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Fuzzy Soc–Small T–ABSO Submodules and Related Concepts

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***Abstract**: In this work, the interaction between the fuzzy small-T-ABSO submodule, small prime submodule, and fuzzy socle are compared. We provide two new concepts fuzzy socle small prime submodule and fuzzy socle small T-ABSO submodule to address this question. The research uncovered a number of traits that supported the novel theories. Using simple algebraic methods, it is also discovered that fuzzy socle small T-ABSO and fuzzy socle T-ABSO have a link, for fuzzy direct sum and homomorphic images, in additione, we looked into the fuzzy socle smallT-ABSO submodule structure. The compatibility of fuzzy socle small T-ABSO submodule with other fuzzy submodule kinds is also discussed. The results of this study are crucial for defining a new fuzzy socle small TABSO.*

Keywords: $Fuzzy\ Soc -$ small prime submodules**;** $Fuzzy\ Soc - T\text{–}\ ABSO$ submodules**;** $Fuzzy\ small\ \ T\text{–}\ ABSO$ submodules; $Fuzzy\ \ Soc\text{–}\ Small\ T\text{–}\ ABSO\ ideal;$

1. INTRODUCTION

A fuzzy subset A of X was first defined by Zadeh [1] in 1965 as a function from X to the unit interval [0,1]. Since then, other domains have added to the theory of fuzzy sets. One of the earliest branches of pure mathematics, algebra, introduced the idea of fuzzy sets. Negoita and Ralescu first presented the ideas of fuzzy modules and fuzzy submodules in 1975 [2]. Fuzzy fundamental submodules were constructed by Saikia and Kalita [3] who then examined their properties. Additionally, research this In this article, all rings are unitary and commutative with identity. Rabi [4] proposed a prime fuzzy submodule as a: Given that W-module G has a fuzzy module X, The term fuzzy submodule U of X if $r_b m_t \subseteq U$,with r_b is fuzzy singleton of W and m_t⊆ X, implying that either $m_t \subseteq U$ or $r_b \subseteq [U : X]$ for each $t, b \in [0, 1]$. The concept of a fuzzy small submodule was introduced by Rahman and Saikia [5]. Let X be a fuzzy module of a W-module G, and K be a fuzzy submodule of X. If K means that K+P≠X for any proper fuzzy submodule P of X, K is referred to be a fuzzy small submodule of X. The concept of a small prime fuzzy submodule was introduced by Khalaf and Hanoon [6]. It is defined as a proper fuzzy submodule K of a fuzzy module X of a W-module G if whenever fuzzy singleton $a_s of\ W$ and $x_v \subseteq X$, $\forall s, v \in L$, with $\langle x_v \rangle \ll X$ and $a_s x_v \subseteq K$, implies either $x_v \subseteq K\ or\ a_s \subseteq (K: X)$. In 2022, Marie [7] made the original fuzzy socle prime submodule notion proposal, A $fuzzy\ Socle\ prime$ (also known as a F-Soc-prime) submodule of X if whenever $r_b m_t \subseteq U$,with r_b is $fuzzy\ singleton$ of W and m_t⊆ X, implying that either $m_t \subseteq U + F - Soc(X)$ or $r_b \subseteq [U + F - Soc(X) : X]$ for each $t, b \in [0, 1]$. In addition, Hanoon and Khalaf [8] proposed the T-ABSO fuzzy submodule: Let X be a fuzzy module of a W-module G, A $fuzzy\ submodule\ U\ of\ X$ is called T-ABSO if $q_s r_b m_t \subseteq U$,with q_s , r_b are $fuzzy\ singletons\ of\ W$ and m_t is fuzzy $singleton\ of\ X$, implying that either $q_s m_t \subseteq U$ or $r_b m_t \subseteq U$ or $q_s r_b \subseteq [U :_R X]$ for each $t, b, s \in [0; 1]$. A small TABSO fuzzy submodule was also suggested by Wafaa [6] in (2019): Let X be a $fuzzy\ module\ of\ an\ W - module\ G$, A $fuzzy\ submodule\ K\ of\ X$ is called small T-ABSO if $q_s\, r_n\ \ m_t \subseteq U$,with q_s, $r_n\ fuzzy\ singleton\ of\ W$ and $< m_t > \ll X$, implying that either $q_s m_t \subseteq P$ or $r_n m_t \subseteq P$ or $q_s\, r_n \subseteq [P : X]$ for each $t, s, n \in [0;\ 1]$. The initial description of the fuzzy socle T-Absorbing submodule was made by Marie in 2022 [7]: A fuzzy socle T-Absorbing (F-Soc-T-ABSO) submodule of X is referred to as a proper fuzzy submodule U of a F-module X of a W-module G if and when $r_b s_q m_t \subseteq U$,with r_b, s_q is $fuzzy\ singletons\ of\ R$ and m_t⊆ X, implying that either $r_b m_t \subseteq U + F - Soc(X)$ or $s_q m_t \subseteq U + F - Soc(X)$ or $r_b \subseteq [U + F - Soc(X) : X]$ for each $t, b, q \in [0, 1]$.

The concepts of a small prime fuzzy submodule and a small T-ABSO fuzzy submodule are generalized in this article to a fuzzy socle small prime submodule and afuzzy socle small T-ABSO submodule, respectively. This article is divided into two parts. We present some fundamental definitions and features that we will require in the first part. The fuzzy Socle smallT-ABSO submodule's many essential characteristics, outcomes, and outputs are examined in section two.

Note: o.w., $F - set, F - submod, F - ideal, F - module, F - T - ABSO, F - singleton,$ and F-Socle T-ABSO submodule are abbreviations for $otherwise$, fuzzy sets, submodules, ideals, modules, and singletons.

2. PRELIMINARIES

This section discusses the several fundamental concepts as well as any prerequisites they may have for the following section.

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Definition 2.1 [1]

Let I be the closed interval of real numbers between [0, 1], and let D be a set that isn't empty. A function from D into I is known as a F-set B in D (or a F-subset of D).**Definition 2.2** [1]

If $B(x) = t, \forall x \in D, t \in I$,then an F-set B of a set D is said to be F-constant.

Definition 2.3 [1]

Let $x_1: D \to I$ be a F-set in D, where $x \in D$, $t \in I$ defined by:

$$x_t\,(\mathrm{y}) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$

for every y ∈ D . x_t is described as an F-singleton or F-point in D.

If x =0 and v =1, then $0_1(y) = \begin{cases} 1 & if\ y = 0 \\ 0 & if\ y \neq 0 \end{cases}$, [9]

Definition 2.4 [10]

Let B be a F-set in D, The set $B_t = \{x \in D; B(X) \geq t\}$ is referred to as a level subset of B for all $t \in I$. if B is an F-set in D. A subset of D in the ordinary sense is B_t; keep that in mind.

Remark 2.5 [1]

If two F-sets A and B exist in S, then:

1- $A = B\ iff\ A(x) = B(x)$.
2- $A \subseteq B\ iff\ A(x) \leq B(x)$.
3- $A = B\ iff\ A_t = B_t$

If $A < B$ and $x \in S$ exist such that $A(x) < B(x)$, then A is a proper F-subset of B and is denoted by the symbol $A < B$.

Part (2) allows us to infer that $x_t \subseteq A$ if and only if $A(x) \geq t$

Definition 2.6 [9]

If G be a W-module. A F-set X of G is called F-module of a W-module G if:

1- $X\,(x - y) \geq \min(X\,(x), X\,(y), forallx, y \in G\}$.
2- $X(rx) \geq X\,(x) forallx \in G\ and\ r \in W$.
3- $X(0) = 1$.

Definition 2.7 [11]

Let X and A be two $F - modules\ of\ W - module\ G$. A is said to be a $F - submod\ of\ X$ if $A \subseteq X$.

Proposition 2.8 [12]

Make A a F-set of a W-module of G. Afterward, the level subset $A_t, t \in I$ is a $submod\ of\ G\ if\ A\ is\ a\ F - submod\ of\ X$ where X is a $F - module\ of\ a\ W - module\ G$.

Lemma 2.9 [1]

If A is a F-module of a W-module G and r_t be a F-singleton of W. Next, for any $w \in G$: $(r_t A)(w) = \begin{cases} \sup\{\inf(t, A(x))\}: if w = rx\} & \text{for some } x \in G \\ 0 & o.w. \end{cases}$

Definition 2.10 [9]

A $F - subset\ K$ of a ring W is called a $F - ideal\ of\ W$, if $\forall x, y \in R$:

1) $K(x - y) \geq min\,\{K(x), K(y)\}$.

2) $K(xy) \geq \max\{K(x), K(y)\}$.

Definition 2.11 [9]

Let A and B be two $F-submods\ of\ a\ F-module\ X$. The residual quotient of A and B denoted by $(A : B)$ is the $F-subset\ of\ W\ defined\ by$:

$(A : B)(r) = \sup\{t \in [0:1]: r_t B \subseteq A\}, for\ all\ r \in W. That\ (A : B) = \{r_t: r_t B \subseteq A; r_t\ is\ a\ F-singleton\ of\ W\}$.

$If\ B = < x_k >, then\ (A :< x_k >) = \{r_t: r_t\, x_k \subseteq A: r_t\ is\ a\ F-singleton\ of\ W\}$.

Proposition 2.12 [9]

Let A and B represent two F-submodules of a W-module G's F-module X. Consequently,$(A : B)$ is a F-ideal of W.

Lemma 2.13 [9]

Let A be a F-submod of F-module X, $(A_t : X_t) \supseteq (A : X)_t, For\ all\ t \in I$.

It follows that if, $X = A \oplus B, for\ A, B \leq X\ then\ X_t = (A \oplus B)_t = A_t \oplus B_t$, **[13].**

Definition 1.14 [11]

Let f be a mapping from set G to set N, and let A and B represent the F-sets in set G and set N, respectively. The F-set in N is the image of A denoted by f (A), specified by:

$$f(A)(y) = \begin{cases} \sup\{A(z): z \in f^{-1}(y) \neq \emptyset, \text{for all } y \in N\ otherwise \\ 0 \qquad\qquad\qquad\qquad o.w. \end{cases}$$

Keep in mind that if f is a bijective mapping, then $f(A)(y) = A\big(f^{-1}(y)\big)$

Proposition 2.15 [14]

Let K and P be F-submods of F-module X and Y respectively, and f be a mapping from a set into a set then f (K) and f^{-1}(P) are F-submod of Y and X respectively.

Definition 2.16 [1]

Let G be a W-module and A and B be two F-submods of G. A+B is defined as an addition. by: $(A + B)(x) = \sup\{inf\ \{A(y), B(z)\} with\ x = y + z, forall\ x, y, z \in G\}$.

Furthermore, $A + B$ is a F-submod of a W-module G.

Corollary 2.17 [15]

For all F-singleton r_k of W, if X is a F-module of a W-module G and $x_t \subseteq X$, then, $r_k\, x_t = (rx)_\lambda$, where $\lambda = \min\{t, k\}$.

Definition 2.18 [16]

Let X be an F-module of a W-module M; X is said to be F-simple if and only if X has no proper F-submods (in actuality, X is F-simple if and only if X has just 0_1 and itself)

Definition 2.19 [17]

If X is the sum of simple F-submods of X, then X is said to be semi-simple.

Additionally, the F-Socle of X is represented by $F - Soc(X)$, and is the sum of the simple F-submod of X. X is referred to as semi-simple if it equals $F - Soc\ (X)$.

Lemma 2.20 [7]

$(F - Soc(X))_t = Soc(X_t)$ for any F-module X for each $t \in I$ with $(F - Soc(X))_t \neq X_t$

Proposition 2.21 [7]

Let $f: G \to N$ be a W-isohomomorphisim, If X_1 and X_2 are F-modules of a W-modules G and N respectively.
then $f\left(F - Soc(X_1)\right) \subseteq F - Soc(X_2)$.

Definition 2.22 [9]

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If $F-annX=0_1$ and X be a F-module of a W-module G, X is said to be faithful. Where $F-annX=\{r_t: r_t\, x_l = 0_1; forall\ x_l \subseteq X\ and\ r_t\ be\ a\ F-singleton\ of\ W\}$

Definition 2.23 [18]

Make X a F-module of G as W-module, If and only if a $F-ideal$ K of W exists such that $A = KX$, then X is referred to as a multiplication F-module..

Proposition 2.24 [18]

Every non-empty $F-submod\ A\ of\ a\ F-module\ X\ of\ a\ W-module\ G$ such that $A = (A:X)X$, if and only if X is multiplication F-module.

Lemma 2.25 [7]

Make X the F-module of G as W-module. If G be a faithful multiplication W-module, then F- $Soc(W)X = F-Soc(X)$

Definition 2.26 [4]

A F-module X of a W-module G is called a cyclic F-module if there exists $x_v \subseteq X$ such that $y_k \subseteq X$ written as $y_k = r_l\, x_v$ for some F-singleton r_l of R, where $k,l,v \in$ L in this case, write $X=< x_v >$ to denote the cyclic F-module generated by x_v.

Definition 2.27 [19]

If X is a F- module of a W-module G, then $X\ is\ called\ a\ finitely\ generated\ F-\ module$ if there exists $x_1, x_2, x_3, \ldots \subseteq X$ such that $X=\{a_1(x_1)_{v_1} + a_2(x_2)_{v_2} + \cdots + a_n(x_n)_{v_n}\}$, where $a_i \in W\ and\ a(x)_v = (ax)_v$, $\forall v \in$L.

Where $(ax)_v(y) = \begin{cases} v & if\ y = ax \\ 0 & o.w. \end{cases}$

Definition 2.28 [5]

A F-ideal H of W with $H(0) = 1$ is referred to F-small ideal of W, if it is F-small submod of a F-module X of a W-module W.

Theorem 2.29 [5]

Let K be a proper F-submod of F-module X of a W-module G, then $K \ll X$ if and only if $K_t \ll G$ for all $t \in I$.

Theorem 2.30 [20]

Let K, P be proper F-submods of F-module X of a W-module G, then $K \ll X, P \ll X$ if and only if $K + P \ll X$.

Proposition 2.31 [20]

Let K be a proper F-submod of F-module X of a W-module G. If $K \ll X$, then $(a_s K) \ll X$ for all F-singleton $a_s \subseteq X$

Proposition 2.32 [7]

$(A \oplus B) + F-Soc(X_1 \oplus X_2) = (A + F-Soc(X_1) \oplus (B + F-Soc(X_2)$ for every F-submods A and B of F-submods of $X_1\ and\ X_2$ respectively

Proposition 2.33[7]

In the event when G is a faithful multiplication a W-module and X is a F-module of G, then $F-Soc(X) = F-Soc(W)X$

Remark 2.34 [7]

Let P be F-submod of a faithful multiplication F-module X of a W-module G, then $(P:X) + F-Soc(W) \subseteq (P + F-Soc(X):X)$

Theorem 2.35 [20]

Let $X_1\ and\ X_2$ be F-module of W-modules $G_1 and\ G_2$ respectively and let $f: X_1 \to X_2$be F-homomorphism if $P \ll X_1$, then $f(P) \ll X_2$.

Definition 2.36 [19]

A F-module of a W-module G, is called a cancellation F-module if $KX = LX$ where K and L are F-ideals of W, then $K = L$.

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3. $F-Soc-$ Small $T-$ ABSO Submod

Definition 3.1

A proper F-submod K of a F-module X of a W-module G is referred to as F-Soc-small prime submod if whenever F-singleton s_a of W and $x_v \subseteq X, \forall a, v \in I$, with $< x_v > \ll X$ and $s_a x_v \subseteq K$ implies either $x_v \subseteq K + F - Soc(X)$ or $s_a \subseteq (K + F - Soc(X): X)$

Example 3.2

Let $X: Z_8 \to I$ such that $X(y) = \begin{cases} 1 & if\ y \in Z_8 \\ 0 & o.w \end{cases}$

It is evident that X is a F-module of Z_8 as Z-module

Let $K: Z_8 \to I$ such that $K(y) = \begin{cases} \frac{1}{2} & if\ y \in (\bar{2}) \\ 0 & if\ y \notin (\bar{2}) \end{cases}$

It is obvious that K is a F-submod of X

$$F - Soc(X)(y) = \begin{cases} 1 & if\ y \in (\bar{4}) \\ \frac{1}{3} & if\ y \notin (\bar{4}) \end{cases}$$

$$(K + F - Soc(X))(y) = \begin{cases} 1 & if\ y \in (\bar{2}) \\ \frac{1}{3} & if\ y \notin (\bar{2}) \end{cases}$$

$$(K + F - Soc(X): X)(y) = \begin{cases} \frac{1}{2} & if\ y \in 2Z \\ 0 & if\ y \notin 2Z \end{cases}$$

Now, K is F-Soc-small prime submod since for $2_{\frac{1}{3}}$ is F-singleton of Z and $2_{\frac{1}{6}} \subseteq X$ where $< 2_{\frac{1}{6}} > \ll X$ since $< 2 > \ll Z_8$, $2_{\frac{1}{3}} \cdot 2_{\frac{1}{6}} = 4_{\frac{1}{6}} \subseteq K$ since $K(4) = \frac{1}{2} > \frac{1}{6}$, then $2_{\frac{1}{6}} = \subseteq K + F - Soc(X)$ since $K + F - Soc(X)(2) = 1 > \frac{1}{6}$ and $2_{\frac{1}{3}} \subseteq (K + F - Soc(X): X)$ since $(K + F - Soc(X): X)(2) = \frac{1}{2} > \frac{1}{3}$

Definition 3.3

A proper F-submod K of a F-module X of a W-module G is referred to as F-Soc-small T-ABSO submod if whenever F-singleton s_a, r_b of W and $x_v \subseteq X$ such that $< x_v > \ll X$, $s_a r_b x_v \subseteq K$, implies either $s_a x_v \subseteq K + F - Soc(X)$ or $r_b x_v \subseteq K + F - Soc(X)\ or\ s_a r_b \subseteq (K + F - Soc(X): X)$

Farthermore, **:** A proper F-ideal H of a ring W is called a F-Soc-small T-ABSO ideal if whenever F-singleton s_a, r_b, q_c of W such that $< q_c >$ is F- small ideal of W and $s_a r_b q_c \subseteq H$, implies either $s_a q_c \subseteq H + F - Soc(W)$ or $r_b q_c \subseteq H + F - Soc(W)\ or\ s_a r_b \subseteq H + F - Soc(W)$

Remarks and examples 3.4

1. Every F-Soc-prime (F-Soc-small prime) submod of a F-module X of a W-module G is a F-Soc-small T-ABSO submod of X. However the converse incorrect, for example:

   Let $X: Z_{16} \to I$ such that $X(y) = \begin{cases} 1 & if\ y \in Z_{16} \\ 0 & o.w \end{cases}$

   It is obvious that X is a F-module of Z_{16} as Z-module

   Let $K: Z_{16} \to I$ such that $K(y) = \begin{cases} \frac{2}{3} & if\ y \in (\bar{8}) \\ 0 & if\ y \notin (\bar{8}) \end{cases}$

   It is clear K is F-submod of X

$$F - Soc(X)(y) = \begin{cases} 1 & if\ y \in (\bar{8}) \\ \frac{1}{3} & if\ y \notin (\bar{8}) \end{cases}$$

$$(K + F - SocX)(y) = \begin{cases} 1 & if\ y \in (\bar{8}) \\ \frac{1}{3} & if\ y \notin (\bar{8}) \end{cases}$$

$$(K + F - SocX{:}X)(y) == \begin{cases} 1 & if\ y \in 8Z \\ 0 & if\ y \notin 8Z \end{cases}$$

Where $((\bar{8}) + Soc(Z){:}Z) = ((\bar{8}){:}Z) = 8Z,\ K\ is$ a F-Soc-Small T-ABSO submod since $< 4 > \ll Z_{16}, then < 4_{\frac{1}{6}} > \ll X, 2_{\frac{1}{2}} \cdot 1_{\frac{1}{5}} \cdot 4_{\frac{1}{6}} = 8_{\frac{1}{6}} \subseteq K$, since $K(8) = \frac{2}{3} > \frac{1}{6}$, implies that $2_{\frac{1}{2}} \cdot 4_{\frac{1}{6}} = 8_{\frac{1}{6}} \subseteq K + F - Soc(X)$, since $K + F - Soc(X)(8) = 1 > \frac{1}{6}$, but K is not F-Soc-prime (not F-Soc-small prime) submod since $2_{\frac{1}{6}} \cdot 4_{\frac{1}{2}} = 8_{\frac{1}{6}} \subseteq K$ $(2_{\frac{1}{6}} \cdot 4_{\frac{1}{2}} = 8_{\frac{1}{6}} \subseteq K, < 4_{\frac{1}{6}} > \ll Z_{16})$, since $K(8) = \frac{3}{2} > \frac{1}{6}, 4_{\frac{1}{2}} \nsubseteq K + F - Soc(X)$ since $K + F - Soc(X)(4) = \frac{1}{3} \ngtr \frac{1}{2}$ and $2_{\frac{1}{6}} \nsubseteq \left(K + F - Soc(X)\right)$ since $(K + F - \ Soc(X){:}X)(2) = 0 \ngtr \frac{1}{6}$

2. Every F-Soc-quasi prime submod is F-Soc-small T-ABSO submod, but the converse incorrect in general, for example:

Let $X{:}Z \to I$ such that $X(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w \end{cases}$

It is obvious that $X\ is\ F - module\ of\ Z\ as\ Z - module$

Let $K{:}Z \to I$ such that $K(y) = \begin{cases} \frac{3}{4} & if\ y \in 36Z \\ 0 & y \notin 36Z \end{cases}$

It is clear K is F-submod of X

$$F - Soc(X) = \begin{cases} 1 & if\ y = 0 \\ \frac{1}{4} & if\ y \neq 0 \end{cases}$$

$$K + F - Soc(X)(y) = \begin{cases} 1 & if\ y \in 36Z \\ \frac{1}{4} & if\ y \notin 36Z \end{cases}$$

K is F-Soc-small T-ABSO submod since 0_1 is only F-small submod of X, so that $a_s b_y 0_1 = 0_1 \subseteq K$, then $a_s 0_1 = 0_1 \subseteq K + F - Soc(X)$ and $b_y 0_1 = 0_1 \subseteq K + F - Soc(X)$. But it is not F-Soc-quasi prime submod, take $6_{\frac{1}{2}}, 2_{\frac{1}{3}}$ are F-singletons of Z and $3_{\frac{1}{3}} \subseteq X$, where $6_{\frac{1}{2}} \cdot 2_{\frac{1}{3}} \cdot 3_{\frac{1}{2}} = 36_{\frac{1}{3}} \subseteq K$ since $K(36) = \frac{3}{4} > \frac{1}{3}$, but $6_{\frac{1}{2}} \cdot 3_{\frac{1}{3}} = 18_{\frac{1}{3}} \nsubseteq K + F - Soc(X)$ where $K + F - Soc(X)(18) = \frac{1}{4} \ngtr \frac{1}{3}$ and $2_{\frac{1}{3}} \cdot 3_{\frac{1}{2}} = 6_{\frac{1}{3}} \nsubseteq K + F - Soc(X)$ where $K + F - Soc(X) = \frac{1}{4} \ngtr \frac{1}{3}$

3. Every F-small T-ABSO submod of a F-module X of a W-module G, is a F-Soc-small T-ABSO submod of X. However the converse incorrect, for example:

Let $X{:}Z_{24} \to I$ such that $X(y) = \begin{cases} 1\ if\ y \in Z_{24} \\ 0 \quad o.w \end{cases}$

It is obvious that X is a F-module of Z_{24} as Z-module

Let $K{:}Z_{24} \to I$ such that $K(y) = \begin{cases} \frac{2}{3} & if\ y \in (\bar{8}) \\ 0 & if\ y \notin (\bar{8}) \end{cases}$

It is clear K is F-submod of X

$$F - Soc(X)(y) = \begin{cases} 1 & if\ y \in (\bar{4}) \\ \frac{1}{2} & if\ y \notin (\bar{4}) \end{cases}$$

$(K+F-SocX)(y)=\begin{cases}1 & if\ y\in(\bar{4})\\ \frac{1}{2} & if\ y\notin(\bar{4})\end{cases}$

$(K+F-Soc(X):X)(y)=\begin{cases}1 & if\ y\in 4Z\\ 0 & if\ y\notin 4Z\end{cases}$

$(K:X)(y)=\begin{cases}1 & if\ y\in 8Z\\ 0 & if\ y\notin 8Z\end{cases}$

Where $\left((\bar{8})+Soc(Z_{24}):Z_{24}\right)=\left((\bar{4}):Z_{24}\right)=4Z$, K is a F-Soc small T-ABSO submod since $<6>\ll Z_{24}$, then $<6_{\frac{1}{6}}>\ll X$, $2_{\frac{1}{2}}\cdot 2_{\frac{1}{5}}\cdot 6_{\frac{1}{6}}=0_{\frac{1}{6}}\subseteq K$ since $K(0)=\frac{2}{3}>\frac{1}{6}$, implies that $2_{\frac{1}{2}}\cdot 6_{\frac{1}{6}}=12_{\frac{1}{6}}\subseteq K+F-Soc(X)$ and $2_{\frac{1}{2}}\cdot 2_{\frac{1}{5}}=4_{\frac{1}{5}}\subseteq (K+F-\ Soc(X):X)$ since $(K+F-Soc(X):X)(4)=1>\frac{1}{5}$, but K is not F-small T-ABSO submod since $2_{\frac{1}{2}}\cdot 6_{\frac{1}{6}}=12_{\frac{1}{6}}\nsubseteq K$ where $K(12)=0\ngtr\frac{1}{6}$ and $2_{\frac{1}{2}}\cdot 2_{\frac{1}{5}}=4_{\frac{1}{5}}\nsubseteq (K:X)$ since $(K:X)(4)=0\ngtr\frac{1}{5}$.

4. Every F-T-ABSO (F-Soc T-ABSO) submod of a F-module X of a W-module G is a F-Soc-small T-ABSO submod of X. However the converse incorrect, for example:

Let $X:Z_{24}\to I$ such that $X(y)=\begin{cases}1 & if\ y\in Z_{24}\\ 0 & o.w\end{cases}$

It is obvious that X is a F-module of Z_{24} as Z-module

Let $K:Z_{24}\to I$ such that $K(y)=\begin{cases}\frac{2}{3} & if\ y\in(\overline{12})\\ 0 & if\ y\notin(\overline{12})\end{cases}$

It is clear K is F-submod of X

$F-Soc(X)(y)=\begin{cases}1 & if\ y\in(\bar{4})\\ \frac{1}{7} & if\ y\notin(\bar{4})\end{cases}$

$K+F-SocX(y)=\begin{cases}1 & if\ y\in(\bar{4})\\ \frac{1}{7} & if\ y\notin(\bar{4})\end{cases}$

$(K+F-Soc(X):X)(y)=\begin{cases}1 & if\ y\in 4Z\\ 0 & if\ y\notin 4Z\end{cases}$

$(K:X)(y)=\begin{cases}1 & if\ y\in 12Z\\ 0 & if\ y\notin 12Z\end{cases}$

Where $\left((\overline{12})+Soc(Z_{24}):_Z Z_{24}\right)=\left((\bar{4}):Z_{24}\right)=4Z$, K is a F-Soc small T-ABSO submod since $<6_{\frac{1}{5}}>\ll Z_{24}$, $2_{\frac{1}{3}}\cdot 2_{\frac{1}{4}}\cdot 6_{\frac{1}{5}}=0_{\frac{1}{5}}\subseteq K$ since $K(0)=\frac{2}{3}>\frac{1}{5}$, implies that $2_{\frac{1}{4}}\cdot 6_{\frac{1}{5}}=12_{\frac{1}{5}}\subseteq K+F-Soc(X)$ and $2_{\frac{1}{3}}\cdot 2_{\frac{1}{4}}=4_{\frac{1}{4}}\subseteq (K+F-Soc(X):X)$ since $(K+F-Soc(X):X)(4)=1>\frac{1}{4}$, but K is not F-small T-ABSO submod, $2_{\frac{1}{3}}\cdot 2_{\frac{1}{4}}\cdot 3_{\frac{1}{5}}=12_{\frac{1}{5}}\subseteq K$ since $K(12)=\frac{2}{3}>\frac{1}{5}$, but $2_{\frac{1}{3}}\cdot 3_{\frac{1}{5}}=6_{\frac{1}{5}}\nsubseteq K$, since $K(6)=\frac{1}{7}\ngtr\frac{1}{5}$ and $2_{\frac{1}{3}}\cdot 2_{\frac{1}{4}}=4_{\frac{1}{4}}\nsubseteq (K:X)$ since $(K:X)(4)=0\ngtr\frac{1}{4}$.

Proposition 3.5

Let $X\ be\ a\ F-\ module$ of a $W-module$ G with $X(g)=1$ for each $g\in G$, if U is a $F-\ submod\ of\ X$ defined by

$U:G\to[I]$ such that $U(g)=\begin{cases}1 & if\ g\in U_t\\ c & if\ g\notin U_t\end{cases}$ with $0<c<1, t\in I$

Where U_t is a $submod\ of\ G.\ Then\ U\ is\ a\ F-Soc-small\ T-ABSO\ submod\ \ of\ X$ if and only if U_t is Soc-small T-ABSO $submod\ of\ G$

Proof: We must define F-Soc(X)

$F-Soc(X)(g)==\begin{cases}1 & if\ g\in Soc(G)\\ h & if\ g\notin Soc(G)\end{cases}$

Now, we define $U+F-Soc(X)$ such that $\big(U+F-Soc(X)\big)(g) == \begin{cases} 1 & if\ g \in U_t + Soc(G) \\ d & if\ g \notin U_t + Soc(G) \end{cases}$,

$d = \max\{c, h\}$

Now suppose U_t is Soc-small T-ABSO $submod\ of\ G$, to prove that U is a F-Soc-small T-ABSO $submod$ of X, let $a_s r_b x_k \subseteq U\ for\ F-\ singleton\ a_s, r_b$ of W and $< x_k > \ll X\ where\ b,k \in I$, that is either $arx \in U_t$ or $arx \notin U_t$

1) If $arx \in U_t$, then either $ax \in U_t + Soc(G)\ or\ rx \in U_t + Soc(G)\ or\ ar \in (U_t + Soc(G): G)$.
If $ax \in U_t + Soc(G)$, then $\big(U+F-Soc(X)\big)(ax) = 1$, implies $a_s x_k = (ax)_t \subseteq (ax) \subseteq U+F-Soc(X)\ where\ t = \min\{s,k\}$, by similarty method if $rx \in U_t + Soc(G)$, we get $r_b x_k \subseteq U+F-Soc(X)$

If $ar \in (U_t + F - Soc(G): G)$, let $y_p \subseteq X$ with $p \in I,\ then\ a_s r_b y_p = (ary)_\lambda$ where $\lambda = \min\{s,b,p\}$, but $ary \in U_t + Soc(G)$, that is $(ary)_\lambda \subseteq (ary)_1 \subseteq U+F-Soc(X)$, thus $a_s r_b y_p \subseteq U+F-Soc(X)$, hence $a_s r_b \subseteq (U+F-Soc(X): X)$
2) I $arx \notin U_t$, then $U(arx) = c$ with $rx \notin U_t$, hence $U(rx) = c$. Since $a_s r_b x_k \subseteq U$, so that $(arx)_\lambda \subseteq U$ where $\lambda = \min\{s,b,k\}$, that is $U(arx) \geq \lambda$, thus $c \geq \lambda$, Now, if $\lambda = k$, implies $(rx)_k \subseteq (rx)_c \subseteq U \subseteq U+F-Soc(X)$, if $\lambda = a, U(m) \geq k,\ for\ any\ m \in G$, and $(a_s r_b X_G)(m) = \begin{cases} b & ifm = arn \text{ for some } n \in G \\ 0 & o.w \end{cases}$
then, we get $(a_s r_b X_G)(m) \leq U(m)$, hence $a_s r_b X_G \subseteq U \subseteq U+F-Soc(X)$, that is mean $a_s r_b \subseteq (U+F-Soc(X): X)$ thus U is a F-Soc-smallT ABSO Submod of X
on the contrary, assume $U\ is\ a\ F-Soc-small\ T-ABSO\ submod\ of\ X$, let $arx \in U_t$ with $a,r \in W\ and\ x \in X_t$, implies $(ar\,x)_t \subseteq U$, then $a_s r_b x_k \subseteq U$where $t = \min\{s,b,k\}$, but $U\ is\ a\ F-Soc-small\ T-ABSO\ submod\ of\ X$, then either $a_s x_k \subseteq U+F-Soc(X)$ or $r_b x_k \subseteq U+F-Soc(X)\ or\ a_s r_b \subseteq (U+F-Soc(X): X)$ thus either $\big(U+F-Soc(X)\big)(ax) \geq t\ or\ \big(U+F-Soc(X)\big)(rx) \geq t\ or\ (U+F-Soc(X): X)(ar) \geq t$, then by Lemma 2.13 and Lemma 2.20 either $ax \in \big(U+F-Soc(X)\big)_t = U_t + \big(F-Soc(X)\big)_t = U_t + Soc(X_t)\ or\ rx \in \big(U+F-Soc(X)\big)_t = U_t + \big(F-Soc(X)\big)_t = U_t + Soc(X_t)\ \ or\ \ ar \in (U+F-Soc(X): X)_t \subseteq \big(U+F-Soc(X)\big)_t : X_t) = (U_t + \big(F-Soc(X)\big)_t : X_t) = U_t + Soc(X_t): X_t)$. Thus, we have either $ax \in U_t + Soc(X_t)\ \ or\ \ rx \in U_t + Soc(X_t)\ or\ \ ar \in (U_t + Soc(X_t): X_t)$, so that U_t is Soc-small T-ABSO submod of $X_t = G$.

Proposition 3.6

Let K be a proper F-submod of F-module X of a W-module G, then K is a F-Soc-small T-ABSO submod if and only if whenever for F-singletons m_s, y_z of W, $P \ll X$, $m_s y_z P \subseteq K$, then either $m_s P \subseteq K+F-Soc(X)$ or $y_z P \subseteq K+F-Soc(X)$ or $m_s y_z \subseteq (K+F-Soc(X): X)$.

Proof: (⇒) Suppose that $m_s y_z P \subseteq K$, but $m_s P \nsubseteq K+F-Soc(X)$ and $y_z P \nsubseteq K+F-Soc(X)$, so there exist F-singletons$x_r, h_y \subseteq P$ such that $m_s h_y \subseteq P$ such that $m_s x_r \nsubseteq K+F-Soc(X)$ and $y_z h_y \nsubseteq K+F-Soc(X)$, then $< x_r > \ll X$ and $< h_y > \ll X$ since $x_r, h_y \subseteq P$ and $P \ll X$. Now $m_s y_z x_r \subseteq K$ and $m_s x_r \nsubseteq K+F-Soc(X)$, hence either $y_z x_r \subseteq K+F-Soc(X)$ or $m_s y_z \subseteq (K+F-Soc(X): X)$. If $m_s y_z \subseteq (K+F-Soc(X): X)$ then we are done. If $y_z x_r \subseteq K+F-Soc(X)$. Meditate $m_s y_z (x_r + h_y) \subseteq K\ and\ (x_r + h_y) \ll X$ since $< x_r > \ll X$ and $< h_y > \ll X$ by Theorem 2.30. Since K is a F-Soc-small T-ABSO submod, then either $m_s(x_r + h_y) \subseteq K+F-Soc(X)$ or $y_z(x_r + h_y) \subseteq K+F-Soc(X)$ or $m_s y_z \subseteq (K+F-Soc(X): X)$. If $m_s y_z \subseteq K+F-Soc(X)$ and $m_s x_r \nsubseteq K+F-Soc(X)$, hence $m_s h_y \nsubseteq K+F-Soc(X)$. But $m_s y_z \subseteq K, m_s h_y \nsubseteq K+F-Soc(X)$ and $y_z h_y \nsubseteq K+F-Soc(X)$ so that $m_s y_z \subseteq (K+F-Soc(X): X)$. If $y_z(x_r + h_y) \subseteq K+F-Soc(X)$, then similary $m_s y_z \subseteq (K+F-Soc(X): X)$.

(⇐) it is evident.

Corollary 3.7

Let H be a proper F-ideal of W, then H is a F-Soc-small T-ABSO ideal if and only if whenever for singletons $m_a y_b$ of W and F-small ideal S of W such that $m_a y_b S \subseteq H$, then either $m_a S \subseteq H+F-Soc(W)$ or $y_b S \subseteq H+F-Soc(W)$ or $m_a y_b \subseteq H + F-Soc(W)$

Proof: It is similar to proof Proposiyion 3.6, only in this corollary use the ideals instead the submodules.

Proposition 3.8

Let K be a proper F-submod of a F-module X of a W-module G, if $K+F-Soc(X)$ is a F-small T-ABSO submod of X, then $(K+F-Soc(X):_X J)$ is a F-small T-ABSO submod for each F-ideal J of W, $JX \nsubseteq K+F-Soc(X)$.

Proof: Let $m_s y_z x_r \subseteq (K + F - Soc(X) :_X J)$ and $< x_r > \ll X$ for F-singleton m_s, y_z of W, so $m_s y_z < x_r > \subseteq (K + F - Soc(X) :_X J)$, hence $m_s y_z J < x_r > \subseteq K + F - Soc(X)$, but $< x_r > \ll X$, then $< J x_r > \ll X$ and since $K + F - Soc(X)$ is F-small T-ABSO submod, then either $m_s < J x_r > \subseteq K + F - Soc(X)$ or $y_z < J x_r > \subseteq K + F - Soc(X)$ or $m_s y_z \subseteq (K + F - Soc(X): X)$ by Proposiyion 3.6, so that $m_s x_r \subseteq (K + F - Soc(X) :_X J)$ or $y_z x_r \subseteq (K + F - Soc(X) :_X J)$ or $m_s y_z \subseteq (K + F - Soc(X): X) \subseteq \left((K + F - Soc(X) :_X J): X\right)$.

Lemma 3.9

Let K be F-submod of a faithful multiplication and cancelation F-module X of a W-module G, then $(K: X) + F - Soc(W) = (K + F - Soc(X): X)$

Proof: by Remark 2.35, we have $(K: X) + F - Soc(W) \subseteq (K + F - Soc(X): X)$

Now, to prove $(K + F - Soc(X): X) \subseteq (K: X) + F - Soc(W)$

Let $r_s \subseteq (K + F - Soc(X): X)$, hence $r_s X \subseteq K + F - Soc(X)$,but X multiplication F-module, then $K = (K: X)X$ by Proposition 2.24, and $F - Soc(X)$= X $F - Soc(W)$ by Proposition 2.38, So that $r_s X \subseteq (K: X)X + \ F - Soc(W)X$, hence $r_s X \subseteq ((K: X) + + \ F - Soc(W))X$, then $r_s \subseteq (K: X) + F - Soc(W)$, that is $(K + F - Soc(X): X) \subseteq (K: X) + F - Soc(W)$.

Thus, $(K: X) + F - Soc(W) = (K + F - Soc(X): X)$

Proposition 3.10

If K is a F-Soc-small T-ABSO submod of a F. module X of a W-module G, then $(K: X)$ is a F-Soc-small T-ABSO ideal of W.

Proof: Let $m_s n_r v_q \subseteq (K: X)$ and $< v_q >$ is a F-small ideal of W, for F.singletons m_s, n_r, v_q of W. suppose that $m_s v_q \nsubseteq (K: X)$ and $n_r v_q \nsubseteq (K: X)$. Now for any F.singleton $x_z \subseteq X$, define $f: \lambda_w \to X$ by $f(1_z) = 1_z x_s = x_s$. It is clear that f is well-defined and F -homomorphism. Since $< v_q >$ is a small F.ideal of W, then $< v_q x_v > \ll X$ ... (1). By assumption there exist $h_y, g_b \subseteq X$ such that $m_s v_q h_y \nsubseteq K$ and $n_r v_q g_b \nsubseteq K$. But $m_s n_r \left(v_q h_y + v_q g_b\right) \subseteq K$ and by (1) $< v_q h_y > \ll X, < v_q g_b > \ll X$, then $< v_q h_y + v_q g_b > \ll X$ by Theorem 2.31. Then either $m_s(v_q h_y + v_q g_b) \subseteq K + F - Soc(X)$ or $n_r\left(v_q h_y + v_q g_b\right) \subseteq K + F - Soc(X)$ or $m_s n_r \subseteq (K + F - Soc(X): X)$.

If $m_s n_r \subseteq (K + F - Soc(X): X) = (K: X) + F - Soc(W)$ by Lemma 3.9, then we are done.

If $m_s\left(v_q h_y + v_q g_b\right) \subseteq K + F - Soc(X)$, $m_s v_q h_y \nsubseteq K + F - Soc(X)$, we get $m_s v_q g_b \nsubseteq K + F - Soc(X)$. But $m_s n_r v_q g_b \subseteq K + F - Soc(X)$, $< v_q g_b > \ll X$ and $n_r v_q g_b \nsubseteq K + F - Soc(X)$. Thus $m_s n_r \subseteq (K + F - Soc(X): X) = (K: X) + F - Soc(W)$ by Lemma 3.9.

By the same method, if $n_r\left(v_q h_y + v_q g_b\right) \subseteq K + F - Soc(X)$, then $m_s n_r \subseteq (K + F - Soc(X): X) = (K: X) + F - Soc(W)$ by Lemma 3.9.

Proposition 3.11

Let K be a proper F. submod of a faithful finitely generated multiplication F.module X of a W-module G. if $(K: X)$ is a F-Soc-small T-ABSO ideal, then K is a F-Soc-small T-ABSO submod of X.

Proof: Let $m_s n_r x_z \subseteq K$ and $< x_z > \ll X$ for F.singleton $m_s n_r$ of W. But X is a faithful finitely generated multiplication F.module, then $< x_z > = \hat{J} X$ for a small F.ideal $\hat{J}$ of W (since if $\hat{J} + U = \lambda_W$ for F.ideal U of W, $\hat{J} X + UX = X$. hence $< x_z > + UX = X$. But $< x_z > \ll X$, so that $UX = X$, thus $U = \lambda_W$ $this$ $contradication$) . Hence $m_s n_r \hat{J} X \subseteq K$. So $m_s n_r \hat{J} \subseteq (K: X)$. Since $(K: X)$ is a F-Soc-small T-ABSO ideal, then either $m_s \hat{J} \subseteq (K: X) + F - Soc(W) = (K + F - Soc(X): X)$ or $n_r \hat{J} \subseteq (K: X) + F - Soc(W) = (K + F - Soc(X): X)$ or $m_s n_r \subseteq (K: X) + F - Soc(W) = (K + F - Soc(X): X)$ by Lemma 3.9 and Corollary 3.7 . So that $m_s \hat{J} X \subseteq K + F - Soc(X)$ or $n_r \hat{J} X \subseteq K + F - Soc(X)$ or $m_s n_r \subseteq (K + F - Soc(X): X)$, then $m_s x_z \subseteq K + F - Soc(X)$ or $n_r x_z \subseteq K + F - Soc(X)$ or $m_s n_r \subseteq (K + F - Soc(X): X)$. Thus, K is a F-Soc-small T-ABSO submod of X.

***Proposition* 3.12**

Let X_1, X_2 be F-modules of a W-module G_1, G_2 respectiviely. Let $f: G_1 \longrightarrow G_2$ be an isomorphism and K is $F - Soc -$ small $T - ABSO$ submod of X_1 such that $F - \ker f \subseteq K$, then f(K) is F-Soc-small T-ABSO submod of X_2.

Proof: It is clear that $f(K)$ a proper F.submod of X_2 since K is a proper F.submod of X_1Let $a_v r_s z_b \subseteq f(K)$ for $F -$ singleton a_v, r_s of W and $< z_b > \ll X_2$, since f is onto, so $z_b = f(x_n)$ for some F-singleton $x_n \subseteq X_1$, hence$< f(x_n) > \ll X_2$ by Theorem 2.35, then $a_v r_s f(x_n) = f(s_a)$ for a F-singleton $s_a \subseteq K$. then $a_v r_s x_n - s_a \subseteq F - \ker f \subseteq K$, thus $a_v r_s x_n \subseteq K$. But K is a F-Soc-

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small T-ABSO submod, hence either $a_v x_n \subseteq K + F - Soc(X_1)$ or $r_s x_n \subseteq K + F - Soc(X_1)$ or $a_v\, r_s \subseteq (K + F - Soc(X_1): X_1)$. If $a_v x_n \subseteq K + F - Soc(X_1)$, then $a_v f(x_n) \subseteq f(K) + F - Socf(X_1)$ by Proposition 2.21 , hence $a_v z_b \subseteq f(K) + F - Soc(X_2)$. By the same method , if $r_s x_n \subseteq K + F - Soc(X_1)$, we get $r_s z_b \subseteq f(K) + F - Soc(X_2)$.

Now, if $a_v\, r_s \subseteq (K + F - Soc(X_1): X_1)$, then $a_v\, r_s X_1 \subseteq K + F - Soc(X_1)$, hence $a_v\, r_s f(X_1) \subseteq f(K) + F - Socf(X_1)$ by Proposition 2.21, then $a_v\, r_s\, X_2 \subseteq f(K) + F - Soc(X_2)$, since f is onto, thus $a_v\, r_s \subseteq (f(K) + F - Soc(X_2): X_2)$.

Thus, f(K) is F-Soc-smallT-ABSO submod of X_2.

Proposition 3.13

Let $f: G_1 \to G_2$ be an epimorphism and X_1, X_2 are F. modules of a W-module G_1, G_2resp. If K is a F-small T-ABSO submod of X_2, then $f^{-1}(K)$ is a F-Soc-small T-ABSO submod of X_1.

Proof: It is clear that $f^{-1}(K)$ a proper F.submod of X_1 since K is a proper F.submod of X_2. Let $m_s n_r v_x \subseteq f^{-1}(K)$ and $< v_x > \ll X_1$ for F-singleton m_s, n_r of W, then $m_s n_r f(v_x) \subseteq K$. But $< f(v_x) > \ll X_2$ by Theorem 2.35, and K is a F-small T-ABSO submod, then either $m_s f(v_x) \subseteq K$ or $n_r f(v_x) \subseteq K$ or $m_s n_r \subseteq (K: X_2)$. Hence $m_s v_x \subseteq f^{-1}(K) + F - Soc(X_1)$ or $n_r v_x \subseteq f^{-1}(K) + F - Soc(X_1)$ or $m_s n_r \subseteq (f^{-1}(K): f^{-1}(X_2)) \subseteq (f^{-1}(K) + F - Soc(X_1): X_1)$.Thus, $f^{-1}(K)$ is a F-Soc-small T-ABSO submod of X_1.

Now, we give the following definition since it is needed in the next proposition

Definition 3.14

F-submod K of a F-module X of a W-module G is referred to as F-distributive if, $(P \cap S) + K = (P + K) \cap (S + K)$ for all F-submods P, S of X. A F- module X is said to be F-distributive if, all F- submods of X are F-distributive.

Proposition 3.15

Let P and K be F-Soc-small T-ABSO submods of a F-module of a W-module G, such that $(P + F - Soc(X): X) = (K + F - Soc(X): X)$ and $F - Soc(X)$ is F-distributive submod , then $P \cap K$ is F-Soc-small T-ABSO submod of X.

Proof: Let $r_s a_v x_n \subseteq P \cap K$, for F-singletons r_s, a_v of W and $< x_n > \ll X$, then $r_s a_v x_n \subseteq P$ and $r_s a_v x_n \subseteq K$, but P and K are F-Soc-small T-ABSO submods, then $r_s x_n \subseteq P + F - Soc(X)$ or $a_v x_n \subseteq P + F - Soc(X)$ or $r_s a_v \subseteq (P + F - Soc(X): X)$ and $r_s x_n \subseteq K + F - Soc(X)$ or $a_v x_n \subseteq K + F - Soc(X)$ or $r_s a_v \subseteq (K + F - Soc(X): X)$. Hence $r_s x_n \subseteq P + F - Soc(X)$ and $r_s x_n \subseteq K + F - Soc(X)$ or $a_v x_n \subseteq P + F - Soc(X)$ and $a_v x_n \subseteq K + F - Soc(X)$ or $r_s a_v \subseteq (P + F - Soc(X): X) = (K + F - Soc(X): X)$, which implies $r_s x_n \subseteq (P + F - Soc(X)) \cap (K + F - Soc(X))$. or $a_v x_n \subseteq (P + F - Soc(X)) \cap (K + F - Soc(X))$or $r_s a_v \subseteq ((P + F - Soc(X)) \cap (K + F - Soc(X)): X)$, hence $r_s x_n \subseteq P \cap K + F - Soc(X)$. or $a_v x_n \subseteq P \cap K + F - Soc(X)$ or $r_s a_v \subseteq (P \cap K + F - Soc(X): X)$ since $F - Soc(X)$ is F-distributive submod. Thus, $P \cap K$ is F-Soc-small T-ABSO submod of X.

Proposition 3.16

If K is F-Soc-small T-ABSO submod of a F-module X of a W-module G, such that $F - Soc(X) \subseteq K$ and H is any F-ideal of W, then $(K:_X H)$ is F-small T-ABSO submod of X.

Proof: Let $r_s a_v x_n \subseteq (K:_X H)$, for F-singletons r_s, a_v of W and $< x_n > \ll X$, then $r_s a_v x_n H \subseteq K$, that is $r_s a_v x_n h_m \subseteq K, \forall h_m$ F-singleton of H. But $< x_n h_m > \subseteq X$ and since $< x_n > \ll X$, implies that $< x_n h_m > \ll X$ by Proposition 2.31 . Then either $r_s x_n h_m \subseteq K + F - Soc(X)$ or $a_v x_n h_m \subseteq K + F - Soc(X)$ or $r_s a_v \subseteq (K + F - Soc(X): X) \subseteq \left((K + F - Soc(X):_X H): X\right)$, but $F - Soc(X) \subseteq K$, so that either $r_s x_n \subseteq (K:_X H)$ or $a_v x_n \subseteq (K:_X H)$ or $r_s a_v \subseteq (K:_X H): X)$. Thus, $(K:_X H)$ is F-small T-ABSO submod of X.

Proposition 3.17

Let X_1, X_2 be two F-modules of a W-modules G_1 *and* G_2 respectively. If $K_1 \oplus K_2$ is F-Soc-small T-ABSO submod of $X_1 \oplus X_2$, then K_1 *and* K_2 are F-Soc-small T-ABSO submods of X_1 *and* X_2 respectiviely.

Proof: Suppose that $r_s a_v x_n \subseteq K_1$ and $r_s a_v y_m \subseteq K_2$, for F-singletons r_s, a_v of W and $< x_n > \ll X_1$ and $< y_m > \ll X_2$, then $r_s a_v (x_n, y_m) \subseteq K_1 \oplus K_2$, such that $< x_n > \oplus < y_m > \ll X_1 \oplus X_2$ by Proposition 2.31. But $K_1 \oplus K_2$ is F-Soc-small T-ABSO submod of $X_1 \oplus X_2$, then either $r_s (x_n, y_m) \subseteq K_1 \oplus K_2 + F - Soc(X_1 \oplus X_2)$ or $a_v (x_n, y_m) \subseteq K_1 \oplus K_2 + F - Soc(X_1 \oplus X_2)$ or $r_s a_v \subseteq (K_1 \oplus K_1 + F - Soc(X_1 \oplus X_2): X_1 \oplus X_2) = (K_1 + F - Soc(X_1): X_1) \cap (K_2 + F - Soc(X_2): X_2)$, hence $r_s x_n \subseteq K_1 + F - Soc(X_1)$ or $a_v\, x_n \subseteq K_1 + F - Soc(X_1)$ or $r_s a_v \subseteq (K_1 + F - Soc(X_1): X_1)$ and $r_s y_m \subseteq K_2 + F - Soc(X_2)$ or $a_r y_m \subseteq K_2 + F -$

$Soc(X_2)$ or $r_s a_v \subseteq (K_2 + F - Soc(X_2): X_2)$ by Proposition 2.32. Thus, $K_1\ and\ K_2$ are F-Soc-small T-ABSO submods of $X_1 and\ X_2$ respectively.

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