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Generalization of the distributive semimodules

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Abstract: *A semimodule M is considered distributive if the lattice formed by its subsemimodules is distributive. In this study, we aim to study the concept of a weak distributive. Specifically, a subsemimodule A is termed weak distributive if $A = A \cap H + A \cap K$ for all H and K belonging to the lattice of M such that $H + K = M$. A semimodule M is classified as weakly distributive if each subsemimodule A belonging to the lattice of M satisfies the conditions of being a weak distributive subsemimodule.*

Keywords: Weak distributive, k-reguler, supplemented extending, subtractive, closed subsemimodules.

1.Introduction

Many authors have extensively studied distributive modules and their properties see [1], ,[2], [3], [4],[5], and [6]. Subsequently, Some studies have clarified the term distribution property like [7], [8],[9] and [10]. In the context of the module, researchers E. Büyükaşık and Y. M. Demirci [11]studied the concept of weakly distributive. In our research, following its introduction in relation to the semimodule, we will explore some of the properties that arise after its generalization. In the realm of semimodules, consider a semi ring R. A **left R-semimodule** be established as a commutative monoid $(M, +)$ with an additive identity 0. It incorporates a function from $R \times M$ to M, represented as (r, m) $\boldsymbol{if}$ $then\, rx$, , where r is an element from R and m is an element from M. This scalar multiplication operation adheres to the following conditions for all elements r, r' in R and all elements m, m' in M: $m, m' \in M$: $(rr')x = r(r'm); r(m + m') = rm + rm'; (r + r')m = rm + r'm; 0_M = 0_M = 0_R m$. let N anonempty subset of a left R-semimodule M which is said a **subsemimodule** of M in case N it was closed in addition and standard multiplication so N was an R-semimodule (we symbolize it as $N \leq M$). we denoted by $\mathrm{L}(M)$ to the set of all subsemimodules of M Let D and M be R - semimodules and we called to the map $f: M \to D$ a **homomorphism** if for each $n, e \in s, M \in R, f(e + n) = f(e) + f(n)$ **and** $f(s\, n) = sf(n)$ [12]. to a homomorphism of R – semimodules $f: M \to D$, we defined $f(M) = \{ f(e) | \, e \in M\}$,$\ker(f) = \{ e \in M \mid f(e) = 0\}$,$Im(f) = \{k \in D | k + f(e) = f(é)$ for some $e, é \in M$ }, f is ***i*-regular** if $f(M) = Im(M)$, and f is ***k*-regular**, if $f(e) = f(é)$ implies $e + h = é + h́$ for some $h, h́ \in \ker(f)$ [13]. A **subtractive** subsemimodule (or k −subsemimodule) N be a subsemimodule of M , if $z, z + w \in N$, then $w \in N$ [14]. A subsemimodule E of R - semimodule M is **superfluous** or **small** (denoted by $E \ll M$). If any $K \in \mathrm{L}(M)$, $K + E = M$ implies $K = M$ [15]. We said to the subsemimodule T a **supplement** of Q in M if $T + Q = M$, and T is minimal with this property[16]. A subsemimodule K of semimodule M is called **closed** if K has no proper essential extension in M [10].

2. Weakly distributive semimodules

Let M be an R– semimodule, and $A \in L(M)$. let A a weak distributive subsemimodule of M if $A = A \cap E + A \cap W$, $\forall\, E, W \in L(M)$ so that $E + W =$ M. when M be a weakly distributive if $\forall\;\; A \in L(M)$ is a weakly distributive subsemimodule, example of this any semiring with identity over itself is weakly distributive, but on the contrary, if $M \neq \{0\}$ be any R – semimodule, so $\mathcal{B} = M \coprod M$ be not a weakly distributive R – semimodule .

Lemma 2.1. Let $\mathcal{D}$ be a R– semimodule, and $\mathcal{T}, \mathcal{L} \in L(\mathcal{D})$ with $\mathcal{T}$ be a weak distributive subsemimodule of $\mathcal{D}$. Then $\mathcal{T}$ is a supplement of $\mathcal{L}$ $if\ and\ only\ if$ $\mathcal{D} = \mathcal{L} + \mathcal{T}$, $\mathcal{T} \cap \mathcal{L}$ it was be a small in $\mathcal{T}$.

Proof. let $\mathcal{T}$ be a supplement to $\mathcal{L}$ and $\mathcal{X} \in L(\mathcal{T})$, $\mathcal{T} \cap \mathcal{L} + \mathcal{X} = \mathcal{T}$.

$\mathcal{D} = \mathcal{L} + \mathcal{T}$ from define of supplement we get $\mathcal{D} = \mathcal{L} + (\mathcal{T} \cap \mathcal{L} + \mathcal{X}) = \mathcal{L} + \mathcal{X}$,$\mathcal{X} = \mathcal{T}$ by the minimality of $\mathcal{T}$.

From other side, we suppose that $= \mathcal{L} + \mathcal{T}$, $\mathcal{T} \cap \mathcal{L}$ small in $\mathcal{T}$. we prove $\mathcal{T}$ be a supplement of $\mathcal{L}$, if $C \subseteq \mathcal{T}$, $\mathcal{D} = \mathcal{L} + C$, so $\mathcal{T} = \mathcal{T} \cap \mathcal{D} = \mathcal{T} \cap (\mathcal{L} + C) = \mathcal{T} \cap \mathcal{L} + \mathcal{T} \cap C = \mathcal{T} \cap \mathcal{L} + C$ and $C = \mathcal{T}$ since $\mathcal{T} \cap \mathcal{L}$ be small in $\mathcal{T}$. so $\mathcal{T}$ be a supplement of $\mathcal{L}$.

Proposition 2.2. we suppose that $\mathcal{D}$ is an $R-$semimodule, $\mathcal{Q}, \mathcal{F}, \mathcal{A} \in L(\mathcal{D})$. If $\mathcal{Q}$ be a weak distributive subsemimodule and supplement of $\mathcal{F}, \mathcal{A}$.

so $\mathcal{Q}$ be a supplement of $\mathcal{F} \cap \mathcal{A}$.

Proof.

By hypothesis $\mathcal{D} = \mathcal{Q} + \mathcal{F} = \mathcal{Q} + \mathcal{A}$, by Lemma 2.1 $\mathcal{Q} \cap \mathcal{F}$, $\mathcal{Q} \cap \mathcal{A}$ be small in $\mathcal{Q}$, if $\mathcal{Q} \cap \mathcal{F} \cap \mathcal{A} \subseteq \mathcal{Q} \cap \mathcal{F}$, so $\mathcal{Q} \cap \mathcal{F} \cap \mathcal{A}$ be small in $\mathcal{Q}$.

from Lemma 2.1 $\mathcal{Q}$ is a supplement of $\mathcal{F} \cap \mathcal{A}$.

Lemma 2.3.[10] we suppose that U , A are R – semimodules , $\mu \in Hom\ (U, A)$, $M \in L(A)$, then $\mu\left(\mu^{-1}(M)\right) = M \cap \mu(U)$.

Proposition 2.4. Let M and H be R-semimodules, g a k-regular homomorphism from M to H. If $g(M)$ is a weakly distributive subsemimodule of A, W and N subsemimodules of A with $E + N = H$ and $g^{-1}(E) + g^{-1}(N)$ is subtractive in M, then $g^{-1}(E) + g^{-1}(N) = M$.

Proof. It is clear that $g^{-1}(E) + g^{-1}(N) \subseteq M$. Let $m \in M$, then $g(m) \in (E + N) \cap g(M)$. Since $g(M)$ is a weakly distributive subsemimodule, $g(m) \in [E \cap g(M) + N \cap g(M)]$, so Lemma 2.3 implies $g(m) \in [g\left(g^{-1}(E)\right) + g(g^{-1}(N))]$ this mean $g(m) = g(m_1) + g(m_2)$ with $m_1 \in g^{-1}(E)$, $m_2 \in g^{-1}(N)$ as a result, we get $g(m) = g(m_1 + m_2)$.

By hypothesis, g is a k-regular, so $m + k_1 = m_1 + m_2 + k_2$ for $k_1, k_2 \in kerg$. Now, $m_1 + m_2 + k_2 \in g^{-1}(E) + g^{-1}(N)$ (since $erg \leq g^{-1}(N)$), so $m + k_1 \in g^{-1}(E) + g^{-1}(N)$. But $k_1 \in kerg \subseteq g^{-1}(E) + g^{-1}(N)$, then $m \in g^{-1}(E) + g^{-1}(N)$(by subtractive property).

Remark 2.5.[6] Let $\eta: \mathcal{B} \rightarrow \mathcal{D}$ be homomorphism of R – semimodules , then $\eta(\mathcal{B})$ is a subtractive subsemimodule of $\mathcal{D}$ if and only if η is i-regular.

Lemma 2.6.[13] we suppose that U , A is R – semimodules. If $\omega, \mu \in Hom\ (U, A)$

where ω is i-regular, then $(\omega + \mu)^{-1}\left(\omega\ (U)\right) = \mu^{-1}\left(\omega(U)\right)$.

Proof. $(\omega + \mu)^{-1}\left(\omega\ (U)\right) = \{ e \in U: (\omega + \mu)(e) \in \omega(U)\}$

$= \{ e \in U: \omega(e) + \mu(e) \in \omega(U)\}$

$= \{ e \in U: \mu(e) \in \omega(U)\}$ (By Remark 2.5)

$= \mu^{-1}\left(\omega(U)\right)$.

Corollary 2.7. Let U be an R - semimodule, A is a subtractive R – semimodule and $\mu, \omega \in Hom\ (U, A)$. If $(\mu + \omega)(U)$ is a weakly distributive subsemimodule in A and $\mu(U) + \omega(U) = A$, then $\omega^{-1}\left(\mu(U)\right) + \mu^{-1}\left(\omega(U)\right) = U$.

Proof.

By Proposition 2.4 $(\mu + \omega)^{-1}\left(\mu(U)\right) + (\mu + \omega)^{-1}\left(\omega(U)\right) = U$. Since A is a subtractive then the subsemimodules $\mu(U)$ and $\omega(U)$ are subtractive, by Remark 2.5 μ and ω are i-regular, hence by Lemma 2.6 $U = (\mu + \omega)^{-1}\left(\mu(U)\right) + (\mu + \omega)^{-1}\left(\omega(U)\right) = \omega^{-1}\left(\mu(U)\right) + \mu^{-1}\left(\omega(U)\right)$.

If we designate the subsemimodule $\mathcal{A}$ that fulfills the condition: $\mathcal{A} + (\mathcal{H} \cap \mathcal{K}) = (\mathcal{A} + \mathcal{H}) \cap (\mathcal{A} + \mathcal{K})\ \forall\, \mathcal{H}, \mathcal{K} \in L(\mathcal{D})$ so $\mathcal{H} + \mathcal{K} = \mathcal{D}$ as the distributive subsemimodule, we can derive the following property:

Proposition 2.8. Let M and E be R-semimodules, f a k-regular homomorphism from M to E. If $kerf$ is a weak distributive subsemimodule in M and $Y + N = M$, then $f(Y \cap N) = f(Y) \cap f(N)$.

Proof. It is clear that $f(Y \cap N) \subseteq f(Y) \cap f(N)$. Now, let $h \in f(Y) \cap f(N)$, then $h = f(\text{y}) = f(n)$ where $\text{y} \in Y$and $n \in N$. Since f is k-regular, there exist $k_1, k_2 \in kerf$ such that $\text{y} + k_1 = n + k_2$.

Now, $y + k_1 \in (Y + kerf) \cap (N + kerf)$. As $kerf$ is a weak distributive subsemimodule, this implies $y + k_1 \in (Y \cap S) + kerf$. Therefor $y + k_1 = z + k_3$ where $z \in Y \cap S$, $k_3 \in kerf$. Then $f(y) = f(z)$ implies $h = f(z) \in f(Y \cap N)$. Therefore $f(Y \cap N) = f(Y) \cap f(N)$.

Proposition 2.9. Let $Z = Z_1 \oplus Z_2 = T_1 + T_2$ be a semimodule, where $T_1 \leq Z_1$. If T_2 is weak distributive subsemimodule and $T_1 \cap T_2$ be small in T_2, $then$ $T_1 \cap T_2$ are small in $Z_1 \cap T_2$.

Proof. we suppose that $Z_1 \cap T_2 = (T_1 \cap T_2) + W$, where W is a subsemimodule of $Z_1 \cap T_2$. Since T_2 is a weak distributive subsemimodule, $T_2 = (Z_1 \cap T_2) \oplus (Z_2 \cap T_2)$. We have $Z = T_1 + T_2 = T_1 + (Z_1 \cap T_2) + (Z_2 \cap T_2) = T_1 + W + (Z_2 \cap T_2)$ and $T_2 = (T_1 \cap T_2) + W + (Z_2 \cap T_2)$.

Since $T_1 \cap T_2$ is small in T_2, we have $T_2 = W \oplus (Z_2 \cap T_2)$. Then, $T_2 = (T_2 \cap Z_1) \oplus (T_2 \cap Z_2)$, and $W \leq Z_1 \cap T_2$ implies $W = Z_1 \cap T_2$. Hence, $T_1 \cap T_2$ is small in $Z_1 \cap T_2$.

Proposition 2.10. Suppose that $S = S_1 \oplus S_2$, where S_1, S_2 be R− semimodules , S distributive. If $K \in L(S)$ be a weak distributive subsemimodule and closed in S, then $K \cap S_h$ is closed in S_h for $h = 1,2$.

Proof. Assume $K \cap S_h \leq^e {S_h}^* \leq S_h$ for $h = 1,2$. Then,

$K \cap S_1 \oplus K \cap S_2 \leq^e {S_1}^* \oplus {S_2}^* \leq S_1 \oplus S_2 = S$, thus $K \leq^e {S_1}^* \oplus {S_2}^*$. Since K is closed in S, it follows that $K = {S_1}^* \oplus {S_2}^*$. Hence, $K \cap S_1 \oplus K \cap S_2 = {S_1}^* \oplus {S_2}^*$ implies that $K \cap S_1 = {S_1}^*$ and $K \cap S_2 = {S_2}^*$. Therefor, $K \cap S_h$ is closed in S_h for $h = 1,2$.

Definition 2.11.[10] An R− semimodule $\mathcal{D}$ we said it a supplemented extending if each closed subsemimodule be supplement

Proposition 2.12. Let $M = M_1 \oplus M_2$ where M, M_1, M_2 be R− semimodules and M weakly distributive, $then$ M is supplemented extending, when M_1, M_2 are supplemented extending.

Proof. Suppose M_1, M_2 are supplement extending R− semimodules, and L is a closed subsemimodule in M. According to Proposition 2.10, $L \cap M_i$ is closed subsemimodule in M_i for $i = 1,2$. Since M_i be supplement extending, $L \cap M_i$ be supplement in M_i. By Lemma 2.1, there exist a subsemimodule K_i of M_i such that $K_i + (L \cap M_i) = M_i$ and $K_i \cap L \cap M_i = K_i \cap L \ll L \cap M_i \ll L$.

Now, consider $(K_1 + K_2) \cap L = (K_1 \cap L) + (K_2 \cap L) \ll L$. Additionally, we have:

$M = [K_1 + (L \cap M_1)] + [K_2 + (L \cap M_2)] \subseteq (K_1 + K_2) + L$.

By Lemma 2.1, L is supplement, concluding the proof.

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