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Interval-valued of Fuzzy AB-subalgebras on AB-Algebra

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***Abstract**: In this paper, the notion of interval-valued fuzzy AB-subalgebras (briefly i-v fuzzy AB- subalgebra) in AB-algebras is introduced and how the homomorphic images and inverse images of i-v fuzzy AB- subalgebras become i-v fuzzy AB- subalgebras in AB-algebras is studied as well.*

Keywords—AB-algebras, fuzzy AB-subalgebra, interval-valued fuzzy AB-subalgebras of AB-algebra.

1. Introduction

A. T. Hameed and et al ([4-7]) introduced a new algebraic structure, called AB-algebra, they have studied a few properties of these algebras, the notion of AB-ideals and fuzzy AB-ideal on AB-algebras was formulated and some of its properties are investigated. In [13,20,21], they defined the notions of α-translation, β-magnified and generalized fuzzy AB-subalgebra, generalized fuzzy AB-ideal of AB-algebra and investigate in some of their properties. The concept of a fuzzy set, was introduced by L.A. Zadeh [27]. A.T. Hameed and et al, ([19]) we introduce the notion of anti-fuzzy AB-ideals of AB-algebras and then we study the homomorphism image and pre-image of anti-fuzzy AB-ideals. S. M. Mostafa and et al. [22,24-26] were introduced a new algebraic structure which is called KUS-algebras and investigated some related properties. In [1-3,8-9,12,15-16], A.T. Hameed and et al. introduced AT-ideals on AT-algebras and introduced the notions fuzzy AT-subalgebras, fuzzy AT-ideals of AT-algebras and investigated relations among them. They introduced the notion of cubic AT-ideals of AT-algebra and they discussed some related properties of it. They also prove that some properties of anti-fuzzy AT-ideals and anti-fuzzy AT-subalgebras. A.T. Hameed and et al, ([1]). In [19], A.T. Hameed and et al., prove that the Cartesian product of anti-fuzzy AB-ideals are anti-fuzzy AB-ideals. In this paper, using the notion of interval-valued fuzzy set, we introduce the concept of an interval-valued fuzzy AB-ideals (briefly, i-v fuzzy AB-ideals) of a AB-algebra, and study some of their properties. Using an i-v level set of an i-v fuzzy set, we state a characterization of an i-v fuzzy AB-ideals. We prove that every AB-subalgebras of an AB-algebra X can be realized as an i-v level AB-subalgebras of an i-v fuzzy AB-subalgebras of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v fuzzy AB- subalgebras become i-v fuzzy AB- subalgebras.

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2. Preliminaries

In this section, we give some basic definitions and preliminaries proprieties of AB-ideals and fuzzy AB-ideals in AB-algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1([4-6]) Let X be a set with a binary operation $*$ and a constant γ. Then $(X;*,\gamma)$ is called **an AB-algebra** if the following axioms satisfied: for all $,y,z \in X$,

(i) $((x*y)*(z*y))*(x*z) = \gamma,$

(ii) $\gamma*x = \gamma,$

(iii) $x*\gamma = x,$

Example 2.2([4-6]) Let $X = \{\gamma, 1, 2, 3, 4\}$ in which $(*)$ is defined by the following table:

*	γ	1	2	3	4
γ	γ	γ	γ	γ	γ
1	1	γ	1	γ	γ
2	2	2	γ	γ	γ
3	3	3	1	γ	γ
4	4	3	4	3	γ

Then $(X;*,\gamma)$ is an AB-algebra.

Remark 2.3([4-6]) Define a binary relation ≤ on AB-algebra $(X;*,\gamma)$ by letting $x \leq y$ if and only if $x*y = \gamma$.

Proposition 2.4([4-6]) In any AB-algebra $(X;*,\gamma)$, the following properties hold: for all $x,y,z \in X$,

(1) $(x*y)*x = \gamma.$

(2) $(x*y)*z = (x*z)*y.$

(3) $(x*(x*y))*y = \gamma.$

Proposition 2.5([4-6]) Let $(X; *, \gamma)$ be an AB-algebra. X is satisfies for all $x, y, z \in X$,

(1) $x \le y$ *implies* $x * z \le y * z$.
(2) $x \le y$ *implies* $z * y \le z * x$.

Definition 2.6([4-6]). Let $(X; *, \gamma)$ be an AB-algebra and let S be a nonempty subset of X. S is called an **AB-subalgebra of** X if $x * y \in S$ whenever $x \in S$ *and* $y \in S$.

Definition 2.7([4-6]). A nonempty subset I of an AB-algebra $(X; *, \gamma)$ is called **an AB-ideal of *X*** if it satisfies the following conditions: for any $x, y, z \in X$,
(I_1) $\in I$,
(I_2) $(x * y) * z \in I$ *and* $y \in I$ *imply* $x * z \in I$.

Proposition 2.9 ([4-6]). Every AB-ideal of AB-algebra is an AB-subalgebra.

Proposition 2.8 ([4-6]). Let $\{I_i \mid i \in \Lambda\}$ be a family of AB-ideals of AB-algebra $(X; *, \gamma)$. The intersection of any set of AB-ideals of X is also an AB-ideal.

Definition 2.9 ([13,20,21]). Let $(X ; *, \gamma)$ and $(Y; *`, \gamma`)$ be nonempty sets. The mapping $f: (X; *, \gamma) \rightarrow (Y; *`, \gamma`)$ is called **a homomorphism** if it satisfies:
$f(x * y) = f(x) *` f(y)$, for all $x, y \in X$. The set $\{x \in X \mid f(x) = \gamma'\}$is called **the kernel of** f denoted by ker f.

Theorem 2.10 ([4-6]). Let $f: (X; *, \gamma) \rightarrow (Y; *`, \gamma`)$be a homomorphism of an AB-algebra X into an AB-algebra Y, then :

A. $f(\gamma) = \gamma'$.
B. f is injective if and only if ker $f = \{\gamma\}$.
C. $x \le y$ implies $f(x) \le f(y)$.

Theorem 2.11 ([4-6]). Let $f: (X; *, \gamma) \rightarrow (Y; *`, \gamma`)$ be a homomorphism of an AB-algebra X into an AB-algebra Y, then:

(F_1) If S is an AB-subalgebra of X, then f (S) is an AB-subalgebra of Y.
(F_2) If I is an AB-ideal of X, then f (I) is an AB-ideal of Y , where f is onto.
(F_3) If H is an AB-subalgebra of Y, then f^{-1} (H) is an AB-subalgebra of .
(F_4) If J is an AB-ideal of Y, then f^{-1} (J) is an AB-ideal of X .
(F_5) ker f is an AB-ideal of X.
(F_6) Im(f) is an AB-subalgebra of Y.

Definition 2.12([27]). Let $(X; *, \gamma)$ be a nonempty set, a fuzzy subset μ of X is a function $\mu: X \rightarrow [\gamma, 1]$.

Definition 2.13 ([27]). Let X be a nonempty set and μ be a fuzzy subset of $(X; *, \gamma)$, for $t \in [\gamma, 1]$, the set $L(\mu, t) = \mu_t = \{x \in X \mid \mu(x) \ge t\}$ is called **a level subset of μ** .

Definition 2.14([7]). Let $(X; *, \gamma)$ be an AB-algebra, a fuzzy subset μ of X is called **a fuzzy AB-subalgebra of *X*** if for all $x, y \in X$, $\mu(x * y) \ge min\{\mu(x), \mu(y)\}$.

Definition 2.15([7]). Let $(X; *, \gamma)$ be an AB-algebra, a fuzzy subset μ of X is called **a fuzzy AB-ideal of *X*** if it satisfies the following conditions, for all $x, y, z \in X$,
(FAB_1) $\mu(\gamma) \ge \mu(x)$,
(FAB_2) $\mu(x * z) \ge min\{\mu((x * y) * z), \mu(y)\}$.

Proposition 2.17([7]).
1- The intersection of any set of fuzzy AB-ideals of AB-algebra is also fuzzy AB-ideal.
2- The union of any set of fuzzy AB-ideals of AB-algebra is also fuzzy AB-ideal where is chain.

Proposition 2.18([7]). Every fuzzy AB-ideal of AB-algebra is a fuzzy AB-subalgebra.

Proposition 2.19([7]).
1- Let μ be a fuzzy subset of AB-algebra $(X; *, \gamma)$. If μ is a fuzzy AB-subalgebra of X if and only if for every t ∈ [γ,1], μ_t is an AB-subalgebra of X .
2- Let μ be a fuzzy AB-ideal of AB-algebra $(X;*, \gamma)$, μ is a fuzzy AB-ideal of X if and only if for every t ∈ [γ,1], μ_t is an AB-ideal of X .

Lemma 2.20([7]). Let μ be a fuzzy AB-ideal of AB-algebra X and if $\le y$, then $\mu(x) \ge \mu(y)$, for all $x, y \in X$.

Definition 2.21 ([27]). Let $f: (X; *, \gamma) \rightarrow (Y; *`, \gamma`)$be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by: $f(\mu)(y) =$
$$\begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if \ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ \gamma & otherwise \end{cases}$$
is said to be **the image of μ under f.**

Similarly if β is a fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$) is called **the pre-image of β under f** .

Definition 2.22 ([27]). A fuzzy subset μ of a set X has sup property if for any subset T of X , there exist $t_0 \in T$ such that $\mu(t_\gamma) = sup\{\mu(t) \mid t \in T\}$.

Proposition 2.23 ([7]). Let $f: (X; *, \gamma) \rightarrow (Y; *`, \gamma`)$ be a homomorphism between AB-algebras X and Y respectively.
1- For every fuzzy AB-subalgebra β of Y, $f^{-1}(\beta)$ is a fuzzy AB-subalgebra of X.
2- For every fuzzy AB-subalgebra μ of X , $f(\mu)$ is a fuzzy AB-subalgebra of Y.
3- For every fuzzy AB-ideal β of Y, $f^{-1}(\beta)$ is a fuzzy AB-ideal of X.
4- For every fuzzy AB-ideal μ of X with sup property, $f(\mu)$ is a fuzzy AB-ideal of Y, where f is onto.”

3. Interval-valued fuzzy AB-subalgebras of AB-algebra

In this section, we will introduce a new notion called interval-valued AB-subalgebras of AB-algebra and study several properties of it.

*	γ	1	2	3
γ	γ	γ	γ	γ
1	1	γ	1	γ
2	2	2	γ	γ
3	3	3	1	γ

Remark 3.1([8,11,23,26]).

An interval number is $\tilde{a} = [\, a^{-}, a^{+}]$, where γ ≤ a^{-}≤ a^{+}≤ 1. Let I be a closed unit interval, (i.e., I = [γ, 1]) . Let D[γ, 1] denote the family of all closed subintervals of I = [γ, 1] , that is, D[γ, 1] = { $\tilde{a} = [\, a^{-}, a^{+}]$ | a^{-}≤ a^{+}, for a^{-}, a^{+}∈ I} .

Now, we define what is known as refined minimum (briefly, rmin) of two element in D[γ,1].

Definition 3.2([8,11,23,26]).

We also define the symbols (≽) , (≼) , (=) , rmin and rmax in case of two elements in D[γ, 1] . Consider two interval numbers (elements numbers) $\tilde{a} = [\, a^{-}, a^{+}]$, $\tilde{b} = [\, b^{-}, b^{+}]$in D[γ, 1] : Then
(1) $\tilde{a}$≽ $\tilde{b}$ if and only if, a^{-}≥b^{-} and a^{+}≥b^{+},
(2) $\tilde{a}$≼ $\tilde{b}$ if and only if, a^{-}≤b^{-} and a^{+}≤b^{+},
(3) $\tilde{a}$= $\tilde{b}$ if and only if, a^{-}=b^{-} and a^{+}=b^{+},
(4) rmin {$\tilde{a}$, $\tilde{b}$}= [min {a^{-}, b^{-}}, min {a^{+}, b^{+}}],
(5) rmax {$\tilde{a}$, $\tilde{b}$}= [max {a^{-}, b^{-}}, max {a^{+}, b^{+}}],

Remark 3.3([8,11,23,26]).

It is obvious that (D[γ, 1] , ≼ , ∨ , ∧) is a complete lattice with $\tilde{\gamma}$ = [γ, γ] as its least element and $\tilde{1}$ = [1, 1] as its greatest element. Let $\tilde{a}_i$ ∈D[γ, 1] where i ∈Λ .
We define $r\ inf_{i\in\Lambda}\tilde{a}$= [$r\ inf_{i\in\Lambda}a^{-}$, $r\ inf_{i\in\Lambda}a^{+}$],
$r\ sup_{i\in\Lambda}\tilde{a}$= [$r\ sup_{i\in\Lambda}a^{-}$, $r\ sup_{i\in\Lambda}a^{+}$].

In what follows, let X denote an AB-algebra unless otherwise specified, we begin with the following definition.

Definition 3.4([8,11,23,26]).

An **interval-valued fuzzy subset** $\tilde{\mu}_A$ **on** $\boldsymbol{X}$ is defined as
$\tilde{\mu}_A$ = {< x, [μ_A^{-} (x) , μ_A^{+} (x)]>| x∈ X} . Where μ_A^{-} (x) ≤ μ_A^{+} (x), for all x∈ X. Then the ordinary fuzzy subsets μ_A^{-}: X → [γ, 1] and μ_A^{+}: X → [γ, 1] are called a **lower fuzzy subset and an upper fuzzy subset** of $\tilde{\mu}_A$ respectively.

Let $\tilde{\mu}_A$ (x) = [μ_A^{-} (x) , μ_A^{+} (x)] , $\tilde{\mu}_A$:X → D[γ, 1], then A = {< x, $\tilde{\mu}_A$ (x) >| x∈ X} .

Definition 3.5. An i-v fuzzy subset A in AB-algebra $(X;*,\gamma)$ is called an **i-v fuzzy AB-subalgebra of** $\boldsymbol{X}$ if $\tilde{\mu}_A$ (x *y) ≽ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}, for all $x, y \in X$.

Example 3.6. Let X = {γ, 1, 2, 3} in which the operation (as in example (*) be define by the following table:

Then $(X;*,\gamma)$ is an AB-algebra. Define a fuzzy subset μ: $X \rightarrow$ $[\gamma, 1]$ by :

$$\mu(x) = \begin{cases} \gamma.7 & if\ x \in \{\gamma, 1\} \\ \gamma.3 & otherwise \end{cases}.$$

I_1={γ,1} is an AB-ideal of X. Routine calculation given that μ is a fuzzy AB-ideal of X. Define $\tilde{\mu}_A$ (x) as follows: $\tilde{\mu}_A$ (x) = $\begin{cases} [\gamma.3, \gamma.9] & if\ x \in \{\gamma, 1\} \\ [\gamma.1, \gamma.6] & otherwise \end{cases}$.

It is easy to check that A is an i-v fuzzy AB-subalgebra.

Proposition 3.7. If A is an i-v fuzzy AB-subalgebra of X, then $\tilde{\mu}_A$ (γ) ≽ $\tilde{\mu}_A$ (x), for all $x \in X$.
Proof. For all $x \in X$, we have $\tilde{\mu}_A$ (γ) = $\tilde{\mu}_A$ (γ * x) ≽ rmin{$\tilde{\mu}_A$ (γ), $\tilde{\mu}_A$ (x)}
= rmin {[μ_A^{-} (γ) , μ_A^{+} (γ)], [μ_A^{-} (x) , μ_A^{+} (x)]}= rmin {[μ_A^{-} (x) , μ_A^{+} (x)]}= $\tilde{\mu}_A$ (x) .⌂

Proposition 3.8. Let A be an i-v fuzzy AB-subalgebra of X, if there exist a sequence { X_n} in X such that $\lim_{n\to\infty} \tilde{\mu}_A(x_n)$ = [1,1] , then $\tilde{\mu}_A$ (γ) = [1, 1].
Proof. By Proposition (3.7), we have $\tilde{\mu}_A$ (γ) ≽ $\tilde{\mu}_A$ (x) , for all $x \in X$. Then
$\tilde{\mu}_A$ (γ) ≽ $\tilde{\mu}_A$ (x_n) , for every positive integer n,
Consider the inequality [1,1] ≥ $\tilde{\mu}_A$ (γ) ≥ $\lim_{n\to\infty} \tilde{\mu}_A(x_n)$ = [1,1]. Hence $\tilde{\mu}_A$ (γ) = [1,1] .⌂

Theorem 3.9. An i-v fuzzy subset A = [μ_A^{-} , μ_A^{+}] of AB-algebra $(X;*,\gamma)$ is an i-v fuzzy AB-subalgebra of X if and only if μ_A^{-} and μ_A^{+} are fuzzy AB-subalgebras of X.
Proof. If μ_A^{-} and μ_A^{+} are fuzzy AB-subalgebras of , for any $x, y \in X$. Observe

μ_A^{-} $(x * y)$ ≥ min{ μ_A^{-} (x), μ_A^{-} (y)} and μ_A^{+} $(x * y)$ ≥ min{ μ_A^{+} (x), μ_A^{+} (y)}. Then

$\tilde{\mu}_A(x*y) = [\mu_A^-(x*y), \mu_A^+(x*y)]$
$\geq$ [min {μ_A^- (x), μ_A^- (y)} , min {μ_A^+ (x) , μ_A^+ (y)}]
= [min {μ_A^- (x), μ_A^+ (x)} , min { μ_A^- (y), μ_A^+ (y)}]
$\succcurlyeq$ rmin {[μ_A^- (x), μ_A^+ (x)], [μ_A^- (y), μ_A^+ (y)]}
= rmin {$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)} .

From what was mentioned above we can conclude that A is an i-v fuzzy AB-subalgebra of X .

Conversely, suppose that A is an i-v fuzzy AB-subalgebra of X. For all $x, y \in X$, we have $[\mu_A^-(x*y), \mu_A^+(x*y)] = \tilde{\mu}_A(x*y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}
= [min{ μ_A^- (x), μ_A^+ (x)}, min {μ_A^- (y), μ_A^+ (y)}]
= [min{ μ_A^- (x), μ_A^- (y)}, min{ μ_A^+ (x), μ_A^+ (y)}]

Therefore , $\mu_A^-(x*y) \geq$ min{ μ_A^- (x), μ_A^- (y)} and
$\mu_A^+(x*y) \geq$ min{ μ_A^+ (x), μ_A^+ (y)}.

|Hence, we get that μ_A^- and μ_A^+ are fuzzy AB-subalgebras of X. ⌂

Theorem 3.10. Let $(X;*,\gamma)$ be an AB-algebra and A be an i-v fuzzy subset of X. Then A is an i-v fuzzy AB-subalgebra of X if and only if, the nonempty set $\widetilde{U}$ (A;[δ_1,δ_2]):={ x∈ X |$\tilde{\mu}_A$ (x) $\succcurlyeq$ [δ_1,δ_2]} is an AB-subalgebra of X, for every [δ_1, δ_2]∈ D[γ, 1] . We call $\widetilde{U}$ (A ; [δ_1, δ_2]) the i-v level AB-subalgebra of A.

Proof. Assume that A is an i-v fuzzy AB-subalgebra of X and let [δ_1, δ_2] ∈ D[γ, 1] be such that x, y ∈$\widetilde{U}$ (A ; [δ_1, δ_2]), then
$\tilde{\mu}_A(x*y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)} $\succcurlyeq$ rmin{[δ_1, δ_2] , [δ_1, δ_2] } = [δ_1, δ_2] and so $(x*y) \in \widetilde{U}$ (A ; [δ_1, δ_2]) . Then $\widetilde{U}$ (A ; [δ_1, δ_2]) the i-v level AB-subalgebra of A.

Conversely, assume that $\widetilde{U}$ (A ; [δ_1, δ_2]) ≠ ∅ is an AB-subalgebra of X, for every [δ_1, δ_2] ∈ D[γ, 1] .In the contrary, suppose that there exist x_0 , y_0 ∈ X , such that
$\tilde{\mu}_A$ (x_0∗ y_0) ≺ rmin{$\tilde{\mu}_A$ (x_0), $\tilde{\mu}_A$ (y_0)} .

Let$\tilde{\mu}_A$ (x_0) = [γ_1, γ_2] ,$\tilde{\mu}_A$ (y_0) = [γ_3, γ_4] and $\tilde{\mu}_A$ (x_0∗ y_0) = [δ_1, δ_2]. If
[δ_1, δ_2] ≺ rmin{ [γ_1, γ_2], [γ_3, γ_4]} = min { min {γ_1, γ_3}, min {γ_2, γ_4}} .
So δ_1 < min {γ_1, γ_3} and δ_2 < min{γ_2, γ_4} . Consider

$$[\lambda_1, \lambda_2] = \frac{1}{2} \{ \tilde{\mu}_A(x_0 * y_0) + r\min\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\} \}$$

We find that

$$[\lambda_1, \lambda_2] = \frac{1}{2} \{[\delta_1, \delta_2] + r\min\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\}\}$$

$$= \frac{1}{2} \{(\delta_1 + \min\{\gamma_1, \gamma_3\}), (\delta_2 + \min\{\gamma_2, \gamma_4\})\}.$$

Therefore min {γ_1, γ_3} > λ_1 = $\frac{1}{2}$ (δ_1+ min{γ_1, γ_3}) > δ_1 ,

min {γ_2, γ_4} > λ_2 = $\frac{1}{2}$ (δ_2+ min{γ_2, γ_4}) > δ_2 .

Hence [min {γ_1, γ_3}, min {γ_2, γ_4}] > [λ_1 , λ_2] > [δ_1 , δ_2] = $\tilde{\mu}_A$ (x_0* y_0) ,
so that, (x_0∗ y_0) ∉ $\widetilde{U}$ (A ; [λ_1, λ_2]) . which is a contradiction , since
$\tilde{\mu}_A$ (x_0*y_0) = [γ_1, γ_2] ≥ [min{γ_1, γ_3} , min {γ_2, γ_4}] > [λ_1, λ_2] .
$\tilde{\mu}_A$ (y_0) =[γ_3, γ_4] ≥ [min{γ_1, γ_3} , min {γ_2, γ_4}] > [λ_1, λ_2] , imply that
x_0 , y_0 ∈$\widetilde{U}$ (A ; [λ_1, λ_2]). Then
$\tilde{\mu}_A(x*y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}, for all $x, y \in X$. ⌂

Theorem 3.13. Every AB-algebra of an AB-algebra $(X;*,\gamma)$ can be realized as an i-v level AB-subalgebra of an i-v fuzzy AB-subalgebra of X.

Proof. Let Y be an AB-subalgebra of X and let A be an i-v fuzzy subset on X defined by

$$\tilde{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & if x \in Y \\ [\gamma, \gamma] & otherwise \end{cases}.$$

Where α_1 , α_2 ∈ (γ, 1] with α_1 < α_2 . It is clear that $\widetilde{U}$ (A ; [α_1 , α_2]) = Y . We show that A is an i-v fuzzy AB-subalgebra of X. Let $x, y, z \in X$,
If x , y ∈ Y, then $(x*y) \in Y$, and therefore
$\tilde{\mu}_A(x*y)$ = [α_1 , α_2] $\succcurlyeq$ rmin{[α_1 , α_2],[α_1 ,α_2]} =rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}.
If x , y ∉Y , then $\tilde{\mu}_A$ (x) = [γ,γ] =$\tilde{\mu}_A$ (y) and so
$\tilde{\mu}_A(x*y)$ ≥ [γ,γ] = rmin{[γ,γ],[γ,γ]} $\succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)} ,
If x∈ Y and y ∉ Y , then $\tilde{\mu}_A$ (x) =[α_1 , α_2] and $\tilde{\mu}_A$ (y)= [γ,γ] , then $\tilde{\mu}_A(x*y) \succcurlyeq$ [γ,γ] =rmin{[α_1 , α_2],[γ,γ]} = rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}.
Similarly for the case x ∉ Y and y ∈ Y we get
$\tilde{\mu}_A(x*y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)}.

Therefore A is an i-v fuzzy AB-subalgebra of X, the proof is complete. ⌂

Theorem 3.14. If A is an i-v fuzzy AB-subalgebra of AB-algebra $(X;*,\gamma)$, then the set $X_{\tilde{M}_A}$ = {x∈ X |$\tilde{\mu}_A$ (x)=$\tilde{\mu}_A$ (γ)} is an AB-subalgebra of X.

Proof. Let x, y $\in X_{\tilde{M}_A}$. Then $\tilde{\mu}_A$ (x) = $\tilde{\mu}_A$ (γ) = $\tilde{\mu}_A$ (y) , and so $\tilde{\mu}_A$ $(x * y) \succcurlyeq$ rmin{$\tilde{\mu}_A$ (x), $\tilde{\mu}_A$ (y)} = rmin{$\tilde{\mu}_A$ (γ), $\tilde{\mu}_A$ (γ)} = $\tilde{\mu}_A$ (γ).

By Proposition (3.7), we get $\tilde{\mu}_A$ $(x * y)$ =$\tilde{\mu}_A$ (γ), that is $(x * y) \in X_{\tilde{M}_A}$. Hence $X_{\tilde{M}_A}$ is an AB-subalgebra of X. ⌂

4. Homomorphism of AB-algebra

Definition 4.1 ([26]). Let $f : (X;\ *, \gamma) \rightarrow (Y;\ *', \gamma')$ be a mapping from set X into a set Y. let B be an i-v fuzzy subset of Y. Then the inverse image of B, denoted by f^{-1} (B) , is an i-v fuzzy subset of X with the membership function given by

$\mu_{f^{-1}(B)}$ (x) =$\tilde{\mu}_B$ (f (x)), for all $x \in X$.

Proposition 4.2 ([26]). Let $f : (X;\ *, \gamma) \rightarrow (Y;\ *', \gamma')$ be a mapping from set X into a set , let $\tilde{m}$ = [m^- ,m^+],
and $\tilde{n}$ = [n^- ,n^+] be i-v fuzzy subsets of X and Y respectively. Then

(1) f^{-1} ($\tilde{n}$) = [f^{-1} (n^-), f^{-1} (n^+)] ,

(2) f ($\tilde{m}$) = [f (m^-), f (m^+)] .

Theorem 4.3. Let $f : (X;\ *, \gamma) \rightarrow (Y;\ *', \gamma')$ be homomorphism from an AB-algebra X into an AB-algebra Y. If B is an i-v fuzzy AB-subalgebra of Y, then the pre-image f^{-1} (B) of B is an i-v fuzzy AB-subalgebra of X.

Proof. Since B= [μ_B^-, μ_B^+] is an i-v fuzzy AB-subalgebra of Y, it follows that from Theorem (3.9), that (μ_B^-) and (μ_B^+) are fuzzy AB-subalgebras of Y. Using Proposition (2.23(1)), we know f^{-1} (μ_B^-) and f^{-1} (μ_B^+) are fuzzy AB-subalgebras of X. Hence by Proposition (4.2), we conclude that f^{-1} (B) = [f^{-1} (μ_B^-) , f^{-1} (μ_B^+)] is an i-v fuzzy AB-subalgebra of X. ⌂

Definition 4.4 ([26]). Let $f : (X;\ *, \gamma) \rightarrow (Y;\ *', \gamma')$ be a mapping from a set X into a set Y . let A be a an i-v fuzzy set of X, then the image of A, denoted by f (A), is the i-v fuzzy subset of Y with membership function denoted by :

$$\tilde{\mu}_{f(A)}(x)=\begin{cases} sup_{z\in f^{-1}(y)}\,\tilde{\mu}_A(z) & if\ \ f^{-1}(y)\neq\varphi, y\in Y \\ [\gamma,\gamma] & otherwise \end{cases},$$

where f^{-1} (y):={x∈ X | f (x) = y}.

Theorem 5.5. Let f be an epimorphism from an AB-algebra X into an AB-algebra Y . If A is an i-v fuzzy AB-subalgebra of X with sup property, then f (A) of A is an i-v fuzzy AB-subalgebra of Y.

Proof. Assume that A= [μ_A^-, μ_A^+] is an i-v fuzzy AB-subalgebra of X. it follows that from Theorem (3.9), that (μ_A^-) and (μ_A^+) are fuzzy AB-subalgebras of X. Using Proposition (2.23(2)), that the images f (μ_A^-) and f (μ_A^+) are fuzzy AB-subalgebras of Y. Hence by Proposition (4.2), we conclude that f (A)= [f (μ_A^-),f (μ_A^+)] is an i-v fuzzy AB-subalgebra of Y .⌂

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