International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 10, October - 2023, Pages: 56-65

Survey Study of Generalized RK Integrators for Solving Ordinary Differential Equation.

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Abstract: *The main focus of our work is to present a survey of generalized Runge–Kutta integrals (RK, RKN, RKD, RKT, RKFD, RKTF, and RKM) to solve for first, second, third, fourth, fifth, seventh- even tenth order. Ordinary differential equations for the development of RK-type explicit direct integration tools for the solution of computational development equations are the major contributions of these papers. To solve the generalized RK equations to solve the ODE equations, the order terms (OCs) of the proposed integrators are derived using Taylor expansion, direct numerical methods with different phases are derived based on these conditions, . Some implementation computation methods are also tested to show the new methods are compatible with the existing RK methods and save a large amount of computational time and require less functional evaluations. The use of modern scientific programs and systems such as MATLAB and MAPEL has the most important role in our study.*

Keyword: Runge-Kutta method (RK); Runge-Kutta Nystrom method (RKN); Direct method (RKD); RKM; RKT; RKFD; Integrators; Class of tenth-order; Stage; Ordinary differential equations; Order conditions; Taylor expansion.

(Introduction)

We can say without exaggeration that in all branches of applied sciences DEs occupies a prominent place in that it is a major tool for analyzing a wide range of real-world phenomena such as economics, chemistry, and eng-systems. It is also an important mathematical model for physical phenomena, and this has been confirmed in many studies such as [1], [2] and [3]. We also find this in studies [4] and [5] in various other fields, such as polymer production, medicine, pharmaceuticals, communication technology, and the study of fluid physics plasma, and to try to learn more properties of differential equations and methods of solving them, although The difficulty of obtaining the solution is sometimes because some of them are not solvable sometimes or their solutions are not always an easy process or dependent on the basis of numerical approximation before the existence of computers, and this is why we offer a study and derivation of more direct numerical methods. The beginning was from the time of Newton, Euler and Taylor, but many methods do not contain any closed form solutions. And in order to apply an indirect numerical method to solve for higher-order DEs, the equation must be transformed into a system of first-order DEs. The rates of change are recalculated through the derivatives.

Using RK to solve First-Order (IVP)

The first-order (IVP) defined as follows:[6]

$$\bar{U}'(\mathbb{T}) = \acute{R}\,(\mathbb{T}, \bar{U}\,(\mathbb{T}))\,,\ \mathring{a} \leq \mathbb{T} \leq Y\,, \tag{1}$$

(IC):

$$\bar{U}\,(\mathring{a}) = \Omega \tag{2}$$

where,

$$\acute{R} : \check{R} \times \check{R}^{N} \longrightarrow \check{R}^{N},$$

$$\bar{U}\,(\mathbb{T}) = [\bar{U}1'(\mathbb{T})\,, \bar{U}2'(\mathbb{T})\,, \ldots\ldots\ldots, \bar{U}N'(\mathbb{T})\,]^{\mathbb{T}}\,, \tag{3}$$

$$\acute{R}\,(\mathbb{T}, \bar{U}) = [\acute{R}1(\mathbb{T}, \bar{U})\,, \ldots.\ \acute{R}N(\mathbb{T}, \bar{U})\,]^{\mathbb{T}} \tag{4}$$

and,

$$\Omega = [\Omega_{1,}\ \Omega_{2\,,\ldots\ldots\ldots}\Omega_{N}].$$

R-K Method

Now, RK method is defined by:[6]

$$\bar{U}\,n{+}1 = \bar{U}\,n + z \sum_{\ddot{I}=1}^{s} Y_{\ddot{I}} v_{\ddot{I}}\,, \tag{5}$$

where,

$$v1 = \acute{R}\,(\mathbb{T}\,n; \bar{U}\,n); \tag{6}$$

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ISSN: 2643-640X
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$$v\,Ï = Ŕ\,(\mathbb{T}\,n + x_{Ï} z\,,\,Ū\,n + z\sum_{j=1}^{s} å_{Ïj} v_j)\,, \quad (7)$$

$Ï$ = 2, 3, ..., m,

C.o.n

$$X\,Ï = \sum_{j=1}^{s} å_{Ïj}\,,$$

Butcher tableau for RK method as Table 1 or Table 2

$$\begin{array}{c|c} X & k \\ \hline & Y^{\mathbb{T}} \end{array}$$

Or

$$\begin{array}{c|cccc} X_1 & å_{1\,1} & å_{1\,2} & \ldots & å_{1\,s} \\ X_2 & å_{2\,1} & å_{2\,2} & \ldots & å_{2\,s} \\ \vdots & & & \vdots & \\ X_s & å_{s\,1} & å_{s\,2} & \ldots & å_{s\,s} \\ \hline & Y_1 & Y_2 & & Y_s \end{array}$$

$s \times s$ Matrix k by

$x = [x1, x2, \ldots, xS]^{\mathbb{T}}$,

$Y = [Y1, Y2, \ldots., YS]^{\mathbb{T}}$,

k = [å Ï j]:

If å $Ï\,j$ = 0, for j and $Ï$ = 1, 2,..., S

Classical R-K

"classical R-K method" as [7] is given.

$$Ū'n{+}1 = Ū'n + \frac{1}{6}\,(v_1 + 2(v_2 + v_3) + v_4)\,, \quad (8)$$

$$z = \frac{Y - å}{m}$$

Using RK to solve 2nd-Order (IVP)

Now, The 2nd-order (IVP) is defined as follows:[8]

$Ū'' (𝕋) = Ŕ (𝕋, Ū (𝕋))$, $å ≤ 𝕋 ≤ Y$, (9)

(IC):

$Ū (å) = 'Ω$, $Ū'' (å) = 'Ω1$ (10)

where,

$Ŕ : Ř × Ř^N ⟶ Ř^N$,

$Ū (𝕋) = [Ū1'(𝕋), Ū2'(𝕋), \ldots\ldots, ŪN'(𝕋)]^{𝕋}$, (11)

$Ŕ (𝕋, Ū=[Ŕ1(𝕋, Ū), \ldots\ldots ŔN(𝕋, Ū)]^{𝕋}$

and,

$$'Ω = ['Ω_{1,} 'Ω_{2,\ldots\ldots} 'Ω_N].$$
$$'Ω1 = ['Ω_{1,} 'Ω_{2,\ldots\ldots} 'Ω_N].$$

R-K-N Method

Now, **R-K-N Method**
is defined by:[8]

$$Ū ђ +1 = Ū ђ + z \sum_{Ï=1}^{s} Y_{Ï} v_{Ï} , \quad (12)$$

where,

$$v\ 1 = Ŕ (𝕋 ђ; Ū ђ); \quad (13)$$
$$v\ Ï = Ŕ (𝕋 ђ + x_{Ï}z , Ū ђ + z \sum_{j=1}^{s} å_{Ïj} v_j) , \quad (14)$$
$$v\ Ï' = Ŕ (𝕋 ђ + x_{Ï}z , Ū ђ + z \sum_{j=1}^{s} å_{Ïj} v'_j) ,$$

$$Ï = 2, 3, ..., m,$$

C.o.n
,

$$X\ Ï = \sum_{j=1}^{s} å_{Ïj} ,$$

x	k
	$Y^{𝕋}$
	$Y'^{𝕋}$

Using RK to solve 3nd-Order (IVP)

Now, The 3nd-order (IVP) is defined as follows:[8]

$Ū''' (𝕋) = Ŕ (𝕋, Ū (𝕋))$, $å ≤ 𝕋 ≤ Y$, (15)

(IC):

$Ū (å) = 'Ω$, $Ū'' (å) = 'Ω1$, $Ū''' (å) = 'Ω2$

where,

$Ŕ : Ř × Ř^N ⟶ Ř^N$,

$$\bar{U}\ (\mathbb{T}) = [\bar{U}1'(\mathbb{T})\ , \bar{U}2'(\mathbb{T})\ , \ldots\ldots\ldots, \bar{U}N'(\mathbb{T})\ ]^{\mathbb{T}}\ , \tag{16}$$

$$Ŕ\ (\mathbb{T}, \bar{U} = [Ŕ1(\mathbb{T}, \bar{U})\ , \ldots\ldots Ŕ N(\mathbb{T}, \bar{U})\ ]^{\mathbb{T}} \tag{4}$$

and,

$$'\Omega\ = \left['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}\right].$$

$$'\Omega 1\ = \left['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}\right].$$

$$'\Omega 2\ = \left['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}\right].$$

Now, **R-K-D Method**

is defined by:[8]

$$\bar{U}\ \mathfrak{h} + 1 = \bar{U}\ \mathfrak{h} + z \sum_{\ddot{I}=1}^{s} Y_{\ddot{I}} v_{\ddot{I}}\ , \tag{17}$$

where,

$$v1 = Ŕ\ (\mathbb{T}\ \mathfrak{h};\ \bar{U}\ \mathfrak{h}); \tag{18}$$

$$v\ \ddot{I} = Ŕ\ (\mathbb{T}\ \mathfrak{h} + x_{\ddot{I}} z\ , \bar{U}\ \mathfrak{h} + z \sum_{j=1}^{s} å_{\ddot{I}j} v_j)\ , \tag{19}$$

$$v\ \ddot{I}\,' = Ŕ\ (\mathbb{T}\ \mathfrak{h} + x_{\ddot{I}} z\ , \bar{U}\ \mathfrak{h} + z \sum_{j=1}^{s} å_{\ddot{I}j} v\,'_j)\ ,$$

$$v\ \ddot{I}\,'' = Ŕ\ (\mathbb{T}\ \mathfrak{h} + x_{\ddot{I}} z\ , \bar{U}\ \mathfrak{h} + z \sum_{j=1}^{s} å_{\ddot{I}j} v\,''_j)\ ,$$

$$\ddot{I} = 2, 3, \ldots$$

the condition,

$$X\ddot{I} = \sum_{j=1}^{s} å_{\ddot{I}j}\ ,$$

x	k
	$Y^{\mathbb{T}}$
	$Y'^{\mathbb{T}}$
	$Y''^{\mathbb{T}}$

Using RK to solve 4nd-Order (IVP)

Now, The 4nd-order (IVP) is defined as follows:[8-9]

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ISSN: 2643-640X
Vol. 7 Issue 10, October - 2023, Pages: 56-65

$$\bar{U}'''(\mathbb{T}) = \acute{R}(\mathbb{T}, \bar{U}(\mathbb{T})), \mathring{a} \leq \mathbb{T} \leq Y, \quad (20)$$

(IC):

$$\bar{U}(\mathring{a}) = \Omega \quad , \quad \bar{U}''(\mathring{a}) = \Omega 1 \quad , \quad \bar{U}'''(\mathring{a}) = \Omega 2 \, , \quad \bar{U}''''(\mathring{a}) = \Omega 2$$

where,

$$\acute{R} : \check{R} \times \check{R}^{N} \longrightarrow \check{R}^{N},$$

$$\bar{U}(\mathbb{T}) = [\bar{U}1'(\mathbb{T}), \bar{U}2'(\mathbb{T}), \ldots\ldots\ldots, \bar{U}N'(\mathbb{T})]^{\mathbb{T}} \, , \quad (21)$$

$$\acute{R}(\mathbb{T}, \bar{U} = [\acute{R}1(\mathbb{T}, \bar{U}), \ldots\ldots \acute{R}N(\mathbb{T}, \bar{U})]^{\mathbb{T}}$$

and,

$$\Omega = [\Omega_{1}, \Omega_{2}, \ldots\ldots\ldots \Omega_{N}].$$

$$\Omega 1 = [\Omega_{1}, \Omega_{2}, \ldots\ldots\ldots \Omega_{N}].$$

$$\Omega 2 = [\Omega_{1}, \Omega_{2}, \ldots\ldots\ldots \Omega_{N}].$$

$$\Omega 3 = [\Omega_{1}, \Omega_{2}, \ldots\ldots\ldots \Omega_{N}].$$

Now, **R-K-M Method**

is defined by:[8]

$$\bar{U}\,\mathfrak{h}+1 = \bar{U}\,\mathfrak{h} + z\sum_{\ddot{I}=1}^{s} Y_{\ddot{I}} v_{\ddot{I}} \, , \quad (22)$$

where,

$$v1 = \acute{R}(\mathbb{T}\,\mathfrak{h}; \bar{U}\,\mathfrak{h}); \quad (23)$$

$$v\ddot{I} = \acute{R}(\mathbb{T}\mathfrak{h} + x_{\ddot{I}} z, \bar{U}\,\mathfrak{h} + z\sum_{j=1}^{s} \mathring{a}_{\ddot{I}j} v_{j}) \, , \quad (24)$$

$$v\ddot{I}' = \acute{R}(\mathbb{T}\,\mathfrak{h} + x_{\ddot{I}} z, \bar{U}\,\mathfrak{h} + z\sum_{j=1}^{s} \mathring{a}_{\ddot{I}j} v'_{j}) \, ,$$

$$v\ddot{I}'' = \acute{R}(\mathbb{T}\,\mathfrak{h} + x_{\ddot{I}} z, \bar{U}\,\mathfrak{h} + z\sum_{j=1}^{s} \mathring{a}_{\ddot{I}j} v''_{j}) \, , \quad v\ddot{I}''' = \acute{R}(\mathbb{T}\mathfrak{h} + x_{\ddot{I}} z, \bar{U}\,\mathfrak{h} + z\sum_{j=1}^{s} \mathring{a}_{\ddot{I}j} v'''_{j}) \, ,$$

$$\ddot{I} = 2, 3, \ldots$$

the condition,

$$X\ddot{I} = \sum_{j=1}^{s} \mathring{a}_{\ddot{I}j} \, ,$$

x	k

$Y^{\mathbb{T}}$

$Y'^{\mathbb{T}}$

$Y''^{\mathbb{T}}$

$Y'''^{\mathbb{T}}$

. **Using RK to solve 5nd-Order (IVP)**

Now, The 4nd-order (IVP) is defined as follows:[8-9]

Ū$''''$ (𝕋) = Ŕ (𝕋, Ū (𝕋)) , å ≤ 𝕋 ≤ Y , (15)

(IC):

Ū (å) = ʼΩ , Ū$''$ (å) = ʼΩ1 , Ū$'''$ (å) = ʼΩ2 , Ū$''''$ (å) = ʼΩ3 Ū$'''''$ (å) = ʼΩ4,

where,

Ŕ : Ř × Ř$^{\mathrm{N}}$ ⟶ Ř$^{\mathrm{N}}$,

Ū (𝕋) = [Ū$1'$(𝕋) , Ū$2'$(𝕋) , … … … , ŪN'(𝕋)]$^{\mathbb{T}}$, (25)

Ŕ (𝕋, Ū=[Ŕ1(𝕋, Ū) , … … ŔN(𝕋, Ū)]$^{\mathbb{T}}$

and,

$$ʼΩ = [ʼΩ_{1,}\, ʼΩ_{2\,,\ldots\ldots\ldots}ʼΩ_{N}].$$

$$ʼΩ1 = [ʼΩ_{1,}\, ʼΩ_{2\,,\ldots\ldots\ldots}ʼΩ_{N}].$$

$$ʼΩ2 = [ʼΩ_{1,}\, ʼΩ_{2\,,\ldots\ldots\ldots}ʼΩ_{N}].$$

$$ʼΩ3 = [ʼΩ_{1,}\, ʼΩ_{2\,,\ldots\ldots\ldots}ʼΩ_{N}].$$

$$ʼΩ4 = [ʼΩ_{1,}\, ʼΩ_{2\,,\ldots\ldots\ldots}ʼΩ_{N}].$$

Now, **R-K-M Method**

is defined by:[9]

Ū ђ +1 = Ū ђ + z $\sum_{Ï=1}^{s} Y_{Ï}$v$_{Ï}$, (26)

where,

v1 = Ŕ (𝕋 ђ; Ū ђ); (27)

v Ï = Ŕ (𝕋 ђ $+\, x_{Ï} z$, Ū ђ $+\, z \sum_{j=1}^{s}$ å$_{Ïj} v_j$) , (28)

υ Ï ′= Ŕ (𝕋 𝔥 $+ x_{ï}z$, Ū 𝔥 $+ z\sum_{j=1}^{s}$ å$_{ïj}\upsilon\,'_{j}$) ,

υ Ï ′′= Ŕ (𝕋 𝔥 $+ x_{ï}z$, Ū 𝔥 $+ z\sum_{j=1}^{s}$ å$_{ïj}\upsilon\,''_{j}$) , υ Ï ′′′= Ŕ (𝕋 𝔥 $+ x_{ï}z$, Ū 𝔥 $+ z\sum_{j=1}^{s}$ å$_{ïj}\upsilon\,'''_{j}$) ,

υ Ï ′′′′= Ŕ (𝕋𝔥 $+ x_{ï}z$, Ū 𝔥 $+ z\sum_{j=1}^{s}$ å$_{ïj}\upsilon\,''''_{j}$) ,

Ï = 2, 3, ...

the condition,

$X\,Ï = \sum_{j=1}^{s}$ å$_{ïj}$,

x	k
	$Y^{\mathbb{T}}$
	$Y'^{\mathbb{T}}$
	$Y''^{\mathbb{T}}$
	$Y'''^{\mathbb{T}}$
	$Y''''^{\mathbb{T}}$

.

Using RK to solve 6-9 nd-Order (IVP)

Now, The 4nd-order (IVP) is defined as follows:[10-14]

Ū$^{6-9'}$ (𝕋) = Ŕ (𝕋, Ū (𝕋)) , å ≤ 𝕋 ≤ Y , (15)

(IC):

Ū (å) = ′Ω , Ū′′ (å) = ′Ω1 , Ū′′′ (å) = ′Ω2 , Ū′′′′ (å) = ′Ω3, Ū′′′′′ (å) = ′Ω4, , Ū′′′′′′ (å) = ′Ω5 , Ū′′′′′′′ (å) = ′Ω6 , Ū′′′′′′′ (å) = ′Ω7 ,…….

where,

$Ŕ : Ř \times Ř^N \longrightarrow Ř^N$,

$\bar{U}\ (\mathbb{T}) = [\bar{U}1'(\mathbb{T})\ , \bar{U}2'(\mathbb{T})\ , \ldots\ldots\ldots , \bar{U}N'(\mathbb{T})\ ]^{\mathbb{T}}$, (29)

$Ŕ\ (\mathbb{T}, \bar{U}=[Ŕ1(\mathbb{T}, \bar{U})\ , \ldots\ldots Ŕ N(\mathbb{T}, \bar{U})\ ]^{\mathbb{T}}$

and,

$$'\Omega\ = ['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}].$$

$$'\Omega 1\ = ['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}].$$

$$'\Omega 2\ = ['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}].$$

$$'\Omega 3\ = ['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}].$$

$$'\Omega 4\ = ['\Omega_{1,}\ '\Omega_{2,\ldots\ldots\ldots}'\Omega_{N}].$$

$$'\Omega \mathbf{5}\ = ['\Omega_{\mathbf{1},}\ '\Omega_{\mathbf{2},\ldots\ldots\ldots}'\Omega_{\boldsymbol{N}}].$$

$$'\Omega \mathbf{6}\ = ['\Omega_{\mathbf{1},}\ '\Omega_{\mathbf{2},\ldots\ldots\ldots}'\Omega_{\boldsymbol{N}}].$$

$$'\Omega \mathbf{7}\ = ['\Omega_{\mathbf{1},}\ '\Omega_{\mathbf{2},\ldots\ldots\ldots}'\Omega_{\boldsymbol{N}}].$$

$$'\Omega \mathbf{8}\ = ['\Omega_{\mathbf{1},}\ '\Omega_{\mathbf{2},\ldots\ldots\ldots}'\Omega_{\boldsymbol{N}}].$$

$$'\Omega \mathbf{9}\ = ['\Omega_{\mathbf{1},}\ '\Omega_{\mathbf{2},\ldots\ldots\ldots}'\Omega_{\boldsymbol{N}}].$$

Now, **R-K-M Method**

is defined by:[9-14]

$\bar{U}\ ђ +1 = \bar{U}\ ђ + z \sum_{ĭ=1}^{s} Y_{ĭ} v_{ĭ}$, (30)

where,

$v1 = Ŕ\ (\mathbb{T}\ ђ;\ \bar{U}\ ђ)$; (31)

$v\ Ï = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v_j)$, (32)

$v\ Ï\ ' = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ '_j)$,

$v\ Ï\ '' = Ŕ\ (\mathbb{T}ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ ''_j)$,

$v\ Ï\ ''' = Ŕ\ (\mathbb{T}ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ '''_j)$,

$v\ Ï\ '''' = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ ''''_j)$,

$v\ Ï\ ''''' = Ŕ\ (\mathbb{T}ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ '''''_j)$

$v\ Ï\ '''''' = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ ''''''_j)$

$v\ Ï\ ''''''' = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ '''''''_j)$

$v\ Ï\ '''''''' = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ ''''''''_j)$

$v\ Ï\ ''''''''' = Ŕ\ (\mathbb{T}\ ђ + x_{ĭ}z\ , \bar{U}\ ђ + z\sum_{j=1}^{s} å_{ĭj} v\ '''''''''_j)$

Ï = 2, 3, ...

the condition,

$X \, Ï = \sum_{j=1}^{s} å_{Ïj}$,

x	k
	$Y^{\mathbb{T}}$
	$Y'^{\mathbb{T}}$
	$Y''^{\mathbb{T}}$
	$Y'''^{\mathbb{T}}$
	$Y''''^{\mathbb{T}}$
	:
	:
	:

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