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Approximating Functions in Different Spaces

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Abstract: *Many researchers have presented studies on approximating functions in various spaces according to the nature of the function used and to obtain the best approximation. Here we will mention a brief overview of the most famous spaces used in approximating functions.*

Keywords. Approximation, L_p space, sobolev space, Banach space,

Introduction.

Functional analysis is an important branch of mathematics concerned with the study of functions and their properties. It provides a study on the properties of functions such as continuity, integration, regression, etc., and also uses mathematical tools such as integrals, derivatives, etc. to analyze functions and find accurate representations for them. The importance of rounding functions lies for several reasons, including:

It facilitates the calculation process and dealing with complex functions. It is also used to estimate the values of functions in regions far from known points, and to calculate derivatives and integrals easier and faster. In addition, it is used to analyze the behavior of functions at edges, corners, and irregular points**.** many researchers have presented studies on approximating functions in different spaces such as **[1],[2],[3],[4],[5],[6],[7],[8],[9],[10].**

- **Approximation of functions in L^p_space**

  The first thing that is important to know is the definition of L^p-space:

  Definition .1. [1] Let (X, A, μ) be a measure space and $1 \leq p < \infty$. and let $f: X \rightarrow R$ so we can definite The space $L^p(x)$.

$$\int |f|^p d\mu < \infty,$$

  And the $L^p -$ norm of $f \in L^p(x)$ is defined by

$$\|f\|_{L^p} = (|f|^p d\mu)^{\frac{1}{p}}.$$

***L^p* spaces come in various types, including:**

1- L^1: includes all absolutely integrable functions.

2- L^2: includes all square-integrable functions.

3- L^∞: includes all bounded functions.

4- L^p: includes all functions that are integrable to the power of p and absolute.

5- $L^p(\Omega)$: includes all functions that are integrable to the power of p and absolute in a specified region Ω**.**

6- $L^p(\Omega, \mu)$: includes all functions that are integrable to the power of p and absolute in a specified region Ω, in addition to the presence of measure μ.

Many researchers have presented studies on approximating functions in L^p-space, so we will review one of those papers, which is a paper entitled(An Approximation Theorem For Continuous Functions on $L^p(1 \leq p < \infty)$spaces Including Representation)**[2]** For the researcher N.U. AHMED, The researcher presented an approximation theory similar in content to Stone-Weierstrass theorem for

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continuous functions defined $L^p(1 \leq p < \infty)$spaces, which produces useful results for approximating physical systems whose outputs depend on their input data. Then he represented the approximation of continuous functions over $L^p(1 \leq p < \infty)$spaces by using the Volterra functional series, which gave results. Similar results results have been reported in Porter, Clark, and Desantis (1974) and Desantis and Porter (1973) for C(X, R), where X is a compact subset of a Hitbert space.

- **Approximation of functions in sobolev space**

Sobolev spaces were used to analyze and approximate functions containing partial derivatives. This possibility has expanded the analytical and approximate capabilities of mathematicians and scientists in different fields.

Definition .2.[8] let Ω be an open subset of R^n. The sobolev space $W^{r,m}(\Omega)$ and $u \in L_p(\Omega)$ such that for every multi-index α with $|\alpha| \leq m$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L_p(\Omega)$. thus

$$W^{r,m}(\Omega) = \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega), |\alpha| \leq m\}.$$

If $u \in W^{r,m}(\Omega)$ and the $W^{r,m}(\Omega) -$ norm of is defined by

$$\|u\|_{W^{r,m}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p dx\right)^{\frac{1}{p}},\ 1 \leq p \leq \infty.$$

Sobolev spaces come in various types, including:

1- Sobolev space: a space that contains all continuous and differentiable functions up to order n in a specific domain.

2- Sobolev-Hölder space: a space that contains all continuous and differentiable functions up to order n in a specific domain, but is equipped in a special way to be a Banach space.

3- Sobolev-Fischer space: a space that contains all continuous and differentiable functions up to order n in a specific domain, but is equipped in a special way to be a Hilbert space.

The Sobolve space was used to approximate functions in an accurate manner, as it can be used to identify functions that belong to the specified space and that can be perfectly and accurately approximated using multiple functions. Sobolve space also allows calculation of high differentials of functions with high precision, which makes it useful in many mathematical and scientific applications.

- **Approximation of functions in Banach space**

In 1910, German engineer Friedrich Levy developed the idea of smooth spaces, which represent areas of continuous functions that can be approximated in linear ways. Then, in 1920, the Polish engineer Stefan Banach developed the idea of Banach spaces, which are considered a generalization of the idea of smooth spaces and include spaces of functions that can be approximated in nonlinear ways. In 1930, the American mathematician Norman Schwartz developed the idea of coupled set spaces (Scchauder Spaces). Since then, our understanding of linear and nonlinear approximation of functions has continued to evolve, and new ideas have developed about Banach spaces and other mathematical spaces related to approximation and functional analysis. This development has contributed to improving our understanding of many different areas of mathematics and physics.

To define the Banach space, we need to know the norm

Definition .3.[2],[9] $f \in \{R, \mathbb{C}\}$, suppose that X be f - vector space. a map $\|.\|: x \to [0, \infty)$ is called norm if

a. $\|x\| = 0 \leftrightarrow x = 0$
b. $\|\lambda x\| = |\lambda| \|x\|\ for\ all\ \lambda \in f, x \in X$
c. $\|x + y\| \leq \|x\| + \|y\|\ for\ all\ x, y \in X$
Then $(X, \|.\|)$ Is called a normed space

Then we can define the Banach space if $(X, d_{\|.\|})$ is a complete matric space then the normed $(X, \|.\|)$ is called a banach space.

banach spaces come in various types, including:

There are many types of Banach spaces used to approximate functions, including:

1. Specific Banach spaces: These are spaces in which a specific type of convergence can be identified, such as strong convergence or uniform convergence.

2. Banach metric spaces: These are the spaces in which the distance between points can be determined, and they are used to approximate functions by metric approximation.

3. Indefinite Banach spaces: These are spaces in which a specific type of convergence is not specified, and are used to approximate functions in different ways.

4. Homogeneous Banach spaces: These are spaces that contain mathematical similarity between all points in them, and are used to approximate functions in special ways.

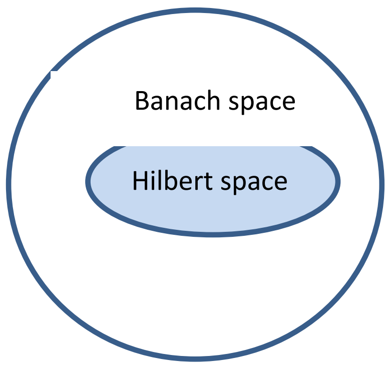

Figure.1:the Banach space And Hilbert space

Also, the Jacobian free form is a type of indefinite Banach space, which is used to approximate functions in a way different from the metric or strong approximation. Jacobian free form is used in data analysis and artificial intelligence, and is based on transforming data into a high-dimensional space. The Jacobian free form is determined by a matrix that is randomly determined, which means that there is no restriction on the shape or size of the matrix. This matrix is used to transform the data into a new space, and then indeterminate approximation techniques are applied to analyze the data.

Recently, deep neural networks have been used with great success to approximate functions. These deep neural spaces rely on the formation of multiple layers of neural units to approximate functions more accurately. Deep neural networks are a powerful and widespread method for approximating functions in computational science, mathematics, and many other fields.

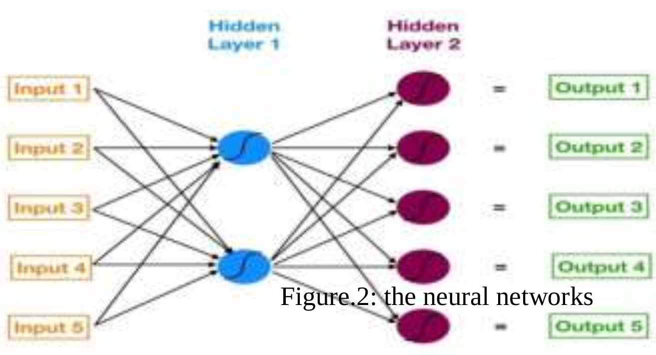

Figure.2: the neural networks

It is difficult to single out one specific type of space as the best space for approximating functions in the easiest and fastest way, as this depends on the nature of the function and the intended use of the approximation. It is possible to use different spaces, such as Hilbert spaces, Banach spaces, etc., depending on the specific case and the purpose of the approximation. Therefore, the appropriate space must be chosen based on the specific situation and desired purpose.

Conclusion

In conclusion, the choice of space for approximating functions depends on the specific requirements and goals of the application. There are many different types of spaces available, including Banach spaces, Hilbert spaces, and Jacobian free form spaces. Each space has its own advantages and disadvantages, and the appropriate space must be chosen based on the specific situation and desired purpose. The development of new spaces and techniques for approximating functions is an ongoing field of research, and it is likely that new and improved methods will continue to be developed in the future.

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