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New Class of Harmonic Univalent Functions Defined by Dziok-Srivastava Operator

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Abstract: *In this paper, we use Dziok-Srivastava operator $H_{q,s}[\alpha_1]$ to introduce a new subclass $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$ of harmonic univalent functions in unit disk $U = \{z : |z| < 1\}$. Here, we give the coefficient bounds, extreme points, closure theorem, convex combinations and integral operator.*

Keywords: Harmonic univalent functions, extreme points, closure theorem, convex combinations.

1_Introduction:

A continuous complex valued functions $f = u + iv$ which is define in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [4]).

Denote by G_H, the class of functions f of the form (1.2) that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in G_H$, we may express

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, |b_1| < 1, \tag{1.2}$$

where the analytic functions h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, g(z) = \sum_{k=1}^{\infty} |b_k| \overline{z^k}, |b_1| < 1, \tag{1.3}$$

In 1984 Clunie and Sheil-Small [4] investigated the class G_H as well as its geometric subclass and obtained some coefficient bounds.

For positive real values of $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ $(\beta_j \notin z_0^- = \{0, -1, -2, \ldots\};\ j = 1, 2, \ldots, s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [6])

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!}, (q \leq s+1; q, s \in N_\circ = N \cup \{0\}, N = \{1, 2, \ldots\}; z \in U)$$

where $(a)_n$ is the pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & n = 0 \\ a(a+1)\ldots(a+n-1) & n \in N. \end{cases}$$

Corresponding to the function $h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ defined by

$$h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = z\ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z),$$

which is defined by following Hadamard product (or convolution) for $\varphi(z)$ in the form:

$$H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)\varphi(z) = h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * \varphi(z),$$

or

$$H(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)Q(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1)\emptyset_k z^k,$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_s)_{k-1}}\frac{1}{(1)_{k-1}}, (k \geq 2), \qquad (1.4)$$

If, for convenience, we write

$$H_{q,s}[\alpha_1] = H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s). \qquad (1.5)$$

The linear operator $H_{q,s}[\alpha_1]$ (see [3]), was introduced and studied by Dziok and Srivastava [5].

Al-kharsani and Al-khal [2] defined the modified Dziok-Srivastava operator of the harmonic function $f = h + \overline{g}$, where h and g given by (1.3) as follows:

$$H_{q,s}[\alpha_1]f(z) = H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}. \qquad (1.6)$$

Also let $G_{\overline{H}}$ denote the subclass of G_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form:

$$h(z) = z - \sum_{k=2}^{\infty}|a_k|z^k, g(z) = -\sum_{k=1}^{\infty}|b_k|\overline{z^k}, |b_1| < 1, \qquad (1.7)$$

By using the modified Dziok-Srivastava operator $H_{q,s}[\alpha_1]$ defined by (1.6), Al-khal [1] introduced and studied the class $G_H(\alpha_1,\lambda,\delta,\beta)$, consisting of functions $f = h + \overline{g}$ such that h and g are given by (1.3) and f satisfies the condition:

$$Re\left\{\frac{z(H_{q,s}[\alpha_1]f(z)'' + \lambda(H_{q,s}[\alpha_1]f(z))'}{(H_{q,s}[\alpha_1]f(z))' + \delta z(H_{q,s}[\alpha_1]f(z))''}\right\} \geq \beta, \qquad (1.8)$$

where $0 \leq \lambda \leq 1, 0 \leq \delta \leq 1, 0 \leq \beta < 1$ *and* $\alpha \in R$.

Let $G_{\overline{H}}(\alpha_1,\lambda,\delta,\beta)$ be the subclass of $G_H(\alpha_1,\lambda,\delta,\beta)$,

where

$G_{\overline{H}}(\alpha_1,\lambda,\delta,\beta) = G_{\overline{H}} \cap G_H(\alpha_1,\lambda,\delta,\beta)$.

First, we obtain a sufficient bound for harmonic functions in the class $G_H(\alpha_1,\lambda,\delta,\beta)$.

2- Coefficient bounds:

In the following theorem, we find a coefficient bounds for functions in the class $G_H(\alpha_1,\lambda,\delta,\beta)$.

Theorem (1): Let $f = h + \overline{g}$ be given (1.3).

If

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\Gamma_k(\alpha_1)|b_k| \leq 1, \qquad (2.1)$$

where $0 \leq \delta \leq 1, 0 \leq \beta < 1$ and $\alpha \in R$.

Then f is harmonic univalent sense preserving in U and $f \in G_H(\alpha_1, \lambda, \delta, \beta)$.

Proof: Suppose that the inequality (2.1) holds true and $|z| = r < 1.$

Using the fact that $Re(w) \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$|A(z) + (1-\beta)\beta(z)| - |A(z) - (1+\beta)\beta(z)| \geq 0$$

where

$A(z) = z(H_{q,s}[\alpha_1]f(z)'' + \lambda(H_{q,s}[\alpha_1]f(z))',$

and

$B(z) = (H_{q,s}[\alpha_1]f(z))' + \delta z(H_{q,s}[\alpha_1]f(z))''.$

Substituting for A(z) and B(z) in above inequality, we get

$|A(z) + (1-\beta)\beta(z)| - |A(z) - (1+\beta)\beta(z)| =$

$$\Bigg|(\lambda-\beta+1) - \sum_{k=2}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)a_k z^{k-1}$$

$$-\sum_{k=1}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)b_k z^{-k-1}\Bigg|$$

$$-\Bigg|(\lambda-\beta-1) - \sum_{k=2}^{\infty} k[k+\lambda-1-(1+\beta)(1+\delta(k-1))]\,\Gamma_k(\alpha_1)a_k z^{k-1}$$

$$-\sum_{k=1}^{\infty} k[k+\lambda-1-(1+\beta)(1+\delta(k-1))]\,\Gamma_k(\alpha_1)b_k z^{-k-1}\Bigg|$$

$$\leq (\lambda-\beta+1) - \sum_{k=2}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)|a_k||z|^{k-1}$$

$$-\sum_{k=1}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)|b_k||z|^{-k-1}$$

$$-(\lambda-\beta-1) + \sum_{k=2}^{\infty} k\big[k+\lambda-1-(1+\beta)\big(1+\delta(k-1)\big)\big]\,\Gamma_k(\alpha_1)|a_k||z|^{k-1}$$

$$+\sum_{k=1}^{\infty} k[k+\lambda-1-(1+\beta)(1+\delta(k-1))]\,\Gamma_k(\alpha_1)|b_k||z|^{-k-1}$$

$$= 2 - 2\sum_{k=2}^{\infty} k[\delta k-\delta+1]\,\Gamma_k(\alpha_1)|a_k| - 2\sum_{k=1}^{\infty} k[\delta k-\delta+1]\,\Gamma_k(\alpha_1)|b_k| \geq 0$$

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|b_k| \le 1,$$

by hypothesis. Then by maximum modulus theorem, we have $f \in G_H(\alpha_1, \lambda, \delta, \beta)$.

The harmonic function

$$f(z) = z - \sum_{k=2}^{\infty} \frac{x_k}{k[\delta(k-1)+1]} z^k - \sum_{k=1}^{\infty} \frac{y_k}{k[\delta(k-1)+1]} \overline{z^k}, \qquad (2.2)$$

Where

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1,$$

Shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.2) are in the class $G_H(\alpha_1, \lambda, \delta, \beta)$, because

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) \frac{|x_k|}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) \frac{|y_k|}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)}$$

$$= \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$$

Theorem (2): Let $f = h + \overline{g}$ be given by (1.7). Then $f \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$ if and only if

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|b_k| \le 1, \qquad (2.3)$$

where $0 \le \delta \le 1, 0 \le \beta < 1$ *and* $\alpha \in R$.

Proof: Since $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta) \subset G_H(\alpha_1, \lambda, \delta, \beta)$, we only need to prove the "only if " part of theorem. Then by (1.8), we have

$$Re\left\{\frac{z(H_{q,s}[\alpha_1]f(z)'' + \lambda(H_{q,s}[\alpha_1]f(z))'}{(H_{q,s}[\alpha_1]f(z))' + \delta z(H_{q,s}[\alpha_1]f(z))''}\right\} \ge \beta,$$

or, equivalently

$$Re\left\{\frac{(\lambda-\beta+1) - \sum_{k=2}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)|a_k|z^{k-1}}{(\lambda-\beta-1) - \sum_{k=2}^{\infty} k\big[k+\lambda-1-(1+\beta)\big(1+\delta(k-1)\big)\big]\,\Gamma_k(\alpha_1)|a_k|z^{k-1}}\right.$$

$$\left.\frac{-\sum_{k=1}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)|b_k|z^{-k-1}}{-\sum_{k=1}^{\infty} k[k+\lambda-1-(1+\beta)(1+\delta(k-1))]\,\Gamma_k(\alpha_1)|b_k|z^{-k-1}}\right\}\} \ge 0.$$

If we choose z to be real and $z \to 1^-$, we get

$$\frac{(\lambda-\beta+1) - \sum_{k=2}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)|a_k|}{(\lambda-\beta-1) - \sum_{k=2}^{\infty} k\big[k+\lambda-1-(1+\beta)\big(1+\delta(k-1)\big)\big]\,\Gamma_k(\alpha_1)|a_k|}$$

$$\frac{-\sum_{k=1}^{\infty} k[k+\lambda-1+(1-\beta)(\delta k-\delta)+(1-\beta)]\,\Gamma_k(\alpha_1)|b_k|}{-\sum_{k=1}^{\infty} k[k+\lambda-1-(1+\beta)(1+\delta(k-1))]\,\Gamma_k(\alpha_1)|b_k|} \ge 0.$$

This gives (2.3).

3- Extreme Points:

In the following theorem, we obtain extreme points for the class $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

Theorem (3): A function $f(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$ if and only if

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)), \tag{3.1}$$

Where

$$h_1(z) = z, h_k(z) = z - \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} z^k, k = 2,3, \dots and$$

$$g_k(z) = z - \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} z^{-k}, k = 1,2, \dots \ .$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, \quad X_k \geq 0 \ and \ Y_k \geq 0.$$

In particular, the extreme points of $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof: For the function $f(z)$ of the form (3.1), we have

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)) = \sum_{k=1}^{\infty} (X_k + Y_k) \ z - \sum_{k=2}^{\infty} \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} X_k z^k$$

$$- \sum_{k=1}^{\infty} \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} Y_k \overline{z^k}$$

Then

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\Gamma_k(\alpha_1) \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} X_k + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\Gamma_k(\alpha_1) \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} Y_k$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1,$$

and so $f(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

Conversely, suppose that $f(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

Setting

$X_k = k[\delta(k-1)+1]\Gamma_k(\alpha_1)|a_k|, \ k = 2,3, \dots and \ \ Y_k = k[\delta(k-1)+1]\Gamma_k(\alpha_1)|b_k|, k = 1,2, \dots .$

Then note that by Theorem (2),$0 \leq Y_k \leq 1 (k = 1,2, \dots), and \ 0 \leq X_k \leq 1 (k = 2,3, \dots)$ .We define

$$X_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k \right),$$

by Theorem (2), $X_1 \geq 0.$ Therefore, $f(z)$ can be written as

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k - \sum_{k=1}^{\infty} b_k \overline{z^k} \ z = \sum_{k=2}^{\infty} \frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)} X_k z^k$$

$$-\sum_{k=1}^{\infty}\frac{1}{k[\delta(k-1)+1]\Gamma_k(\alpha_1)}Y_k\overline{z^k}$$

$$= z+\sum_{k=2}^{\infty}(h_k(z)-z)X_k \ +\sum_{k=1}^{\infty}(g_k(z)-z)Y_k$$

$$= z\left\{1-\left(\sum_{k=2}^{\infty}X_k \ +\sum_{k=1}^{\infty}Y_k\right)\right\}+\sum_{k=2}^{\infty}h_k(z)\,X_k \ +\sum_{k=1}^{\infty}g_k(z)\,Y_k$$

$$=\sum_{k=2}^{\infty}(X_k\,h_k(z)+Y_k\,g_k(z)).$$

That is the required representation.

4- Closure Theorem:

In next theorem, we get closure theorem for the class $G_{\overline{H}}(\alpha_1,\lambda,\delta,\beta)$.

Theorem (4): Let the function $f_{n,i}$ defined by

$$f_{n,i}(z)=z-\left(\sum_{k=2}^{\infty}|a_{k,i}|z^k \ +\sum_{k=1}^{\infty}|b_{k,i}|\overline{z^k}\right), i=1,2,\dots,l.$$

Be in the class $G_{\overline{H}}(\alpha_1,\lambda,\delta,\beta)$. Then the function $F_n(z)$ defined by

$$F_n(z)=z-\left(\sum_{k=2}^{\infty}|c_k|z^k \ +\sum_{k=1}^{\infty}|e_k|\overline{z^k}\right),$$

is a member of the class $G_{\overline{H}}(\alpha_1,\lambda,\delta,\beta)$, where

$$c_k=\frac{1}{l}\sum_{i=1}^{l}a_{k,i} \quad and \quad e_k=\frac{1}{l}\sum_{i=1}^{l}b_{k,i}.$$

Proof: Since $f_{n,i}\in\ G_{\overline{H}}(\alpha_1,\lambda,\delta,\beta)$, it follows from Theorem (2) that

$$\sum_{k=2}^{\infty}k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|a_{k,i}|+\sum_{k=1}^{\infty}k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|b_{k,i}|\le 1.$$

Hence

$$\sum_{k=2}^{\infty}k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|c_k|+\sum_{k=1}^{\infty}k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|e_k|$$

$$=\frac{1}{l}\sum_{i=1}^{l}\left(\sum_{k=2}^{\infty}k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|a_{k,i}|+\sum_{k=1}^{\infty}k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|b_{k,i}|\right)$$

$$\le\frac{1}{l}\sum_{i=1}^{l}1=1$$

which implies that $F_n(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$. The proof is complete.

5- Convex Combination:

In the following theorem, we get convex combination theorem for the class $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

Theorem (5): The class $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$ is closed under convex combination.

Proof: For $j = 1,2,\ldots,$ suppose that $f_j(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$, where

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k - \sum_{k=1}^{\infty} b_{k,j} \overline{z^k} \tag{5.1}$$

Then, by theorem (2), we have

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) a_{k,j} + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) b_{k,j} \le 1,$$

for $\sum_{j=1}^{\infty} t_j = 1, \quad 0 \le t_j \le 1,$ the convex combination of $f_j(z)$ may be written as:

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} t_j\, a_{k,j} \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} t_j b_{k,j} \right) \overline{z^k} \tag{5.2}$$

Then, by using (5.2), we have

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) \left(\sum_{j=1}^{\infty} t_j\, a_{k,j} \right) + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) (\sum_{j=1}^{\infty} t_j b_{k,j})$$

$$= \sum_{j=1}^{\infty} t_j \left(\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) a_{k,j} + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) b_{k,j} \right)$$

$$\le \sum_{j=1}^{\infty} t_j\,(1) = 1$$

Therefore, $\sum_{j=1}^{\infty} t_j f_j(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

6- Integral Operator:

Let $f = h + \bar{g} \in G_H$ be given by (1.3). Then $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)\,dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)\,dt}, c > -1. \tag{6.1}$$

Theorem (6): Let $f(z) = h(z) + \overline{g(z)} \in G_H$ be given (1.7) and $f(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$, then $F(z)$ defined by (6.1) also belong to $G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

Proof: Let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k - \sum_{k=1}^{\infty} b_k \overline{z^k}\ z$$

be in the class$G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$ then by Theorem (2), we have

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|b_k| \le 1.$$

By definition of $F(z)$, we have

$$F(z) = z - \sum_{k=p+1}^{\infty} \left(\frac{c+1}{c+k} \right) |a_k|\, z^k - \sum_{k=1}^{\infty} \left(\frac{c+1}{c+k} \right) |b_k|\ \overline{z^k}$$

Since $\frac{c+1}{c+k} < 1$, then

$$\sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) \left(\frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1) \left(\frac{c+1}{c+k} |b_k| \right)$$

$$\leq \sum_{k=2}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} k[\delta(k-1)+1]\,\Gamma_k(\alpha_1)|b_k| \leq 1.$$

Thus $F(z) \in G_{\overline{H}}(\alpha_1, \lambda, \delta, \beta)$.

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