International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 12, December - 2023, Pages: 99-109

Truncated Exponential Fréchet distribution: Properties and Applications

Prof. Awad Kadim Shaalan AL-Khalidi 1 and Karam Nasser Hussein Al-Abadi 2

1University of Warit AL-Anbiyaa College
Email:bawade81@gmail.com 2
University of Warit AL-Anbiyaa College
Email: karam.n@s.uokerbala.edu.iq

Abstract: *In this research we suggest a new compound distribution called [0,1]Truncated Exponential Fréchet (TEF). The new four-parameters distribution is generated by using the A Truncated Exponential. In this method, the probability density function and cumulative distribution function of Fréchet distribution are used as a base distribution for Fréchet Distribution. The probability density function and cumulative distribution function(CDF) of the Fréchet distribution are substituted in the Truncated Exponential model to get the new and more flexible lifetime distribution for modelling real-life data. The authors reveal that the hazard rate of the A Truncated Exponential Fréchet distribution is increasing were found including the Quintile Function, the moments, the moments - generating function, By maximum likelihood method was estimated the parameters of the new distribution. They also found that the Truncated Exponential Fréchet distribution gives a much close fit than the Truncated Exponential Fréchet Distribution are evaluated on competing models viz Fréchet Distribution distribution (FD), Weibull Pareto Distribution (WPD), Weibull distribution(WD), Exponential distribution (ED).In this study, a novel probability distribution is introduced. Truncated Exponential Fréchet distribution is capable of modelling upside-down bathtub shaped hazard rates. The model is appropriate to fit the asymmetrical data that are not correctly fitted by other distributions. The said distribution can be applied to different fields like insurance, earthquake data for analysis, reliability.*

Keywords: Truncated Exponential, Reliability Analysis, hazard function, Moments, Parameter Estimation, Fréchet Distribution, MLE .

1. Introduction

Recently, there has been an increased interest among statisticians to progress new extended distributions to be more capable for exhibiting data in different areas such as lifetime investigation, engineering, economics, finance, demography, actuarial, living, and medical sciences. The Fréchet distribution is an imperative distribution developed within the extreme value theory. It has applications in life testing, floods, horse racing, precipitation, queues in supermarkets, sea whitecaps, and wind speeds. Further evidence about the Fréchet distribution and its applications can be explored in Kotz and Nadarajah (2000) , Aiming a more flexible Fréchet distribution, for many years geometricians have been developing various extensions and modified forms of the Fréchet distribution, with different number of parameters. For specimen, the exponentiated Fréchet due to Nadarajah and Kotz (2003), the beta Fréchet due to Nadarajah and Gupta (2004)) and Barreto-Souza, Cordeiro, and Simas (2011)), the transmuted Fréchet due to Mahmoud and Mandouh (2013)), the gamma extended Fréchet due to da Silva et al. (2016)) [15], the Marshall–Olkin Fréchet due to Krishna et al. (2013), the Kumaraswamy Fréchet due to Mead and Abd-Eltawab (2014), the transmuted Marshall–Olkin Fréchet due to Afify et al. (2015), the Kumaraswamy Marshall–Olkin Fréchet due to Afify et al. (2016a), the Kumaraswamy transmuted Marshall–Olkin Fr due to Yousof et al. (2016), the Weibull Fréchet due to Afify et al. (2016b) and the beta exponential Fréchet due to Mead et al. (2017). The probability density function (PDF) and cumulative distribution function (CDF) of the Fréchet distribution are given by (for $x \geq 0$)

$$f(x,k,\gamma)=\frac{\gamma}{k}\left(\frac{k}{x}\right)^{\gamma+1} e^{-\left(\frac{k}{x}\right)^{\gamma}} \quad ; x,k,\gamma>0 \qquad (1)$$

$$F(x,k,\gamma)=e^{-\left(\frac{k}{x}\right)^{\gamma}} \quad ; x,k,\gamma>0 \qquad (2)$$

respectively, where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter.

2. Truncated Exponential Fréchet distribution:

To find the [0,1] Truncated Exponential Fréchet distribution (Ribeiro-Reis, L. D. (2022))we will follow The CDF and PDF of new family are given as:

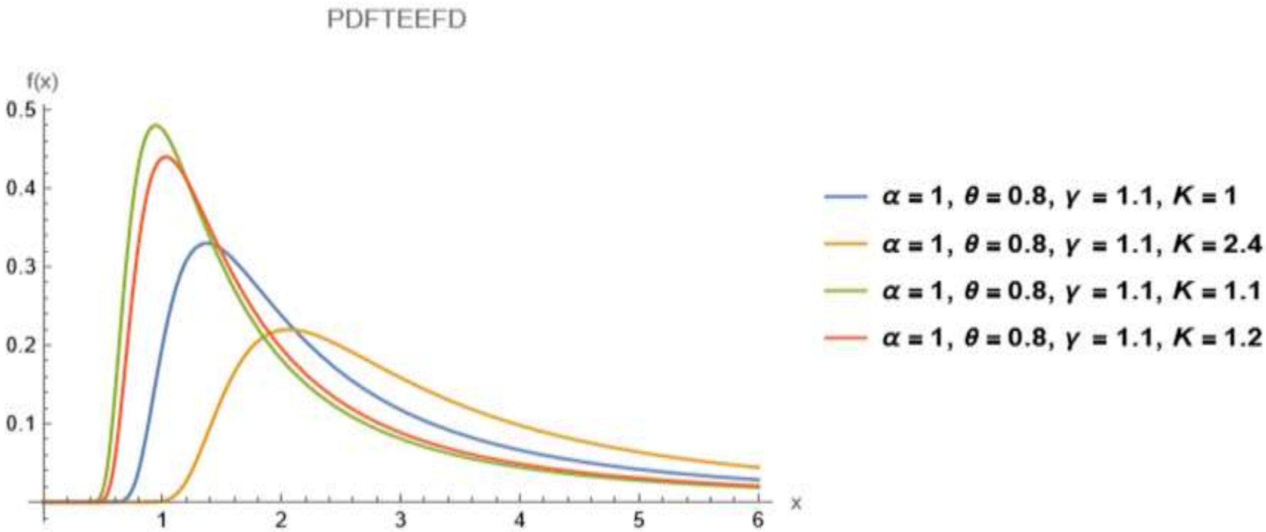
$$F(x)=\frac{1-e^{-\theta(G(X))^{\alpha}}}{1-e^{-\theta}} \qquad (3)$$

$$f(x) = \frac{\alpha\theta g(x) e^{-\theta(G(X))} (G(X))^{\alpha-1}}{1-e^{-\theta}} \qquad (4)$$

And by Substituting the equation (1) in (3) we get the CDF of the new distribution and by deriving the CDF we get the PDF as follows.

$$f(x,\alpha,\theta,\gamma,k) = \frac{\alpha\theta \frac{\gamma}{k}\left(\frac{k}{x}\right)^{\gamma+1} e^{-\left(\frac{k}{x}\right)^{\gamma}} e^{-\theta\left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)} \left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)^{\alpha-1}}{1-e^{-\theta}} \qquad (5)$$

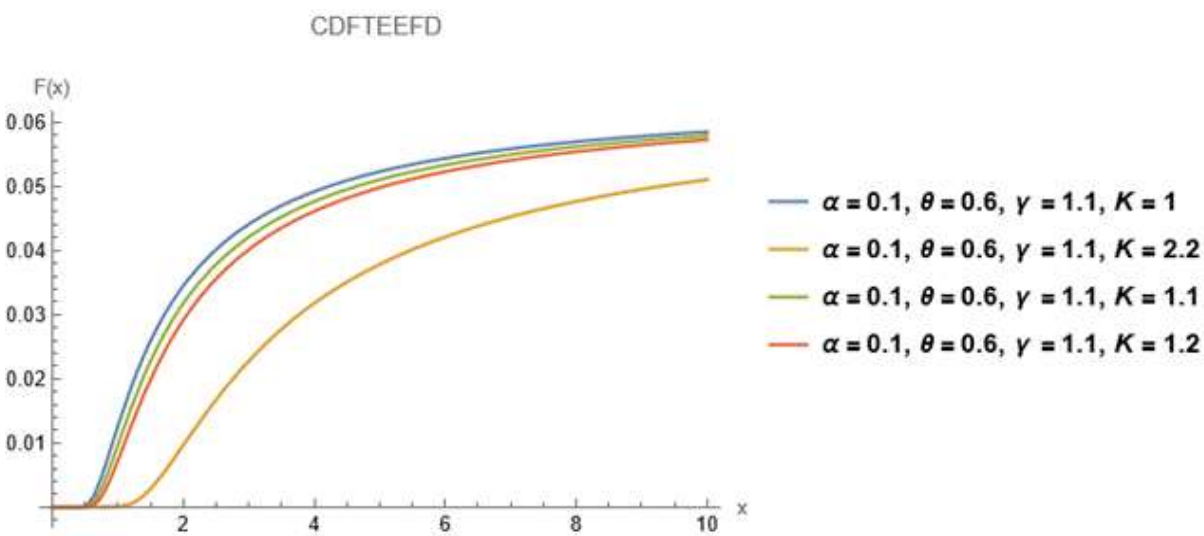
The reasonable shapes of PDF Truncated Exponential Fréchet distribution



The CDF of this distribution is

$$F(x,\theta,\alpha,k,\gamma) = \frac{1-e^{-\theta\left(e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}}\right)}}{1-e^{-\theta}} \qquad (6)$$

The reasonable shapes of CDF Truncated Exponential Fréchet distribution



The Survival function of the new distribution is:

$S(t) = 1 - F(t)$

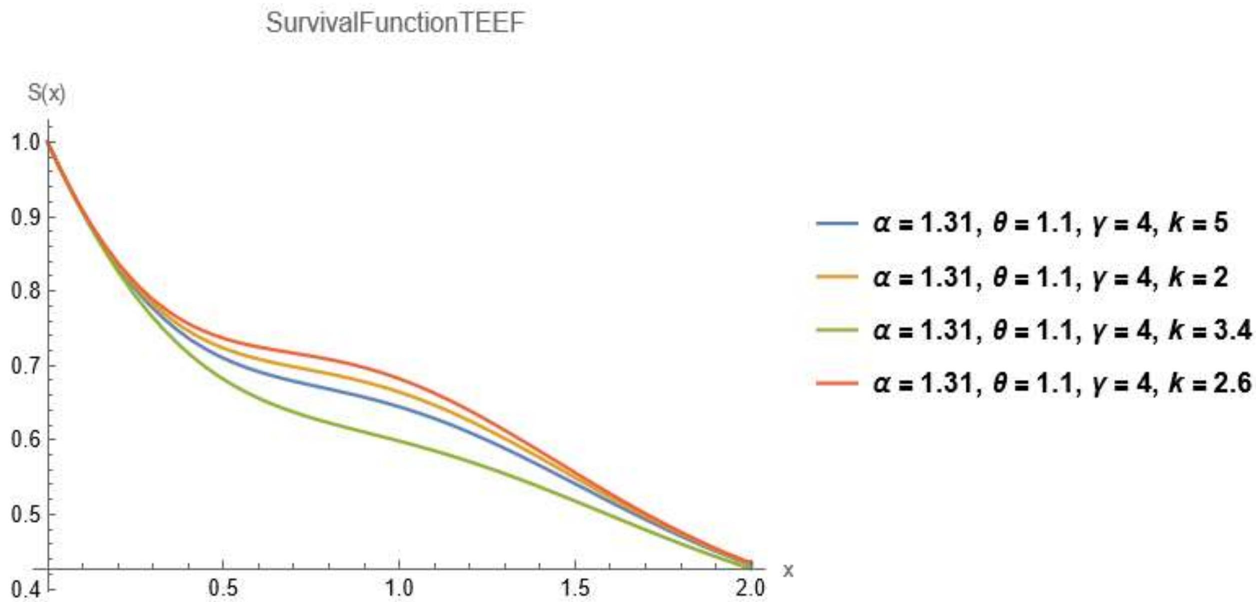
International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 12, December - 2023, Pages: 99-109

$S(t)_{TEF} = 1 - F(t)_{TEF}$

$$S(x,\theta,\alpha,k,\gamma) = 1 - \frac{1-e^{-\theta\left(e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}}\right)}}{1-e^{-\theta}} \quad (7)$$

The reasonable shapes Survival Truncated Exponential Fréchet distribution:

The hazard function of the new distribution is:

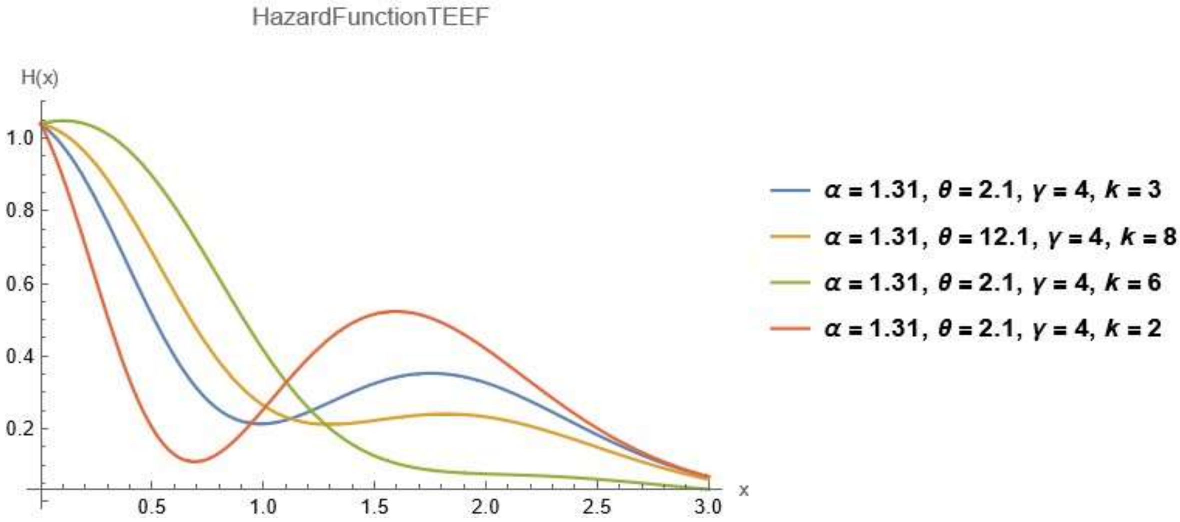


$$h(x,\theta,\alpha,k,\gamma)_{TEF} = \frac{f(x,\theta,\alpha,k,\gamma)_{TEF}}{S(x,\theta,\alpha,k,\gamma)_{TEF}}$$

For any random variable X which follows Truncated Exponential Fréchet distribution, its hazard function is given as:

$$h(x,\theta,\alpha,k,\gamma)_{TEF} = \left[\frac{\frac{\alpha\theta\frac{\gamma}{k}\left(\frac{k}{x}\right)^{\gamma+1} e^{-\left(\frac{k}{x}\right)^{\gamma}} e^{-\theta\left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)}\left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)^{\alpha-1}}{1-e^{-\theta}}}{1-\frac{1-e^{-\theta\left(e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}}\right)}}{1-e^{-\theta}}}\right] \quad (8)$$

The reasonable shapes hazard function Truncated Exponential Fréchet distribution



3 .Statistical Properties

In this section, some of the properties of the Truncated Exponential Fréchet distribution are discussed:

3.1Quantile function

The quantile function or inverse cumulative distribution function. returns the value t such that:

$$t = Q(u) = F^{-1}(u), 0 < u < 1$$

$$u = \left(\frac{1 - e^{-\theta\left(e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}}\right)}}{1 - e^{-\theta}} \right)$$

$$x = k\left(- \frac{\mathrm{Log}[- \frac{\mathrm{Log}[(1-e^{\theta})(\frac{1}{1-e^{\theta}} - u)]}{\theta}]}{\alpha} \right)^{-1/\lambda} \quad (9)$$

3.2Moments

Let x denote the random variable follows Truncated Exponential Fréchet distribution then r^{th} order moment about origin of μ_r is:

$$E(x^r) = \mu'_r = \int_0^{\infty} x^r f(x, \theta, \alpha, k, \gamma) \, . dx \quad (10)$$

$$E(X^r) = \mu_r = \int_0^{\infty} x^r \frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1 - e^{-\theta}} \left[x^{-\gamma-1} e^{-\theta e^{-\left(\frac{k}{x}\right)^{\gamma}}} e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}} \right] . dx$$

$$= \frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1 - e^{-\theta}} \int_0^{\infty} x^r x^{-\gamma-1} e^{-\theta e^{-\left(\frac{k}{x}\right)^{\gamma}}} e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}} . dx$$

$$= \frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1 - e^{-\theta}} \int_0^{\infty} x^{-\gamma-1+r} e^{-\theta e^{-\left(\frac{k}{x}\right)^{\gamma}}} e^{-\alpha\left(\frac{k}{x}\right)^{\gamma}} . dx$$

$$=\frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\int_0^\infty x^{-\gamma-1+r}e^{-\theta e^{-\left(\frac{k}{x}\right)^\gamma}}e^{-\alpha\left(\frac{k}{x}\right)^\gamma}.dx$$

$$z=\left(\frac{k}{x}\right),\ k=zx, x=\frac{k}{z}, dx=\frac{-k}{z^2}dz$$

$$=\frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\int_0^\infty \left(\frac{k}{z}\right)^{-\gamma-1+r} e^{-\theta e^{-z^\gamma}}e^{-\alpha z^\gamma}.\frac{-k}{z^2}\ dz$$

$$=\frac{k^{-\gamma-1+r}k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\int_0^\infty (z^{-1})^{-\gamma-1+r}e^{-\theta e^{-z^\gamma}}e^{-\alpha z^\gamma}.\frac{-k}{z^2dz}$$

$$=\frac{-k^r\alpha\theta\gamma}{1-e^{-\theta}}\int_0^\infty (z^{\gamma+1-r})\ e^{-\theta e^{-z^\gamma}}e^{-\alpha z^\gamma}z^{-2}.dz$$

$$\text{Let } e^{-\alpha e^{x^\beta}}=\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\alpha]^n n^m}{n!\,m!}x^{m\beta}$$

$$\text{Let}^{-\theta e^{-z^\gamma}}=\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}z^{m\gamma}$$

$$=\frac{-k^r\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\int_0^\infty (z^{\gamma+1-r})z^{m\gamma}\ e^{-\alpha z^\gamma}z^{-2}.dz$$

$$=\frac{-k^r\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\int_0^\infty z^{\gamma-1-r+m\gamma}\ e^{-\alpha z^\gamma}.dz$$

$$E(x^r)=\mu'_r=\frac{-k^r\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{r}{\lambda}}\Gamma[1+m-\frac{r}{\lambda}]}{\lambda}\right);r=1,2,3\ldots n \quad (11)$$

Where r=1

$$E(x^1)=\mu'_1=\frac{-k^1\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{1}{\lambda}}\Gamma[1+m-\frac{1}{\lambda}]}{\lambda}\right)$$

Where r=2

$$E(x^2)=\mu'_2=\frac{-k^2\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{2}{\lambda}}\Gamma[1+m-\frac{2}{\lambda}]}{\lambda}\right)$$

Where r=3

$$E(x^3)=\mu'_3=\frac{-k^3\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{3}{\lambda}}\Gamma[1+m-\frac{3}{\lambda}]}{\lambda}\right)$$

Where r=4

$$E(x^4)=\mu'_4=\frac{-k^4\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{4}{\lambda}}\Gamma[1+m-\frac{4}{\lambda}]}{\lambda}\right)$$

3.3moments about the mean

Let x denote the random variable follows Truncated Exponential Fréchet Distribution then moments about the mean order moment about origin of μ_r TEFD is:

$$\mathrm{E}(x-\mu)^r=\int_0^\infty (x-\mu)^r\ f(x).dx \qquad (12)$$

$$\mathrm{E}(x-\mu)^r=\int_0^\infty (x-\mu)^r \frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\left[x^{-\gamma-1}e^{-\theta e^{-\left(\frac{k}{x}\right)^\gamma}}e^{-\alpha\left(\frac{k}{x}\right)^\gamma}\right].dx$$

$$=\frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\int_0^\infty x^j x^{-\gamma-1}e^{-\theta e^{-\left(\frac{k}{x}\right)^\gamma}}e^{-\alpha\left(\frac{k}{x}\right)^\gamma}.dx$$

$$=\frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\int_0^\infty x^{-\gamma-1+j}e^{-\theta e^{-\left(\frac{k}{x}\right)^\gamma}}e^{-\alpha\left(\frac{k}{x}\right)^\gamma}.dx$$

$$=\frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\int_0^\infty x^{-\gamma-1+j}e^{-\theta e^{-\left(\frac{k}{x}\right)^\gamma}}e^{-\alpha\left(\frac{k}{x}\right)^\gamma}.dx$$

$$z=\left(\frac{k}{x}\right),\ k=zx, x=\frac{k}{z}, dx=\frac{-k}{z^2}dz$$

$$=\frac{k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\int_0^\infty \left(\frac{k}{z}\right)^{-\gamma-1+j}e^{-\theta e^{-z^\gamma}}e^{-\alpha z^\gamma}.\frac{-k}{z^2}\ dz$$

$$=\frac{k^{-\gamma-1+j}k^{\gamma+1}\alpha\theta\frac{\gamma}{k}}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\int_0^\infty (z^{-1})^{-\gamma-1+j}e^{-\theta e^{-z^\gamma}}e^{-\alpha z^\gamma}.\frac{-k}{z^2dz}$$

$$=\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\int_0^\infty (z^{\gamma+1-j})\ e^{-\theta e^{-z^\gamma}}e^{-\alpha z^\gamma}z^{-2}.dz$$

$$e^{-\alpha e^{x^\beta}}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\alpha]^n n^m}{n!\,m!}x^{m\beta}$$

$$e^{-\theta e^{-z^\gamma}}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}z^{m\gamma}$$

$$=\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\int_0^\infty (z^{\gamma+1-j})z^{m\gamma}\ e^{-\alpha z^\gamma}z^{-2}.dz$$

$$=\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\int_0^\infty z^{\gamma-1-j+m\gamma}\ e^{-\alpha z^\gamma}.dz$$

$$\mathrm{E}(x-\mu)^r=\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{r}\binom{r}{j}(\mu)^{r-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right); r=1,2,3\ldots n \qquad (13)$$

Now we obtain the first four moments of the Truncated Exponential Fréchet Distribution by putting r = 2,3,4... n in Equation (13) as

Where r=2

$$\sigma^2=\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{2}\binom{2}{j}(\mu)^{2-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right) \qquad (14)$$

$$\sigma = \sqrt{\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{2}\binom{2}{j}(\mu)^{2-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right)} \quad (15)$$

Where r=3

$$E(x-\mu)^3 = \frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{3}\binom{3}{j}(\mu)^{3-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right)$$

Where r=4

$$E(x-\mu)^4 = \frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{4}\binom{4}{j}(\mu)^{4-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!\,m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right)$$

3.4 Coefficient of Variation

The Coefficient of Variation for Truncated Exponential Fréchet Distribution is given by:

$$C\cdot V = \frac{\sigma}{\mu'_1}\times 100\%$$

$$C\cdot V = \frac{\sqrt{\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{2}\binom{2}{j}(\mu)^{2-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right)}}{\frac{-k^1\alpha\theta\gamma}{1-e^{-\theta}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{1}{\lambda}}\Gamma[1+m-\frac{1}{\lambda}]}{\lambda}\right)} \quad (16)$$

3.5 Coefficient of Skewness

Coefficient of Skewness for Truncated Exponential Fréchet Distribution is given by:

$$S.K = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$

$$S.K = \frac{\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{3}\binom{3}{j}(\mu)^{3-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right)}{\left(\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{2}\binom{2}{j}(\mu)^{2-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma[1+m-\frac{j}{\lambda}]}{\lambda}\right)\right)^{\frac{3}{2}}} \quad (17)$$

3.6 Coefficient of Kurtosis

The Coefficient of Kurtosis of for Truncated Exponential Fréchet Distribution is given by:

$$C.K = \frac{E(t-\mu)^4}{\sigma^4}$$

$$C.K = \frac{\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{4}\binom{4}{j}(\mu)^{4-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma\left[1+m-\frac{j}{\lambda}\right]}{\lambda}\right)}{\left[\frac{-k^j\alpha\theta\gamma}{1-e^{-\theta}}\sum_{j=0}^{2}\binom{2}{j}(\mu)^{2-j}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n[\theta]^n n^m}{n!m!}\left(\frac{\alpha^{-1-m+\frac{j}{\lambda}}\Gamma\left[1+m-\frac{j}{\lambda}\right]}{\lambda}\right)\right]^2} \quad (18)$$

4.Moment Generating Function

Let X be random variable follows Truncated Exponential Fréchet Distribution, then the moment generating function (mfg) of x is obtained as:

$$M_X(x) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x, \theta, \alpha, k, \gamma)\,.\,dx \tag{19}$$

$$M_X(t) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots + \frac{(tx)^r}{r!}\right) f(x, \theta, \alpha, k, \gamma).\,dx$$

$$M_X(t) = \int_0^{\infty} \frac{t^r}{r!} x^r f(x, \theta, \alpha, k, \gamma).\,dx$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{-k^r \alpha\theta\gamma}{1 - e^{-\theta}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n [\theta]^n n^m}{n!\,m!} \left(\frac{\alpha^{-1-m+\frac{r}{\lambda}} \Gamma\left[1 + m - \frac{r}{\lambda}\right]}{\lambda}\right) \tag{20}$$

Similarly, the characteristic function of Truncated Exponential Fréchet Distribution, can be obtained as:

$$M_X(ti) = \sum_{r=0}^{\infty} \frac{ti^r}{r!} \frac{-k^r \alpha\theta\gamma}{1 - e^{-\theta}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n [\theta]^n n^m}{n!\,m!} \left(\frac{\alpha^{-1-m+\frac{r}{\lambda}} \Gamma\left[1 + m - \frac{r}{\lambda}\right]}{\lambda}\right) \tag{21}$$

5.Parameter estimation

Let x1 ,x2,x3, x4,........xn be a random sample of size n from Truncated Exponential Fréchet Distribution.

The likelihood function, L of Truncated Exponential Fréchet Distribution is given by:

$$Lf(x, \theta, \alpha, k, \gamma)_{TEFD} = \prod_{i=1}^{n} f(x, \theta, \alpha, k, \gamma)_{TEFD} \tag{22}$$

$$Lf(x, \theta, \alpha, k, \gamma)_{TEFD} = \prod_{i=1}^{n} \left(\frac{\alpha\theta \frac{\gamma}{k} \left(\frac{k}{x}\right)^{\gamma+1} e^{-\left(\frac{k}{x}\right)^{\gamma}} e^{-\theta\left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)} \left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)^{\alpha-1}}{1 - e^{-\theta}}\right)$$

$$= \left(\frac{\gamma\alpha\theta}{k}\right)^n \prod_{i=1}^{n} \left(\frac{\left(\frac{k}{x}\right)^{\gamma+1} e^{-\left(\frac{k}{x}\right)^{\gamma}} e^{-\theta\left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)} \left(e^{-\left(\frac{k}{x}\right)^{\gamma}}\right)^{\alpha-1}}{1 - e^{-\theta}}\right) \tag{23}$$

The log-likelihood function for the vector of parameters can be written as,

$$LogLf(x, \theta, \alpha, k, \gamma)_{TEFD} = n\log\left(\frac{\gamma\alpha\theta}{k}\right) + \sum_{i=1}^{n} \log\left(\frac{k}{x}\right)^{\gamma+1} - \theta \sum_{i=1}^{n} e^{-\left(\frac{k}{x}\right)^{\gamma}} - \sum_{i=1}^{n} \left(\frac{k}{x}\right)^{\gamma} - (\alpha - 1) \sum_{i=1}^{n} \left(\frac{k}{x}\right)^{\gamma} - \sum_{i=1}^{n} \log\left(1 - e^{-\theta}\right) \tag{24}$$

By taking the first partial derivatives of the log-likelihood function with respect to the four parameters $(\lambda, \mathcal{S}, \alpha, \theta)$ as follows

$$\frac{f(x, \theta, \alpha, k, \gamma)_{TEFD}}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} e^{-\left(\frac{k}{x}\right)^{\gamma}} \tag{25}$$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 12, December - 2023, Pages: 99-109

$$\frac{f(x,\theta,\alpha,k,\gamma)_{TEFD}}{d\theta} = -\sum_{i=1}^{n} e^{-(\frac{k}{x})^{\gamma}} n - \frac{e^{\theta}(1-e^{-\theta}(-1+e^{\theta}))n}{-1+e^{\theta}} \quad (26)$$

$$\frac{f(x,\theta,\alpha,k,\gamma)_{TEFD}}{dk} = \frac{n}{x} + \frac{e^{-(\frac{k}{x})^{\gamma}} \sum_{i=1}^{n} (\frac{k}{x})^{-1+\gamma}(-1+\alpha)\gamma}{x} + \frac{\sum_{i=1}^{n} e^{-(\frac{k}{x})^{\gamma}} (\frac{k}{x})^{-1+\gamma}\gamma\theta}{x} - \frac{n}{k} \quad (27)$$

$$\frac{f(x,\theta,\alpha,k,\gamma)_{TEFD}}{d\gamma} = \frac{n}{\gamma} + e^{-(\frac{k}{x})^{\gamma}} \sum_{i=1}^{n} (\frac{k}{x})^{\gamma}(-1+\alpha)\mathrm{Log}[\frac{k}{x}] + e^{-(\frac{k}{x})^{\gamma}} \sum_{i=1}^{n} (\frac{k}{x})^{\gamma}\theta\mathrm{Log}[\frac{k}{x}] \quad (28)$$

he maximum likelihood estimates $(\hat{k},\hat{\gamma},\hat{\theta},\hat{\alpha})$ equations $\frac{dLogL}{d\alpha}=0$, $\frac{dLogL}{d\theta}=0$, $\frac{dLogL}{dk}=0$, $\frac{dLogL}{d\gamma}=0$,The Equation (25),(26) and Equation (27) and (28)cannot be solved as they both are in closed forms. So we compute the parameters of the Transmuted Survival Exponential Pareto Distribution.

6.Application of Truncated Exponential Fréchet Distribution.

The flexibility and performance of Truncated Exponential Fréchet Distribution are evaluated on competing models viz Fréchet Distribution distribution (FD), Weibull Pareto Distribution (WPD), Weibull distribution(WD), Exponential distribution (ED) .Here, the distribution is fitted to data set for the number of hours the patients people with virus (covid-19) were in hospital before death for AL

Hussein Educational Hospital in Karbala, for sample size (n=64) (see table 1.), the performance of the distribution was compared with Fréchet Distribution distribution (FD), Weibull Pareto Distribution (WPD), Weibull distribution(WD), Exponential distribution (ED) .for the data set using Akaike Information Criterion (AIC), Akaike Bayesian Criterion Corrected (BIC). Information Criterion (AIC),Distribution with the lowest AIC, AICC considered the most flexible and superior distribution for a given data. The results are presented in the tables (2).

TABLEI. Data set for the number of hours patients were in hospital before death

0.7, 0.24 , 0.38 , 0.18 , 0.37 , 0.39 , 0.12 , 0.911 , 0.58 , 0.07 , 0.35 , 0.03 ,0.63 , 0.31 , 0.20 ,0.14 , 0.56 , 0.72 , 0.90 , 0.35 , 0.12 , 0.48 , 0.72 , 0.49 , 0.16 , 0.38 , 0.39 , 0.12 , 0.12 , 0.41 ,0.98 , 0.58 ,0.85 , 1.39 , 0.02 , 0.39 , 0.29 , 0.10 , 0.26 , 0.41 , 0.10 , 0.07 , 0.07 , 2.05 , 0.04, 0.81 , 0.29 , 0.10 , 0.13 , 0.13 , 0.11 , 0.08 , 0.10 , 0.06 , 0.31 , 0.44 ,1.03 , 0.84 , 0.64 , 0.42, 0.17 , 0.13 , 0.10 , 0.09

To choose the best model within the set of models that was compared with the new distribution, the best is the model corresponding to the lowest value for Akaike Information Criterion (AIC) and Akaike Information Correct (AIC_c) (see tabul 2.) , the general formula for (AIC) ,(AIC_c) and (BIC) are:

$$AIC = -2\log\left(\frac{\hat{\theta}_{MLE}}{x}\right) + 2K \quad (29)$$

Where:
$\log\left(\frac{\hat{\theta}_{MLE}}{x}\right)$: value of the logarithm maximum likelihood function.
K: Estimated number of parameters.
And

$$AIC_c = AIC + \frac{2K(K+1)}{N-K-1} \quad (30)$$

Where
AIC: Akaike Information Criterion.
K: Estimated number of parameters.
N: sample size

$$BIC = -2\log(\hat{\theta}_{MLE}) + K\log(N) \quad (31)$$

Where
BIC: Bayesian Information Criterion.
K: Estimated number of parameters.
N: sample size

TABLE II:ML Estimates and Criterion Values X^2Anderson-D, Cramer- V, AIC , AIC_C, BIC,and Pearson ,and comparison Truncated Exponential Fréchet Distribution ion with Fréchet Distribution distribution (FD), Weibull Pareto Distribution (WPD), Weibull distribution(WD), Exponential distribution (ED).

Distributions	MLE	X^2Anderson-D		Cramer- V		-2log	AIC	AIC_C	BIC
		statistic	P-Value	statistic	P-Value				
Truncated Exponential Fréchet distribution	$\hat{\alpha} = 1.23435$ $\hat{\theta} = 1.173$ $\hat{\gamma} = 3.6417$ $\hat{k} = 7.471$	0.53714	0.4219	0.6437	0.42206	202.77	210.77	211.447	308.524
Fréchet distribution	$\hat{\gamma} = 2.368$ $\hat{k} = 0.4539$	3.344	0.4742	0.28287	0.93456	332.246	336.246	336.442	465.321
Weibull Pareto Distribution	$\hat{\alpha} = 3.7565$ $\hat{\beta} = 2.47345$ $\hat{\theta} = 0.1945$	1.1529	0.0503	0.674356	0.0632	307.532	313.532	313.931	406.424
Weibull distribution	$\hat{\alpha} = 0.643$ $\hat{\theta} = 0.9325$	2.4259	0.03685	1.2691	0.08533	421.154	425.154	425.609	505.340
Exponential	$\hat{\theta} = 0.457$	3.434	0.35128	0.056181	0.7278	534.413	536.413	536.509	452.094

7.Conclusion

In this paper, a novel probability distribution is introduced. The new distribution is a Transmuted Survival Exponential Pareto Distribution,. Some of the properties are derived and discussed like moments, reliability analysis, and hazard rate. The method of maximum likelihood estimation is used for determining the parameters. The performance of the new model is determined by fitting to real-life data using the goodness of fit criteria such as AIC, AICC and BIC. The value of P-Value tests (Cramer -von misses, Anderson-Darling) is greater than the moral level (0.05) and this leads not to reflect the hypothesis of the notice (the appropriateness of the real data for probability distributions under study It is found that Transmuted Survival Exponential Pareto Distribution gives a better fit to the data set as compared Fréchet Distribution (FD), Weibull Pareto Distribution (WPD), Weibull distribution(WD), Exponential distribution (ED) Further, Truncated Exponential Fréchet Distribution can be applied to various areas. Truncated Exponential Fréchet Distribution and distribution may suitable for most of the lifetime data and provides better outcomes than other well-known distribution.

References :

[1].Afify, A. Z., G. G. Hamedani, I. Ghosh, and M. E. Mead. 2015. The transmuted Marshall–Olkin Fréchet distribution: Properties and applications. International Journal of Statistics and Probability 4:132–48.

[2]. Afify, A. Z., H. M. Yousof, G. M. Cordeiro, Z. M. Nofal, and M. Ahmad. 2016a. The Kumaraswamy Marshall–Olkin Fréchet distribution with applications. Journal of ISOSS 2:41–58.

[3].Afify, A. Z., H. M. Yousof, G. M. Cordeiro, E. M. M. Ortega, and Z. M. Nofal. 2016b. The Weibull Fréchet distribution and its applications. Journal of Applied Statistics 43:2608–26.

[4].Al-Kanani, I. H., and S. A. Jasim. 2012. Estimate survival function for brain cancer disease: On complete data by using Weibull distribution. Saarbrücken: LAP LAMBERT Academic Publishing.

[5].Barreto-Souza, W., G. M. Cordeiro, and A. B. Simas. 2011. Some results for beta Fréchet distribution. Communications in Statistics—Theory and Methods 40:798–811.

[6].da Silva, R. V., T. A. N. de Andrade, D. B. M. Maciel, R. P. S. Campos, and G. M. Cordeiro. 2016. Anew lifetime model: The gamma extended Fréchet distribution. Journal of Statistical Theory and Applications 12:39–54.

[7].Fréchet, M. 1924. Sur la loi des erreurs dIobservation. Bulletin de la Société Mathématique de Moscou 33:5–8. Gupta, R. C., P. L. Gupta, and R. D.

[8].Gupta. 1998. Modeling failure time data by Lehmann alternatives. Communictions in Statistics - Theory and Methods 27:887–904.

[9].Keller, A. Z., and A. R. Kamath. 1982. Reliability analysis of CNC machine tools. Reliability Engineering 3:449–73.

[10].Kotz, S., and S. Nadarajah. 2000. Extreme value distributions: Theory and applications. London: Imperial College Press.

[11].Krishna, E., K. K. Jose, T. Alice, and M. M. Ristic. ´ 2013. The Marshall–Olkin Fréchet distribution. Communications in Statistics—Theory and Methods 42:4091–107.

[12]. Mahmoud, M. R., and R. M. Mandouh. 2013. On the transmuted Fréchet distribution.Journal of Applied Sciences Research 9:5553–61.

[13].Mead, M. E., and A. R. Abd-Eltawab. 2014. A note on Kumaraswamy Fréchet distribution. Australian Journal of Basic and Applied Sciences 8:294–300. Mead, M. E., A. Z. Afify, G. G. Hamedani, and I. Ghosh. 2017. The beta exponential Fréchet distribution with applications. Austrian Journal of Statistics 46:41–63.

[14].Nadarajah, S., and A. K. Gupta. 2004. The beta Fréchet distribution. Far East Journal of Theoretical Statistics 14:15–24.

[15].Nadarajah, S., and S. Kotz. 2003. The exponentiated exponential distribution. Retrieved from http://interstat.stat journals.net/YEAR/2003/abstracts/0312001.php.

[16]. Nofal, Z. M., A. Z. Afify, H. M. Yousof, and G. M. Cordeiro. 2017. The generalized transmuted-G family of distributions. Communications in Statistics—Theory and Methods 46:4119–36.

[17]. Ribeiro-Reis, L. D. (2022). Truncated exponentiated-exponential distribution: A distribution for unit interval. Journal of Statistics and Management Systems, 1-12.

[18].Shaw, W. T., and I. R. C. Buckley. 2007. The alchemy of probability distributions: Beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.

[19].Treyer, V. N. 1964. Doklady Acad, Nauk, Belorus, U.S.S.R.

[20].Yousof, H. M., A. Z. Afify, A. N. Ebraheim, G. G. Hamedani, and N. S. Butt. 2016. On six-parameter Fréchet distribution: Properties and applications. Pakistan Journal of Statistics and Operation Research 12:281–99.