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Review of n-inner Product Spaces and Generalized

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Abstract— This article review aims to consider $\mathfrak{n}$ –inner product of the spaces and study some important ideas about it. Also, we offer the notion of generalized $\mathfrak{n}$ – inner spaces of the product.

Keywords— n- spaces of normed , n-inner spaces of the product, inner space of the product, generalized n-inner spaces of the product.

1. Introduction

The idea of n-spaces of normed and $\mathfrak{n}$ –inner spaces of the product has been studied by Gunawan, note for example[7,9]. While Diminnie , Gahler ,and White from early researchers who introduced 2-inner spaces of the product , see for example [3,4]. Generalization n- inner of the product for $\mathfrak{n} \geq 2$ expanded by Misiak [10].

1.1 Note: We will symbolize for:
1) real-vector space by r. v-s; real –valued function by r. v-f ;
2) inner spaces of the product by i-p-s; $\mathfrak{n}$ -inner spaces of the product by $\mathfrak{n}$ -i-p-s;
3) generalized $\mathfrak{n}$ –inner spaces of the product by g- $\mathfrak{n}$ -i-p-s;
4) linearly dependent by l. d and linearly independent by l.id .

1.2 Definition[8]: Suppose that $\mathfrak{n} \in \mathbb{Z}_+$ and $\mathbb{W}$ is a r. v-s of dimension $\mathbb{d} \geq \mathfrak{n}$. A r. v-f $\langle .,.|.,\dots,.\rangle$ on $\mathbb{W}^{\mathfrak{n}+1}$ having the conditions:
(C1) $\langle \mathfrak{p}_1,\mathfrak{p}_1|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle > 0$; $\langle \mathfrak{p}_1,\mathfrak{p}_1|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle = 0 \leftrightarrow \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}$ are l. d;
(C2) $\langle \mathfrak{p}_1,\mathfrak{p}_1|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle = \langle \mathfrak{p}_{\mathfrak{h}_1},\mathfrak{p}_{\mathfrak{h}_1}|\mathfrak{p}_{\mathfrak{h}_2},\dots,\mathfrak{p}_{\mathfrak{h}_\mathfrak{n}}\rangle\ \forall$ permutation $(\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{n})$ of $(1,\dots,\mathfrak{n})$;
(C3) $\langle \mathfrak{p}_1,\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle = \langle \mathfrak{q},\mathfrak{p}_1|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle$;
(C4) $\gamma\langle \mathfrak{p}_1,\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle = \langle \gamma\mathfrak{p}_1,\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle$, $for\ any\ \gamma \in \mathbb{R}$;
(C5) $\langle \mathfrak{p}_1,\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle + \langle \mathfrak{y},\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle = \langle \mathfrak{p}_1+\mathfrak{y},\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle$,
defines an $\mathfrak{n}$ -i-p on $\mathbb{W}$ and $\mathfrak{n}$-i-p-s symbolized by $(\mathbb{W},\langle .,.|.,\dots,.\rangle)$.
The idea of 2- and $\mathfrak{n}$ – spaces of normed were previously studied by Gahler , see for example [5]

1.3 Definition[5]: : A function $\|.,\dots,.\|:\mathbb{W}^\mathfrak{n} \to \mathbb{R}$, with $\mathfrak{n} > 0$ and $\mathbb{W}$ is r. v-s with $\mathbb{d} \geq \mathfrak{n}$, having the following properties
(C1) $\|\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}\| = 0 \leftrightarrow \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}$ l.d;
(C2) $\|\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}\|$ is invariant under permutation ;
(C3) $|\gamma|\|\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}\| = \|\gamma\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}\|, for\ any\ \gamma \in \mathbb{R}$;
(C4) $\|\mathfrak{p}_1,\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\| + \|\mathfrak{q},\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\| \geq \|\mathfrak{p}_1+\mathfrak{q},\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\|$
defines an $\mathfrak{n}$ –times norm on $\mathbb{W}$, whereas n- space of normed is symbolized by $(\mathbb{W},\|.,,.\dots,.\|)$.

1.4 Note[8]: The relation between $\mathfrak{n}$ -i-p-s and $\mathfrak{n}$ –spaces of normed is defined by $\sqrt{\langle \mathfrak{p}_1,\mathfrak{p}_1|\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\rangle} \coloneqq \|\mathfrak{p}_1,\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}\|$

1.5 Definition [8]: Let $(\mathbb{W},\langle .,.|.,\dots,.\rangle_\mathfrak{n})$ be an n-i-p-s , for $\mathfrak{n} \geq 2$. Confirm a l.id set $\{\mathfrak{c}_1,\dots,\mathfrak{c}_\mathfrak{n}\} \in \mathbb{W}$. With reference to $\{\mathfrak{c}_1,\dots,\mathfrak{c}_\mathfrak{n}\}$, known $\forall \mathfrak{r} \in \{1,\dots,\mathfrak{n}-1\}$ the function $\langle .,.|.,\dots,.\rangle_{\mathfrak{n}-\mathfrak{r}}$ on $\mathbb{W}^{\mathfrak{n}-\mathfrak{r}+1}$ by $\langle \mathfrak{p},\mathfrak{q}|\mathfrak{p}_2,\dots,\mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}\rangle \coloneqq \sum_{\{\mathfrak{h}_1,\dots,\mathfrak{h}_\mathfrak{n}\}\subseteq\{1,\dots,\mathfrak{n}\}}\langle \mathfrak{p}|\mathfrak{q},\mathfrak{p}_2,\dots,\mathfrak{p}_{\mathfrak{n}-\mathfrak{r}},\mathfrak{c}_{\mathfrak{h}_1},\dots,\mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\rangle$.

The definition of generalized n-inner spaces of the product was explained by Malceski and Trencevski [11]. Note the following definition

1.6 Definition[11]: Let $\mathbb{W} \geq \mathfrak{n}$ is a r .v-s with $\mathfrak{n} > 0$ and $\langle .,\dots,.|.,\dots,.\rangle$ is a r .v-f on $\mathbb{W}^{2\mathfrak{n}}$ having the following properties:
(C1) If $\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}$ are l.id vectors then $\langle \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}\rangle > 0$;
(C2) $\langle \mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}|\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}\rangle = \langle \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle\ \forall \mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n},\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n} \in \mathbb{W}$;
(C3)$\gamma\langle \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle = \langle \gamma\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle\ for\ any\ \gamma\ \mathbb{R}\ and\ \forall \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n},\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n} \in \mathbb{W}$;
(C4) $-\langle \mathfrak{p}_{\propto(1)},\dots,\mathfrak{p}_{\propto(\mathfrak{n})}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle = \langle \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle, for\ all$ odd permutation $\propto \in \{1,\dots,\mathfrak{n}\}$ and $\forall \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n},\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n} \in \mathbb{W}$;
(C5) $\langle \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle + \langle \mathfrak{y},\mathfrak{p}_2,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle = \langle \mathfrak{p}_1+\mathfrak{y},\mathfrak{p}_2\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle$
$\forall \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n},\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n} \in \mathbb{W}$;
(C6)If$\langle \mathfrak{p}_1,\mathfrak{q}_1,\dots,\mathfrak{q}_{\mathfrak{h}-1},\mathfrak{q}_{\mathfrak{h}+1},\dots,\mathfrak{q}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle = 0\ \ \forall \mathfrak{h} \in \{1,\dots,\mathfrak{n}\}$,then $\langle \mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}|\mathfrak{q}_1,\dots,\mathfrak{q}_\mathfrak{n}\rangle = 0$ for optional vectors $\mathfrak{p}_1,\dots,\mathfrak{p}_\mathfrak{n}$. The function $\langle .,,.,\dots,.|.,.,\dots,.\rangle$ defines g-n-i-p and g-n-i-p-s symbolized by $(\mathbb{W},\langle .,\dots,.|.,\dots,.\rangle)$.

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The relationship between g-n-i-p on $\mathbb{W}$ and n-norms was defined by [10] as follows $\sqrt{\langle \mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{n} | \mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{n} \rangle} \coloneqq \|\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|$.
1.7 Note [8] : In any n-i-p-s when $\mathfrak{n} \geq 2$ have been derived an $(\mathfrak{n}-\mathfrak{r})$-i-p from the n-i-p $\forall\, \mathfrak{r} \in \{1, \dots, \mathfrak{n}-1\}$.
1.8 Notes [2]:
1) A g-n-i-p is sometimes symbolized by $\langle ., ., \dots, . | ., \dots, . \rangle_n$ and an n-norm by $\|., ., \dots, .\|_\mathfrak{n}$.
2) On n-space of normed simulated from g-n-i-p-s ,we have the following result
$\|\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_{\mathfrak{n}-1}\|_{\mathfrak{n}-1} = \sqrt{\sum_{\mathfrak{h}=1}^{\mathbb{d}} \|\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h}\|^2}$ defines an $(\mathfrak{n}-1)$- norm on $\mathbb{W}$.

2. The Main Results

2.1 Fact [8]: The function $\langle ., . | ., \dots, . \rangle_{\mathfrak{n}-\mathfrak{r}}$ determines an $(\mathfrak{n}-\mathfrak{r})$- i-p-s on $\mathbb{W}$ for all $\mathfrak{r} \in$ *the set* $\{1, \dots, \mathfrak{n}-1\}$ .Especially when $\mathfrak{r} = \mathfrak{n}-1$ we have $\langle \mathfrak{p}_1 | \mathfrak{q} \rangle \coloneqq \sum_{\{\mathfrak{h}_2, \dots, \mathfrak{h}_\mathfrak{n}\} \subseteq \{1, \dots, \mathfrak{n}\}} \langle \mathfrak{p}_1 | \mathfrak{q}, \mathfrak{c}_{\mathfrak{h}_2}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{n}} \rangle$ is an i-p on $\mathbb{W}$.
2.2 Corollary[8]: $\forall\, \mathfrak{r} = 1, \dots, \mathfrak{n}-1$, we get all n-i-p-s is an $(\mathfrak{n}-\mathfrak{r})$-i-p – s , especially an n- i-p-s is an i-p –s.
2.3 Corollary [8]: Purport $\|., \dots, .\|_\mathfrak{n}$ is the stimulated n-norms on $\mathbb{W}$. Then $\forall\, \mathfrak{r} \in \{1, \dots, \mathfrak{n}-1\}$, we get $\|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}\| \coloneqq \sqrt{\sum_{\{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r}\} \subseteq \{1, \dots, \mathfrak{n}\}} \|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\|^2}$ ….(*) determines an $(\mathfrak{n}-\mathfrak{r})$- a norm that agrees with $\langle ., . | ., \dots, . \rangle_{\mathfrak{n}-\mathfrak{r}}$ on $\mathbb{W}$. Especially $\|\mathfrak{p}_1\| \coloneqq \sqrt{\sum_{\{\mathfrak{h}_2, \dots, \mathfrak{h}_\mathfrak{n}\} \subseteq \{1, \dots, \mathfrak{n}\}} \|\mathfrak{p}_1, \mathfrak{c}_{\mathfrak{h}_2}, \dots, \mathfrak{c}_{\mathfrak{h}_n}\|^2}$, which determines a norm that agrees with the derived i-p $\langle ., . \rangle$ on $\mathbb{W}$.
2.4 Fact [8]: The derived $(\mathfrak{n}-\mathfrak{r})$-norm described in …(*) accepts
$2\sum_{\{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r}\} \subseteq \{1, \dots, \mathfrak{n}\}} \|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\|^2 + 2\sum_{\{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r}\} \subseteq \{1, \dots, \mathfrak{n}\}} \|\mathfrak{q}, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\|^2 = \sum_{\{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r}\} \subseteq \{1, \dots, \mathfrak{n}\}} \|\mathfrak{p}_1 + \mathfrak{q}, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\|^2 + \sum_{\{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r}\} \subseteq \{1, \dots, \mathfrak{n}\}} \|\mathfrak{p}_1 - \mathfrak{q}, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\|^2$, when the n-norm accepts the parallelogram law $\langle \mathfrak{p}_1 + \mathfrak{q}, \mathfrak{p}_1 + \mathfrak{q} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle + \langle \mathfrak{p}_1 - \mathfrak{q}, \mathfrak{p}_1 - \mathfrak{q} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle = 2\langle \mathfrak{p}_1, \mathfrak{p}_1 | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle + 2\langle \mathfrak{q}, \mathfrak{q} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle$.

2.5 Fact[8]: $\langle ., . | ., \dots, . \rangle_{\mathfrak{n}-\mathfrak{r}}$ determines an $(\mathfrak{n}-\mathfrak{r})$-i-p on $\mathbb{W}$.
2.6 Theorem (Hlawka′s inequality) [1]: The inequality $\left(\sqrt{\langle \mathfrak{q}+\mathfrak{w}, \mathfrak{q}+\mathfrak{w} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle} + \sqrt{\langle \mathfrak{p}_1+\mathfrak{w}, \mathfrak{p}_1+\mathfrak{w} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle} + \sqrt{\langle \mathfrak{p}_1+\mathfrak{q}, \mathfrak{p}_1+\mathfrak{q} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle}\right) \leq \left(\sqrt{\langle \mathfrak{p}_1, \mathfrak{p}_1 | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle} + \sqrt{\langle \mathfrak{q}, \mathfrak{q} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle} + \sqrt{\langle \mathfrak{w}, \mathfrak{w} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle} + \sqrt{\langle \mathfrak{p}_1+\mathfrak{q}+\mathfrak{w}, \mathfrak{p}_1+\mathfrak{q}+\mathfrak{w} | \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \rangle}\right)$ holds , when $\mathfrak{p}_1, \mathfrak{q}, \mathfrak{w}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n} \in \mathbb{W}$ are vectors got from an n-i-p –s *on* $\mathbb{W}$.
2.7 Theorem (Hornich´s inequality)[1]: Suppose that $\mathfrak{p}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}, \mathfrak{q}_1, \dots, \mathfrak{q}_\mathfrak{m} \in \mathbb{W}$ are vectors got from an n-i-p-s on $\mathbb{W}$ .The inequality $\sum_{\mathfrak{r}=1}^{\mathfrak{m}} (\|\mathfrak{q}_\mathfrak{r}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\| - \|\mathfrak{q}_\mathfrak{r}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|) \leq (\mathfrak{m}-2)\|\mathfrak{p}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|$ holds ,when $\sum_{\mathfrak{r}=1}^{\mathfrak{m}} \mathfrak{q}_\mathfrak{r} = -\mathfrak{s}\mathfrak{p}$ $(\mathfrak{s} \geq 1)$.
2.8 Theorem [1]: Suppose that $\mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}, \mathfrak{q}_1, \dots, \mathfrak{q}_\mathfrak{m} \in \mathbb{W}$ are vectors got from an n-i-p-s $\mathbb{W}$. Then $\sum_{1 \leq \mathfrak{h}_1 < \dots < \mathfrak{h}_\mathfrak{r} \leq \mathfrak{m}} \|\mathfrak{q}_{\mathfrak{h}_1} + \dots + \mathfrak{q}_{\mathfrak{h}_\mathfrak{r}}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\| \leq \binom{\mathfrak{m}-2}{\mathfrak{r}-2} \left(\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{r}-1} \sum_{\mathfrak{h}=1}^{\mathfrak{m}} \|\mathfrak{q}_\mathfrak{h}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\| + \|\sum_{\mathfrak{h}=1}^{\mathfrak{m}} \mathfrak{q}_\mathfrak{h}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|\right)$ when $\mathfrak{m} = 3,4, \dots$ *and* $\mathfrak{r} = 2, \dots, \mathfrak{m}-1$.
2.9 Theorem[1] : Suppose that $\mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}, \mathfrak{q}_1, \dots, \mathfrak{q}_\mathfrak{m} \in \mathbb{W}$ are vectors got from an n-i-p-s $\mathbb{W}$ .Then $\sum_{1 \leq \mathfrak{h}_1 < \dots < \mathfrak{h}_\mathfrak{r} \leq \mathfrak{m}} \|\mathfrak{q}_{\mathfrak{h}_1} + \dots + \mathfrak{q}_{\mathfrak{h}_\mathfrak{r}}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|^2 \leq \binom{\mathfrak{m}-2}{\mathfrak{r}-2} \left(\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{r}-1} \sum_{\mathfrak{h}=1}^{\mathfrak{m}} \|\mathfrak{q}_\mathfrak{h}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|^2 + \|\sum_{\mathfrak{h}=1}^{\mathfrak{m}} \mathfrak{q}_\mathfrak{h}, \mathfrak{p}_2, \dots, \mathfrak{p}_\mathfrak{n}\|^2\right)$ when $\mathfrak{m} = 3,4, \dots$ *and* $\mathfrak{r} = 2, \dots, \mathfrak{m}-1$.
2.10 Theorem [2]: Suppose that $(\mathbb{W}, \langle ., ., \dots, . | ., ., \dots, . \rangle_\mathfrak{n})$ is g- n-i-p-s with finite –dimension $\mathbb{d} \geq \mathfrak{n} \geq 2$.
Known the next function $\langle ., \dots, . | ., ., \dots, . \rangle_{\mathfrak{n}-1}$ *on* $\mathbb{W}^{2(\mathfrak{n}-1)}$ with a l.id set $\{\mathfrak{c}_1, \dots, \mathfrak{c}_\mathbb{d}\}$ through $\langle \mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-1} | \mathfrak{q}_1, \dots, \mathfrak{q}_{\mathfrak{n}-1} \rangle = \sum_{\mathfrak{h}=1}^{\mathbb{d}} \langle \mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h} | \mathfrak{q}_1, \dots, \mathfrak{q}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h} \rangle$, which accepts (C6) in definition 1.6 , then the function $\langle ., \dots, . | ., ., \dots, . \rangle_{\mathfrak{n}-1}$ is a g-$(\mathfrak{n}-1)$-i-p-s on $\mathbb{W}$.
2.11 Corollary [2]: Each g- n-i –p –s is g- $(\mathfrak{n}-\mathfrak{r})$ -i-p-s $\forall\, \mathfrak{r} = 1,2, \dots, \mathfrak{n}-1$, through induction for g- $(\mathfrak{n}-\mathfrak{r})$ -i-p-s $\langle \mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}} | \mathfrak{q}_1, \dots, \mathfrak{q}_\mathfrak{n} \rangle = \sum_{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r} \subseteq \{1, \dots, \mathbb{d}\}} \langle \mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}} | \mathfrak{q}_1, \dots, \mathfrak{q}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}} \rangle_\mathfrak{n}$,which accepts (C6) in definition 1.6 ,this case is necessary for $\mathfrak{r} = 1,2, \dots, \mathfrak{n}-2$, which trivial for $\mathfrak{r} = \mathfrak{n}-1$.
In especially,$\langle \mathfrak{p}, \mathfrak{q} \rangle = \sum_{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r} \subseteq \{1, \dots, \mathbb{d}\}} \langle \mathfrak{p}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_{\mathfrak{n}-1}} | \mathfrak{q}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_{\mathfrak{n}-1}} \rangle_\mathfrak{n}$

2.12 Corollary[2]: The function $\|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-1}\|_{\mathfrak{n}-1} = \sqrt{\sum_{\mathfrak{h}=1}^{\mathbb{d}} \|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h}\|^2}$ is an $(\mathfrak{n}-1)$ –norm that agrees with $\langle ., ., \dots, . | ., ., \dots, . \rangle_{\mathfrak{n}-1}$ on $\mathbb{W}$, When $\|., ., \dots, .\|_\mathfrak{n}$ is the stimulated n-norms from a g-n-i-p on $\mathbb{W}$. Adding by induction ,we get $\|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-1}\|_{\mathfrak{n}-\mathfrak{r}} = \sqrt{\sum_{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r} \subseteq \{1, \dots, \mathbb{d}\}} \|\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathfrak{n}-\mathfrak{r}}, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_\mathfrak{r}}\|^2}$.

In especially, $\|\mathfrak{p}_1\| = \sqrt{\sum_{\mathfrak{h}_1, \dots, \mathfrak{h}_\mathfrak{r} \in |1, \dots, \mathbb{d}|} \|\mathfrak{p}_1, \mathfrak{c}_{\mathfrak{h}_1}, \dots, \mathfrak{c}_{\mathfrak{h}_{\mathfrak{n}-1}}\|^2}$ determines a norm that agrees with the derived g- i- p on $\mathbb{W}$.

2.13 Theorem [2]: The $(\mathfrak{n}-1)$ –norm stimulated by g-n-i-p agrees $2\sum_{\mathfrak{h}=1}^{\mathbb{d}} \|\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h}\|^2 + 2\sum_{\mathfrak{h}=1}^{\mathbb{d}} \|\mathfrak{q}, \mathfrak{p}_2, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h}\|^2 = \sum_{\mathfrak{h}=1}^{\mathbb{d}} \|\mathfrak{p}_1 + \mathfrak{q}, \mathfrak{p}_2, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h}\|^2 + \sum_{\mathfrak{h}=1}^{\mathbb{d}} \|\mathfrak{p}_1 - \mathfrak{q}, \mathfrak{p}_2, \dots, \mathfrak{p}_{\mathfrak{n}-1}, \mathfrak{c}_\mathfrak{h}\|^2$. When the n-norm stimulated by g- n-i-p agrees the parallelogram law: $2\langle \mathfrak{p}_1, \ldots, \mathfrak{p}_n | \mathfrak{p}_1, \ldots, \mathfrak{p}_n \rangle + 2\langle \mathfrak{q}, \ldots, \mathfrak{p}_n | \mathfrak{q}, \ldots, \mathfrak{p}_n \rangle = \langle \mathfrak{p}_1 + \mathfrak{q}, \ldots, \mathfrak{p}_n | \mathfrak{p}_1 + \mathfrak{q}, \ldots, \mathfrak{p}_n \rangle + \langle \mathfrak{p}_1 - \mathfrak{q}, \ldots, \mathfrak{p}_n | \mathfrak{p}_1 - \mathfrak{q}, \ldots, \mathfrak{p}_n \rangle$. Adding the derived norm agrees

$$2\sum_{\mathfrak{h}_1,\ldots,\mathfrak{h}_r \in |1,\ldots,\mathbb{d}|} \left\| \mathfrak{p}_1, \mathfrak{c}_{\mathfrak{h}_1}, \ldots, \mathfrak{c}_{\mathfrak{h}_{n-1}} \right\|^2 + 2\sum_{\mathfrak{h}_1,\ldots,\mathfrak{h}_r \in |1,\ldots,\mathbb{d}|} \left\| \mathfrak{q}, \mathfrak{c}_{\mathfrak{h}_1}, \ldots, \mathfrak{c}_{\mathfrak{h}_{n-1}} \right\|^2 = \sum_{\mathfrak{h}_1,\ldots,\mathfrak{h}_r \in |1,\ldots,\mathbb{d}|} \left\| \mathfrak{p}_1 + \mathfrak{q}, \mathfrak{c}_{\mathfrak{h}_1}, \ldots, \mathfrak{c}_{\mathfrak{h}_{n-1}} \right\|^2 + \sum_{\mathfrak{h}_1,\ldots,\mathfrak{h}_r \in |1,\ldots,\mathbb{d}|} \left\| \mathfrak{p}_1 - \mathfrak{q}, \mathfrak{c}_{\mathfrak{h}_1}, \ldots, \mathfrak{c}_{\mathfrak{h}_{n-1}} \right\|^2.$$

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