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Some New Results of Dissipative RDSs on Local Compact Spaces

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Abstract: *In this paper, we generalize some results from the (deterministic) theory of dissipative dynamical systems to the random dynamical systems, especially, in locally compact spaces. It will be shown that the Compact and local dissipativity are equivalent in locally compact spaces on RDS.*

Keywords: Random dynamical system (RDS), Dissipative Random Dynamical System (DRDS), Locally Compact Space, random attracter.

1. Introduction: The study of the asymptotic behavior of dissipative random dynamical systems is now made much easier by the current theory of random attractors. A random attractor is a random invariant compact set which that attracts every trajectory as time becomes infinite.

One of the main concepts for the study of the asymptotic behavior of dissipative dynamical systems is the global attractor (see Hale [3] or Temam [8] and the references therein).
Many papers devoted to the study the dissipativety in RDSs, see, for example, Xiaoying S., Qiaozhen M. [9], Hoang L. [4], Kuehn C., Neamt¸u A. and Pein A. [7], Flandoli F. and Langa J. [2].
In our work, we have generalized some concepts of dissipative of random dynamical system with locally compact phase space. Moreover, it will be shown that the locally completely continuous and weakly dissipative are dynamic properties.

2. RDS, concepts and definitions:

In this section, we recall some basic concepts related to RDS and random attractors for RDS in [igor], which are important for getting our main results.

Definition 2.4[7]*:* *Let* $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. The measurable invariant action $\theta:\mathbb{T}\times\Omega\longrightarrow\Omega$ is called metric dynamical system and will written as MDS.

Definition (2.1)[5]: *Let* $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\theta:\mathbb{T}\times\Omega\longrightarrow\Omega$ is measurable function satisfy the following

$$\theta_0=\text{ id},\ \theta_t\circ\theta_s=\theta_{t+s}\ \text{ for all } t,s\in\mathbb{T};\ \text{ and }\ \theta_t\mathbb{P}=\mathbb{P}\ \text{ for all } t\in\mathbb{T}.$$

A set $B\in\mathcal{F}$ is called θ −invariant if $\theta_t B=B$ for all $t\in\mathbb{T}$. An MDS θ called ergodic under $\mathbb{P}$ if for any θ −invariant set $B\in\mathcal{F}$ we have either $\mathbb{P}(B)=0$ or $\mathbb{P}(B)=1$.

Definition (2.2) [5]: *Let* X be a topological space and $\mathbb{T}$ be a locally compact group. RDS is a pair (θ,φ) involving an MDS θ and a cocycle φ over θ of continuous mappings of X , i.e. a measurable mapping $\varphi:\mathbb{T}\times\Omega\times X\longrightarrow X,\ (t,\omega,x)\mapsto\varphi(t,\omega,x)$,such that

(i) for every $t\in\mathbb{T}$ and $\omega\in\Omega$, the function $x\mapsto\varphi(t,\omega,x)\equiv\varphi(t,\omega)x$ is continuous

(ii) for all $t,s\in\mathbb{T}$ and $\omega\in\Omega$,the function$\varphi(t,\omega):=\varphi(t,\omega,\cdot)$ fulfill:

$$\varphi(0,\omega)=\text{id},\ \varphi(t+s,\omega)=\varphi(t,\theta_s\omega)\circ\varphi(s,\omega)\ .$$

The property (ii) called cocycle property of φ**.**

Definition (2.3) [5]: Let X be a metric space with a metric d. The multifunction $\omega\mapsto D(\omega)\neq\varphi$ is said to be a **random set** if the mapping $\omega\ \mapsto\ \text{dist}_X(x,D(\omega))$ measurable for any $x\in X$. If $D(\omega)$ is closed for each $\omega\in\Omega$ then D is called a **random**

closed set. If $\mathrm{D}(\omega)$ are compact sets for all $\omega \in \Omega$ then D is called a **random compact set**. A random set $\{\mathrm{D}(\omega)\}$ is said to be **bounded** if there exist $x_0 \in \mathrm{X}$ and a random variable $\mathrm{r}(\omega) > 0$ such that

$$\mathrm{D}(\omega) \subset \{x \in \mathrm{X}: \mathrm{d}(\mathrm{x}, \mathrm{x}_0) \leq \mathrm{r}(\omega)\} \quad \text{for all } \omega \in \Omega.$$

For ease of notation we denote the random set $\omega \longmapsto \mathrm{D}(\omega)$ by D or $\{\mathrm{D}(\omega)\}$.

Definition (2.4) [12]: A collection $\mathcal{M}$ of random sets is called a **universe of sets** if

(a) Every members of $\mathcal{M}$ is closed, and

(b) $\mathcal{M}$ is closed with respect to inclusions.

Definition (2.5) [5]: Consider the RDS (θ, φ) and the universe $\mathcal{M}$. Then (θ, φ) is called **dissipative** in $\mathcal{M}$, if for some absorbing set A for the RDS (θ, φ) in $\mathcal{M}$ and some closed random ball $B_{\mathrm{r}(\omega)}(x_0)$ with center $x_0 \in X$ and radius $\mathrm{r}(\omega)$ we have

$$A(\omega) \subset B_{\mathrm{r}(\omega)}(x_0), \text{ for all } \omega \in \Omega..$$

Definition (2.6) (Invariance Property)[38]: Let(θ, φ) be a random dynamical system. A multifunction $\omega \longmapsto D(\omega)$is said to be

(i) Forward invariant with respect to (θ, φ) *if* $\varphi(t, \omega)D(\omega) \subseteq D(\theta_t\omega)$ for all $t > 0$ and $\omega \in \Omega$, *i.e. if* $x \in D(\omega)$implies $\varphi(t, \omega)x \in D(\theta_t\omega)$ for all $t \geq 0$ and $\omega \in \Omega$*;*
(ii) Backward invariant with respect to (θ, φ) *if* $D(\theta_t\omega) \subseteq \varphi(t, \omega)D(\omega)$ *for all* $t > 0$ and $\omega \in \Omega$, *i.e.* for every $t > 0$, $\omega \in \Omega$ and $y \in D(\theta_t\omega)$ there exists $x \in D(\omega)$ such that $\varphi(t, \omega)x = y$*;*

$$\varphi(t, \omega)D(\omega) \subseteq D(\theta_t\omega), t < 0.$$

Definition 2.9(2.7) [2 بحث]:RDS (θ, φ) is said to be **point dissipative** if *for every* $x \in X^{\Omega}$, there is random set K in X so that,

$$\lim_{t \longrightarrow +\infty} d\big(\varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega), K(\omega)\big) = 0$$

Definition 2.5 (2.8) [5]**:** Let $X = \mathbb{R}^d$, Suppose that $r(\omega) \geq 0$ is a **random variable** and $a(\omega)$ is a random vector from $\mathbb{R}^d$. Then the multifunction

$$\omega \longmapsto \mathrm{B}(\omega) = \{x: |x - a(\omega)| \leq \mathrm{r}(\omega)\}$$

is a random compact set. Here $|.|$ is the Euclidean distance in $\mathbb{R}^d$. This fact follows from the formula

$$dis_X\big(y, \mathrm{B}(\omega)\big) = \begin{cases} 0 & if\ y \in \mathrm{B}(\omega) \\ |y - a(\omega) - \mathrm{r}(\omega)| & if\ y \notin \mathrm{B}(\omega) \end{cases}$$

Which implies that $dis_X\big(y, \mathrm{B}(\omega)\big) = \max\{0, |y - a(\omega) - \mathrm{r}(\omega)|\}$. It is also clear that $int\ B(\omega) = \{x: |x - a(\omega)| < \mathrm{r}(\omega)\}$ is a random (open) set.

Definition 2.6(2.9) [5]**:** A random set $\mathrm{D}(\omega)$ is said to be **tempered** with respect to MDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{T}\})$ if there exist a random variable $\mathrm{r}(\omega)$ and an element $y \in \mathrm{X}$ such that

$$\mathrm{D}(\omega) \subset \{x: dis_X(x, \mathrm{y}) \leq \mathrm{r}(\omega)\} \text{ for all } \omega \in \Omega$$

And $\mathrm{r}(\omega)$ is a tempered random variable with respect to θ, i.e

$$\sup_{t \in \mathbb{T}}\{e^{-\lambda|t|}\,|r(\theta_t\omega)|\} < \infty\,, \quad \text{for all } \omega \in \Omega \text{ and } \lambda > 0$$

Definition 2.7(2.10) [5]**:** A random closed set $\{M(\omega)\}$ from a universe $\mathcal{M}$ is called a **random attractor** of RDS (θ, φ) in $\mathcal{M}$ if $B(\omega)$ is proper subset of X *for every* $\omega \in \Omega$ and :

(i) B is an invariant set, i.e. $\varphi(t, \omega)B(\omega) = B(\theta_t\omega)$ for $t \geq 0$, $\omega \in \Omega$;

(ii) B is an attracting in $\mathcal{M}$, i.e. $for\ all\ M \in \mathcal{M}$

$$\lim_{n \to +\infty} d_X\{\varphi(t, \theta_{-t}\omega)M(\theta_{-t}\omega), B(\omega)\} = 0,\ \omega \in \Omega$$

Where $d_X\{A \setminus B\} = sup_{x \in A} dis_X(x, B)$.

Definition 2.8 **(2.11)** **[6]:** Let (X, d) be a metric space, $K \subset X$ is **precompact** or **totally bounded** if every sequence in K admits a subsequence converges to a point of X.

Definition 2.9 **(2.12)** **(Equivalence of RDS)[Igor]:** Let (θ, φ_1) and (θ, φ_2) be two RDS over the same MDS θ with phase spaces X_1 and X_2 respectively. These RDSs (θ, φ_1) and (θ, φ_2) are said to be (topologically) equivalent (or conjugate) if there exists a mapping $T: \Omega \times X_1 \to X_2$ with the properties:

(i)The mapping $T_\omega: X_1 \longrightarrow X_2$ such that $T_\omega(x_1) = T(\omega, x_1)$ is homeomorphism for every $x_1 \in X_1$ and $\omega \in \Omega$;
(ii) The mappings $\beta_1: \Omega \longrightarrow X_2$, such that $\beta_1(\omega) = T(\omega, x_1)$ is measurable for every $x_1 \in X_1$ and $\omega \in \Omega$;
and $\beta_2: \Omega \longrightarrow X_1$, such that $\beta_2(\omega) = T^{-1}(\omega, x_2)$ is measurable for every $x_2 \in X_2$;
(iii) The cocycles φ_1 and φ_2 are cohomologous, i.e.
$\varphi_2\ (t, \omega, T(\omega, x)) = T\ (\theta_t\ \omega, \varphi_1\ (t, \omega, x))$ for any $x \in X_1$.

In this case we will write $(\theta, \varphi_1) \cong_T (\theta, \varphi_2)$. Any property that preserved under the topological equivalent it will be called **dynamical property**.

3. Dissipative RDSs on Locally Compact Spaces:

This section devoted to a study the compact and local dissipativity . The main result is that under certain conditions the weak dissipative, point dissipative, compact dissipative, and local dissipative are equivalent in locally compact spaces. As will be shown below, a stronger statement holds, namely that in locally compact spaces, point dissipativity implies local dissipativity.

In the following definition we will introduce two types of dissipativity in RDS:

Definition 3.1: We will call the RDS (θ, φ)
1. **locally completely continuous** (**locally compact**) if for every random variable $p \in X^\Omega$, there exist a tempered random variable $\delta = \delta(p, \omega) > 0$ and $l = l(p) > 0$ such that

$$\varphi(l, \theta_{-l}\omega)B(\theta_{-l}\omega), \text{ where } B(\omega) \coloneqq B(\theta_{-l}\omega) = \{x \in X: d(x, p(\omega)) < \delta(\omega)\}$$

is precompact;
2. **weakly dissipative** if there exists a nonempty random compact set $K \subseteq X$ such that for every random variable $\varepsilon > 0$ and $x \in X$, there exists $\tau = \tau(\varepsilon, x) > 0$ for which

$$\varphi(\tau, \theta_{-\tau}\omega)x \in B(K(\omega), \varepsilon(\omega))\ .$$

In this case, we will call K a **weak attractor**.

Remark 3.2: Every RDS (θ, φ) defined on the locally compact metric space X is locally completely continuous.

Proof: by Definition 3.1(1).

Lemma 3.3: Let $K \subseteq X$ be a nonempty compact random set, $p_i \in X^\Omega$, and $\delta_i(\omega) > 0$ $(i = 1, \ldots, m), \omega \in \Omega$. If $K \subseteq \cup\{B(p_i, \delta_i(\omega))\ |\ i = 1, \ldots, m\}$, then there exists a tempered random variable $\alpha > 0$ such that

$$B(K, \alpha) \subseteq \cup\{B(p_i, \delta_i(\omega))\ |\ i = 1, \ldots, m\} \qquad (3.1)$$

Proof: Assume that the inclusion (3.1) does not hold for all $\alpha > 0$. i.e.,

$$B(K, \alpha) \nsubseteq \cup\ \{B(p_i, \delta_i(\omega))|\ i = 1, \ldots, m\}, \text{ for all } \alpha > 0.$$

Then there exist $\alpha_n \downarrow 0$ such that

$$B(K, \alpha_n) \nsubseteq \cup\ \{B(p_i, \delta_i(\omega))|\ i = 1, \ldots, m\}.$$

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In other words:

Then there exist sequence $\{\alpha_n\}$ with $\alpha_n \downarrow 0$ and $r_n \in \mathrm{B}(\mathrm{K}, \alpha_n)$ (i.e., $d(r_n, K) = \inf_{q \in K} d(r_n, q) < \alpha_n$)such that

$$r_n \notin \bigcup \{B(p_i, \delta_i(\omega)) \mid i = 1, \ldots, m\}$$

This implies that $r_n \notin B(p_i, \delta_i(\omega))$ for all $i = 1, \ldots, m$, so

$$d(r_n, p) > \delta_i(\omega) \text{ , for all } p \in B(p_i(\omega), \delta_i(\omega)) \text{ and all } i = 1, \ldots, m$$

Since $r_n \in B(K, \alpha_n)$, as $\alpha_n \downarrow 0$, for every point r_n there exists a sequence $\{q_n\}$ such that

$$d(r_n, q_n) < \alpha_n . \quad (3.2)$$

By the compactness of $K(\omega)$, the sequence $\{q_n\}$ can be considered to converge to q (i.e., $\lim_{n \to \infty} d(q_n, q) = 0$). Note that since $K(\omega) \subseteq \cup \{B(p_i(\omega), \delta_i(\omega)) | \, i = 1, \ldots, m\}$, then $q \in B(p_i(\omega), \delta_i(\omega))$ for some $i = 1, \ldots, m$

Then according to (3.2), $r_n \to q$,

$$d(r_n, q) \le d(r_n, q_n) + d(q_n, q)$$

$$< \alpha_n + d(q_n, q).$$

Take a limit as n for arbitrary large :

$$\lim_{n \to \infty} d(r_n, q) < \lim_{n \to \infty} \alpha_n + \lim_{n \to \infty} d(q_n, q) = 0 + 0 = 0.$$

So $\lim_{n \to \infty} d(r_n, q) = 0$.

which contradicts the choice of the sequence $\{r_n\}$.

since $d(r_n, p) > \delta_i$, for all $p \in B(p_i(\omega), \delta_i(\omega))$ and all $i = 1, \ldots, m$. ■

Theorem 3.4: Let (θ, φ) be weakly dissipative and locally completely continuous RDS, and let $K \subset X$ be a random weak attractor of (θ, φ).Then:

1.There exist $a_0 > 0$ and $l_0 > 0$ such that

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \, \omega \in \Omega, \, B(\omega) \coloneqq B(p, \delta) = \{x \in X : d(x, p(\omega)) < \delta(\omega)\}$$

is precompact for every $t \ge l_0$.

2.There exist $L_0 \ge l_0$ such that for all $t \ge L_0$,

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq \varphi(t_0, \theta_{-t_0}\omega)B(\theta_{-t_0}\omega).$$

Proof: Let $x \in K^{\Omega}$. By virtue of the local complete continuity of (θ, φ) for the random variable $x \in K^{\Omega}$, there exist $l(x, \omega) > 0$ and $\delta_x(\omega) > 0$ such that

$\varphi(t, \theta_{-t}\omega)B_x(\theta_{-t}\omega)$, $B_x(\omega) \coloneqq B(x, \delta_x) = \{y \in X : d(y, x(\omega)) < \delta_x(\omega)\}$

is precompact for all $t \ge l(x, \omega)$. It is clear that

$$\{B_x(\omega) | \, x \in K\}$$

is an random open covering of K, and by its compactness, we can extract a finite sub covering $\{B_{x_i}(\omega) : i = 1, \ldots, n\}$ from the constructed covering. Let $l_0(\omega) \coloneqq \max\{l(x_i) | \, i = 1, \ldots, n\}$. According to Theorem 3.2, there exists $a_0 > 0$ such that

$$K \subseteq B_x(\theta_{-t}\omega) \subseteq \cup \{B_{x_i}(\theta_{-t}\omega) \big| \, i = 1, \ldots, n\}.$$

Consequently,

$$\varphi(t, \theta_{-t}\omega)B_x(\theta_{-t}\omega) \subseteq \cup \{\varphi(t, \theta_{-t}\omega)B_{x_i}(\theta_{-t}\omega) \big| \, i = 1, \ldots, n\},$$

which is why the set $\varphi(t,\theta_{-t}\omega)B_K(\theta_{-t}\omega)$ is precompact for all $t \geq l_0$, where $B_K(\omega) := B(K,a_0)$.

Let us now prove the second statement of the lemma. Let $a_0(\omega)$ and $l_0(\omega)$ be a random positive numbers from the previous item. Suppose that the second statement of the theorem is not true. Then we will see that there exist $\{x_k(\omega)\} \subset B_K(\theta_{-t}\omega)$ and $t_k \to +\infty$ such that

(1) $x_k(\omega) \in \overline{B_K(\theta_{-t}\omega)}$.
(2) $\varphi(t,\theta_{-t}\omega)x_k \in X \setminus B_K(\theta_{-t}\omega)$ for any $0 < t < t_k$.
(3) $\varphi(t_k,\theta_{-t_k}\omega)x_k \in B_K(\theta_{-t}\omega)$.
From (2) it follows that

(4) $\varphi(t,\theta_{-t}\omega)\bar{x}_k \in X \setminus B_K(\theta_{-t}\omega)$ for all $0 < t < t_k - l_0$, where $\bar{x}_k(\omega) := \varphi\left(l_0,\theta_{-l_0}\omega\right)x_k$.

By virtue of the relative compactness of $\varphi\left(t_0,\theta_{-t_0}\omega\right)B_K\left(\theta_{-t_0}\omega\right)$, the sequence $\bar{x}_k(\omega) = \varphi\left(l_0,\theta_{-l_0}\omega\right)x_k$ can be considered convergent. Assume $x_0(\omega) = \lim_{k\to+\infty} \bar{x}_k(\omega)$. Then from (4), it follows that

$$\varphi(t,\theta_{-t}\omega)x_0 \in X \setminus B_K(\theta_{-t}\omega) \text{ for all } 0 < t < +\infty,$$

and consequently,

$$\emptyset \neq \Gamma_{x_0}(\omega) \subset X \setminus B_K(\theta_{-t}\omega).$$

So $\Gamma_{x_0}(\omega) \cap K(\omega) = \emptyset$, which contradicts the weak dissipativity of RDS (θ,φ) and the fact that $K(\omega)$ is a weak attractor. This contradiction completes the proof of theorem.

Theorem 3.5: For locally completely continuous RDS (θ,φ) with $\varphi(t,\theta_{-t}\omega): X \to X$ is continuous for every t, the following statements are equivalent

1.weak dissipative,

2. point dissipative,

3. compact dissipative, and

4.local dissipative.
Proof: Suppose that (θ,φ) is locally completely continuous.

(1) ⟹(4):Let $K \neq \emptyset$ be a compact weak random attractor of (θ,φ). Denote by $a_0(\omega)$ the tempered from Theorem 3.4. If $x \in X$, then there exists $\tau > 0$ such that

$\varphi(\tau,\theta_{-\tau}\omega)x \in B(\theta_{-t}\,\omega)$. Let $\epsilon > 0$ be a tempered random variable such that $B(\varphi(\tau,\theta_{-\tau}\omega),\epsilon(\omega)) \subset B(K,a_0(\omega))$. By the continuity of the mapping $\varphi(\tau,\theta_{-\tau}\omega) : X \longrightarrow X$ at the point x, there exists $\alpha > 0$ such that

$$\varphi(\tau,\theta_{-\tau}\omega)B(x,\alpha) \subset B(\varphi(\tau,\theta_{-\tau}\omega)x,\gamma(\omega)) \subset B(K,a_0(\omega)).$$

Assume $M = \overline{B(K,a_0(\omega))}$. According to Theorems 3.5 from [10] and 2.4, the set M is nonempty and compact random set, and $\lim_{k\to+\infty}\beta(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega),M(\omega)) = 0$, So we have constructed a nonempty compact subset $M \subset X$ attracting every point with some α-neighborhood, that is, (θ,φ) is locally dissipative. The theorem is proved.

Theorem 3.6: Let $(\theta,\varphi_1) \cong_T (\theta,\varphi_2)$. If an RDS (θ,φ_1) is locally completely continuous, then so is (θ,φ_2).

Proof: Suppose that (θ,φ_1) is locally completely continuous. Let $p \in X_2^{\Omega}$. Since the mapping $T_\omega: X_1 \longrightarrow X_2$ such that $T_\omega(x_1) = T(\omega,x_1)$ is homeomorphism for every $x_1 \in X_1$ and $\omega \in \Omega$. Then $q := T_\omega^{-1} \circ p: \Omega \longrightarrow X_1$ is random variable in X_1 . By hypothesis there exists $\delta = \delta(q,\omega)$ and $l = l(q)$ such that

$$\varphi_1(l,\theta_{-l}\omega)B_{(q,\delta)}(\theta_{-l}\omega)$$

is precompact. Now, since

$$\varphi_2\ (t, \omega, T(\omega, x)) \ = \ T\ (\theta_t\ \omega, \varphi_1\ (t, \omega, x)) \text{ for any } x \in \mathrm{X}_1 \text{ and every } t \in \mathbb{R}.$$

Then

$$\varphi_2\ (l, \theta_{-l}\omega, T(\theta_{-l}\omega, x)) \ = \ T\ (\theta_l\theta_{-l}\omega, \varphi_1\ (l, \theta_{-l}\omega, x)) \text{ for any } x \in \mathrm{X}_1,$$

or

$$\varphi_2\ (l, \theta_{-l}\omega, T(\theta_{-l}\omega, x)) \ = \ T\ (\omega, \varphi_1\ (l, \theta_{-l}\omega, x)) \text{ for any } x \in \mathrm{X}_1.$$

Since $\varphi_1(l, \theta_{-l}\omega)B_{(q,\delta)}(\theta_{-l}\omega)$ is precompact and $T_\omega: X_1 \longrightarrow X_2$ is homeomorphism, then

$$T_\omega(\varphi_1(l, \theta_{-l}\omega)B_{(q,\delta)}(\theta_{-l}\omega))$$

is precompact. Now,

$$T_\omega(\varphi_1(l, \theta_{-l}\omega)B_{(q,\delta)}(\theta_{-l}\omega) = \varphi_2\ (l, \theta_{-l}\omega, T(\theta_{-l}\omega, B_{(q,\delta)}(\theta_{-l}\omega)))$$

$$= \varphi_2\ (l, \theta_{-l}\omega, T(\theta_{-l}\omega, B_{(q,\delta)}(\theta_{-l}\omega))).$$

Since

$$B_{(q,\delta)}(\theta_{-l}\omega) = \left\{x \in X_1 : d_{X_1}\left(x, q(\theta_{-l}\omega)\right) < \delta(\theta_{-l}\omega)\right\},$$

Then

$$T(\theta_{-l}\omega, B_{(q,\delta)}(\theta_{-l}\omega)) = T_{\theta_{-l}\omega}B_{(q,\delta)}(\theta_{-l}\omega)$$

$$= T_{\theta_{-l}\omega}\left\{x \in X_1 : d_{X_1}\left(x, q(\theta_{-l}\omega)\right) < \delta(\theta_{-l}\omega)\right\}$$

$$= \left\{T_{\theta_{-l}\omega}(x) \in X_2 : d_{X_1}\left(x, q(\theta_{-l}\omega)\right) < \delta(\theta_{-l}\omega)\right\}$$

$$= \left\{T_{\theta_{-l}\omega}(x) \in X_2 : d_{X_2}\left(T_{\theta_{-l}\omega}(x), T_{\theta_{-l}\omega}q(\theta_{-l}\omega)\right) < \delta(\theta_{-l}\omega)\right\}$$

$$= \left\{T_{\theta_{-l}\omega}(x) \in X_2 : d_{X_2}\left(T_{\theta_{-l}\omega}(x), T_{\theta_{-l}\omega}T^{-1}_{\theta_{-l}\omega} \circ p(\theta_{-l}\omega)\right) < \delta(\theta_{-l}\omega)\right\}$$

$$= \left\{T_{\theta_{-l}\omega}(x) \in X_2 : d_{X_2}\left(T_{\theta_{-l}\omega}(x), p(\theta_{-l}\omega)\right) < \delta(\theta_{-l}\omega)\right\}$$

$$= B_{(p,\delta)}(\theta_{-l}\omega)$$

So,

$$T_\omega(\varphi_1(l, \theta_{-l}\omega)B_{(q,\delta)}(\theta_{-l}\omega) = \varphi_2\ (l, \theta_{-l}\omega)B_{(p,\delta)}(\theta_{-l}\omega).$$

This implies that $\varphi_2\ (l, \theta_{-l}\omega)B_{(p,\delta)}(\theta_{-l}\omega)$ is precompact. Consequently (θ, φ_2) is locally completely continuous.
■

Theorem 3.7: Let $(\theta, \varphi_1) \cong_T (\theta, \varphi_2)$. If an RDS (θ, φ_1) is **weakly dissipative**, then so is (θ, φ_2).

Proof: Suppose that (θ, φ_1) is weakly dissipative. Let $y \in X_2$ and ε be a tempered random variable. Then $\coloneqq T^{-1}_\omega(y) \in X_1$. By hypothesis there a nonempty random compact set $K \subseteq X$ and $\tau = \tau(\varepsilon, x) > 0$ such that

$\varphi_1(\tau, \theta_{-\tau}\omega)x \in B(K(\omega), \varepsilon(\omega))$.

Then

$$T_\omega(\varphi_1(\tau,\theta_{-\tau}\omega)x) \in T_\omega(B(K(\omega),\varepsilon(\omega)))$$

But

$$T_\omega(\varphi_1(\tau,\theta_{-\tau}\omega)x) = \varphi_2(\tau,\theta_{-\tau}\omega)T_\omega(x)$$
$$= \varphi_2(\tau,\theta_{-\tau}\omega)T_\omega(T_\omega^{-1}(y)) = \varphi_2(\tau,\theta_{-\tau}\omega)y$$

and

$$T_\omega\left(B(K(\omega),\varepsilon(\omega))\right) = B(T_\omega K(\omega),\varepsilon(\omega)).$$

So

$$\varphi_2(\tau,\theta_{-\tau}\omega)y \in B(T_\omega K(\omega),\varepsilon(\omega)).$$

Now since $T_\omega: X_1 \longrightarrow X_2$ is homeomorphism and $K(\omega)$ is non-empty compact set in X_1, then $T_\omega K(\omega)$ is non-empty compact random set in X_2. Consequently (θ,φ_2) is weakly dissipative. ■

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