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On Operator Theory

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Abstract: *In this essay, bounded linear operators are reviewed. We recollect some general characteristics that have significant propositions.*

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1. Introduction .

The concept of Hilbert Space was first proposed by David Hilbert in his work on quadratic forms with an unlimited number of variables. More specifically, we look at the linear continuous map between Hilbert Spaces. Bounded operators and the "Operator Theory" branch of functional analysis both refer to this.

Recall that $\mathcal{B}(\delta + \varsigma\eta) = \mathcal{B}\delta + \varsigma\mathcal{B}\eta \quad \forall\delta, \eta \in \mho$ and $\in \mathcal{C}$, satisfies the equation for a linear map $\mathcal{B}$ between two complex vector spaces $\mho$ and $\mathcal{G}$. If $\mathcal{B}: \mathcal{X} \to \mathcal{X}$ is a linear map and if there is $r > 0 \ni \|\mathcal{B}\delta\| \le r\|\delta\| \quad \forall\delta \in \mathcal{X}$ then a map $\mathcal{B}$ is a bounded linear operator. $\mathfrak{B}(\mathcal{X})$ stands for the collection of all bounded linear operators on $\mathcal{X}$.

$\forall\, \mathcal{B} \in \mathfrak{B}(\mathcal{X})$, then

$$\|\mathcal{B}\| = \inf\{\|\mathcal{B}\delta\| \le r\|\delta\| \quad \forall\delta \in \mathcal{X}\} = \sup_{\delta\neq 0}\frac{\|\mathcal{B}\delta\|}{\|\delta\|}$$

and refer to it as $\mathcal{B}$'s norm

2. Some of the important types of operators

Definition 2.1. [4] if $\mathcal{B} = \mathcal{B}^*$ ($\mathcal{B}^*$ *is the adjoint of* $\mathcal{B}$) then an operator $\mathcal{B} \in \mathfrak{B}(\mathcal{X})$ is said self adjoint , or $\forall\delta, \eta \in \mathcal{X} \to \langle\delta,\mathcal{B}\eta\rangle = \langle\mathcal{B}\delta,\eta\rangle$.

Definition 2.2. [4] if $M = M^* = M^2$ then an element $M \in \mathfrak{B}(\mathcal{X})$ is an orthogonal projection.

Definition 2.3. [4] if $MM^* = M^*M = 1$ then an element $M \in \mathfrak{B}(\mathcal{X})$ *is* called a unitary operator.

Proposition 2.4. [4] 1) Assume $W \in \mathfrak{B}(\mathcal{X})$ is an isometry . then

i. $\langle W\delta,W\eta\rangle = \langle\delta,\eta\rangle \quad \forall\delta, \eta \in \mathcal{X}$.
ii. $\|W\delta\| = \|\delta\| \quad \forall\delta \in \mathcal{X}$.
iii. $\mathcal{X} \neq \{0\} \to \|W\| = 1$.
iv. $W^*\delta = W^{-1}\delta$ if $\delta \in Ran(W)$ and $W^*\eta = 0$ if $\eta \perp Ran(W)$.

2) $\|V\delta\| = \|V\| \ \forall\delta \in \mathcal{X}$ iff an element $V \in \mathfrak{B}(\mathcal{X})$ be an isometry.

Proposition 2.5. [4] Assume $W \in \mathfrak{B}(\mathcal{X})$ is a partial isometry . then

i. $\forall\delta, \eta \in \mathcal{X} \to VM = V and\ \langle V\delta,V\eta\rangle = \langle M\delta,M\eta\rangle$.
ii. $\|V\| = 1$ if $M \neq 1$.

Definition 2.6. [4] If there is a family $\{F_n\}_{n\in N}$ of finite rank operators $\ni \lim_{n\to\infty}\|H - F_n\| = 0$ then an element $H \in \mathfrak{B}(\mathcal{X})$ is a compact operator.

Proposition 2.7. [4] If $Y(X)$ denotes the set of all compact operators . then

i. $H \in Y(X)$ iff $H^* \in Y(X)$
ii. If $H \in Y(X)$ and $F \in \mathfrak{B}(\mathcal{X}) \to FH\ and\ HF \in Y(X)$.

Definition 2.8. [3] Assume that B is a Banach space with a Schauder basis, if $y \in B$, $y_n\ (Ky)_n \leq 0$ then a p-operator is a bounded linear operator $K: B \to B$with respect to a particular Schauder basis.

Proposition 2.9. [3] on X , if an orthonormal basis $E = \{\varrho_n\}_n$, then the operator QKQ^* be a p-operator to $E' = \{Q\varrho_n\}_n$ iff K be a p-operator to $E = \{\varrho_n\}_n$.

Definition 2.10. [5] On Hilbert space X the function $\delta : X \times X \to \mathcal{C}$ is called a sesquilinear form $\ni\ \forall\, \tau, \sigma, \rho \in X\ and\ \varsigma \in \mathcal{C}$ then

i. $\delta\,(\tau + \sigma, \rho) = \delta\,(\tau, \rho) + \delta\,(\sigma, \rho)$, $\delta\,(\varsigma\,\tau, \rho) = \varsigma\,\delta\,(\tau, \rho)$
ii. $\delta\,(\rho, \tau + \sigma) = \delta\,(\rho, \tau) + \delta\,(\rho, \sigma)$, $\delta\,(\tau, \varsigma\,\rho) = \bar{\varsigma}\,\delta\,(\tau, \rho)$.

Definition 2.11. [5] On Hilbert space X , $\forall$ sesquilinear form δ then

$$\|\delta\| = sup\{|\delta(\tau, \sigma)| \ni \tau, \sigma \in X, \|\tau\| = \|\sigma\| = 1\}.$$

δ is bounded if $\|\delta\| < \infty$.

Theorem 2.12. [5] $K \in \mathfrak{B}\,(\mathcal{X})$ if and only if $\delta_K\,(\tau, \sigma) = \langle K\tau, \sigma \rangle$ where δ_K is a sesquilinear form with $\|K\|$.

Corollary 2.13. [5] $\forall\, \tau \in X$,if $\langle K\tau, \tau \rangle = \langle R\tau, \tau \rangle$ then $K = R$.

Theorem 2.14. [5] let $K, R \in \mathfrak{B}\,(\mathcal{X})$ and $\varsigma \in \mathcal{C}$ then

i. $R^* + K^* = (R + K)^*$
ii. $(\varsigma K)^* = \bar{\varsigma}\, K^*$
iii. $(RK)^* = K^* R^*$
iv. $\|K\| = \|K^*\|$
v. $\|K\|^2 = \|K^* K\|$
vi. $K = K^{**}$.

Proposition 2.15. [5] $\forall\, K \in \mathfrak{B}\,(\mathcal{X})$ then

i. $A(K^*) = \overline{D(K)^{\perp}}$
ii. $A(K)^{\perp} = \overline{D(K^*)}$

Proposition 2.16.[2] $\forall\, \mathrm{M}_1\mathrm{M}_2 \in \mathfrak{B}\,(\mathcal{X})$, $\psi_{\mathrm{M}_1} = \psi_{\mathrm{M}_2}$ iff $\mathrm{M}_1 = \mathrm{M}_2$.

Proposition 2.17. [2] Assume $M \in \mathfrak{B}\,(\mathcal{X})$, then

i. $\forall \tau \in \mathcal{X}, \|M^*\tau\| = \|M\tau\|$ iff M be normal
ii. $\forall \tau \in \mathcal{X}, (M\,\tau, \tau)$ be real iff M be self adjoint.

Definition 2.18. [2] $\forall\, M \in \mathfrak{B}\,(\mathcal{X})$, A subset of $\mathcal{C}$ called $\mathcal{S}\mathfrak{p}M$ is defined by

$(M - \alpha\mathrm{I})$ doesn't have an inverse in $\mathfrak{B}\,(\mathcal{X})$ if and only if $\alpha \in \mathcal{S}\mathfrak{p}M$

Proposition 2.19. [2] Assume the normal $M \in \mathfrak{B}\,(\mathcal{X})$, then

i. $\forall \tau \in \mathcal{X}\ and\ \alpha \in \mathcal{C}$, $M\tau = \alpha\tau \ \to M^*\tau = \bar{\alpha}\,\tau$
ii. If α_1 and $\alpha_2 \in \mathcal{C} \to$ kernel $(M - \alpha_1 I) \perp$ kernel $(M - \alpha_2 I)$.

Corollary 2.20. [2] Assume a normal operator $M \in \mathfrak{B}\,(\mathcal{X})$, then

$$\forall \tau \in \mathcal{X}\ and\ k > 0\,, \|(M - \alpha I)\tau\| > k\|\tau\| \text{ iff } \alpha \notin \mathcal{S}\mathfrak{p}M\,.$$

Corollary 2.21. [2] Assume the normal $M \in \mathfrak{B}\,(\mathcal{X})$, a sequence of unit vectors (τ_m) exists $\ni \|(M - \alpha I)\tau_m\| \to 0\ \ if\ m \to \infty$ iff $\alpha \in \mathcal{S}\mathfrak{p}M$.

Theorem 2.22. [2] Assume the normal $M \in \mathfrak{B}(\mathcal{X})$, then

i. $\mathcal{S}\mathfrak{p}M \subset \mathfrak{K} \Leftrightarrow$ M be self adjoint.
ii. $\mathcal{S}\mathfrak{p}M \subset \mathfrak{K}^{+} \Leftrightarrow M\ is\ not$ negative .

Lemma 2 .23. [1] Considering that there are two Drazin invertible operators M ,D $\in \mathfrak{B}(\mathcal{X})$, then

i. $(M^{*})^{\mathfrak{I}} = \left(M^{\mathfrak{I}}\right)^{*}$.
ii. $(M^{\mathcal{L}})^{\mathfrak{I}} = \left(M^{\mathfrak{I}}\right)^{\mathcal{L}}\ \forall \mathcal{L} = 1,2, \ldots$
iii. $(D^{-1}MD)^{\mathfrak{I}} = D^{-1}M^{\mathfrak{I}}D$.
iv. $DM = MD \rightarrow (DM)^{\mathfrak{I}} = D^{\mathfrak{I}}M^{\mathfrak{I}} = M^{\mathfrak{I}}D^{\mathfrak{I}}, D^{\mathfrak{I}}M = MD^{\mathfrak{I}}\ and\ M^{\mathfrak{I}}D = DM^{\mathfrak{I}}$.
v. $DM = MD = 0 \rightarrow (D+M)^{\mathfrak{I}} = D^{\mathfrak{I}} + M^{\mathfrak{I}}$.

Definition 2.24. [1] Suppose $M \in \mathfrak{B}(\mathcal{X})$ is Drazin invertible. M is known as the $\mathfrak{I}$ -operator if $M^{*2}\left(M^{\mathfrak{I}}\right)^{2} = \left(M^{*}M^{\mathfrak{I}}\right)^{2}$. $[\mathfrak{I}]$ designates a group that includes all $\mathfrak{I}$ -operators.

Proposition 2.25. [1] Suppose $M, D \in [\mathfrak{I}]$,if $[M, D] = [M, D^{*}] = 0 \rightarrow DM \in [\mathfrak{I}]$.

Proposition 2.26. [1] Suppose $M, D \in [\mathfrak{I}]$,if $DM = MD = 0 \rightarrow D + M \in [\mathfrak{I}]$.

Corollary 2.27. [1] $\forall\, m \geq 0\, , M^{m} \in [\mathfrak{I}]\ if\ M \in [\mathfrak{I}]$.

Theorem 2.28. [1] Suppose $M_1, M_2, \ldots, M_m \in [\mathfrak{I}]$, then

i. $M_1 + M_2 + \cdots + M_m \in [\mathfrak{I}]$.
ii. $M_1 \times M_2 \times \ldots \times M_m \in [\mathfrak{I}]$.

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