If $h\big(g(m)\big) \in P + J(W')$, but P is N-E-small T-ABSO submod of W', then $h(m) \in P + J(W')$ or $g(m) \in P + J(W')$ or $h\circ g(W) \subseteq P + J(W')$. If $h(m) \in P + J(W')$, then $f\big(h(m)\big) \in f(P) + f\big(J(W')\big) \subseteq f(P) + J(f(W')) \subseteq P + J(W')$ since P and W' are fully invariant, hence $f(h(m)) \in P + J(W')\ , f^{-1}[f(h(m)) \in P + J(W')\ ]$, so $h(m) \in f^{-1}(P) + f^{-1}(J(W))$, $then\ h(m) \in f^{-1}(P) + J\big(f^{-1}(W')\big) = f^{-1}(P) + J(W)$. It similarty if $g(m) \in P + J(W')$, we get $g(m) \in f^{-1}(P) + J(W)$.
If $h\circ g(W) \subseteq P + J(W')$, then $f(h\circ g(W)) \in f(P) + f\big(J(W')\big) \subseteq f(P) + J(f(W')) \subseteq P + J(W')$ since P and W' are fully invariant, hence $f(h\circ g(W)) \subseteq P + J(W')\ , f^{-1}[f(h\circ g(W)) \subseteq P + J(W')]$, so $h\circ g(W) \subseteq f^{-1}(P) + f^{-1}(J(W')) \subseteq f^{-1}(P) + J(f^{-1}(W')), then\ \ h\circ g(W) \subseteq f^{-1}(P) + J(W)$.

Case3: If $f(h(W) \subseteq P + J(W'),\ f^{-1}[f(h(W) \subseteq P + J(W')\ ]$, then $h(W) \subseteq f^{-1}(P) + f^{-1}(J(W')) \subseteq f^{-1}(P) + J(f^{-1}(W'))$, $so\ h(W) \subseteq f^{-1}(P) + J(W)$. Thus, $f^{-1}(P)$ is N-E-small T-ABSO submod of W

***Proposition* 3. 14**

Let W as G-module and $P \subset W$, Let K be a fully invariant submod of W and contained in P. If $\frac{P}{K}$ is E- small T- ABSO submod of $\frac{W}{K}$, then P is N-E- small T-ABSO submod of W.

Proof

Let $L\ , h\ \in\ EndW\ ,\ x \in\ W$ such that $(\ L\circ h)(x) \in P\ where < x >\ll$ W . Define $L_1\ ,\ h_1 \colon \frac{W}{K} \to \frac{W}{K}$ by $L_1(x + K) = L(x) +\ K\ \ ;\ h_1(x + K) = h(x)\ + K\ \ for\ each\ \ x \in\ W$. It is clear that $L_1\ ,\ h_1$ are well-defined since K is fully invariant.

Now, $(L_1 \circ h_1)(x+K) = L_1\big(h_1(x+K)\big) = L_1(h(x)+K) = L\big(h(x)\big)+K = (L\circ h)(x+K) \in \frac{P}{K}$.
$But\ < x+K > \ll \frac{W}{K}\ and\ \frac{P}{K}$ is E-small -T-ABSO submod of $\frac{W}{K}$ by Proposition 2.10.
Then either $L_1(x+K) \in \frac{P}{K}\ or\ h_1(x+K) \in \frac{P}{K}\ or\ L_1\left(h_1\left(\frac{W}{K}\right)\right) \subseteq \frac{P}{K}\ ,So\ or\ L(x)+K \in \frac{P}{K}+J\left(\frac{W}{K}\right)\ or\ h(x)+K \in \frac{P}{K}+J\left(\frac{W}{K}\right)\ or\ L(h(W)+K) \subseteq \frac{P}{K}+J\left(\frac{W}{K}\right)$, therefore $L(x) \in P+J(W)\ or\ h(x) \in P+J(W) or\ L\big(h(W)\big) \subseteq P+J(W)$.
Thus, P is a N-E-small-T-ABSO submod of W.

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