ζ -Dot Cubic AB-ideals of AB-algebra

Dr. Areej Tawfeeq Hameed[1], Dr. Ahmed Hamzah Abed[2], Huda Adnan Mohammed[3]

[1]Department of Mathematics, Faculty of Education for Girls, University of Kufa, Najaf, Iraq.
E-mail: areej.tawfeeq@uokufa.edu.iq
[2]College of Islamic Sciences, AL Iraqia University, Baghdad, Iraq.
E-mail: ahmedh.abed@uokufa.edu.iq
[3]Department of Mathematics, Faculty of Education for Girls, University of Kufa, Najaf, Iraq.
E-mail: areej238@gmail.com

Abstract*—The paper introduces the concepts of ζ -dot cubic AB-ideals on AB-algebras, and then explores their many features. They are defined, and both the image and inverse image of them in AB-algebras are investigated.*

Keywords— AB-algebras, ζ -dot cubic AB-algebra, ζ -dot cubic AB-ideal, homomorphism of AB-algebra.

.

1. Introduction

K. Is'eki and S. Tanaka [22] looked into the ideals and congruences of BCK-algebras. KUS-algebras are a novel type of algebraic structure that were introduced and explored by S. M. Mostafa and coworkers [26]. L.A. Zadeh [30] first proposed the idea of a fuzzy set. Using the fuzzy set notion, O.G. Xi [28] described some of the characteristics of BCK-algebras. The concept of cubic ideals in BCK-algebras was proposed by Y. B. Jun and coauthors [23], who went on to examine some of the features associated with these ideals. Cubic KUS-ideals of KUS-algebra were first described by A.T. Hameed et al. in [21], and their homomorphisms were subsequently investigated. The concept of cubic AT-ideals of AT-algebra was first presented and some of its features were described by A.T. Hameed and coauthors in [1]. Here, we shall define the notion of ζ -dot cubic of AB-ideals, and we study some of the relations, theorems, propositions and examples of ζ -dot cubic of AB-ideals of AB-algebra.
We define and investigate the homomorphic and inverse images of AB-algebraic cubic AB-ideals.

2. Preliminaries

In this section, we introduce the concepts of AB-ideals and fuzzy AB-ideals in AB-algebra and provide some definitions and preliminary properties of these concepts.

Definition 2.1([2-4]) Let X be a set with a binary operation $*$ and a constant 0. Then $(X;*,\aleph)$ is called **an AB-algebra** if the following axioms satisfied: for all m, y, x ∈X ,
(i) ((m *y) *(x *y)) *(m *x) = ℵ ,
(ii) ℵ *m = ℵ ,
(iii) m *ℵ = m,

Example 2.2([2-4]) Let $X = \{\aleph, 1, 2, 3, 4\}$ in which $(*)$ is defined by the following table:

*	ℵ	1	2	3	4
ℵ	ℵ	ℵ	ℵ	ℵ	ℵ
1 a	1	ℵ	1	ℵ	ℵ
2	2	2	ℵ	ℵ	ℵ
3	3	3	1	ℵ	ℵ
4	4	3	4	3	ℵ

Then $(X;*,\aleph)$ is an AB-algebra.

Remark 2.3([2-4]) Define a binary relation ≤ on AB-algebra $(X;*,\aleph)$ by letting $x \leq y$ if and only if $x * y = 0$.

Proposition 2.4([2-4]) *In any AB − algebra* $(X;*,\aleph)$, *the following properties hold*: for all $x, y, z \in X$,
(1) (m * y) * m = 0.
(2) (m * y) * x = (m * x) * y.

(3) (m * (m * y)) * y = 0.

Proposition 2.5([2-4]) Let (X; *, ℵ) be an AB-algebra. X is satisfies for all m, y, x ∈X,

(1) m ≤ y implies m*x ≤ y*x.
(2) m ≤ y implies x*y ≤ x*m.

Definition 2.6([2-4]). Let (X; *,ℵ) be $an\ AB-algebra\ and\ let\ \ S\ be\ a\ nonempty\ subset\ of\ X.$ S is called an **AB-subalgebra of** X if m *y ∈ S whenever m ∈ S and y ∈ S.

Definition 2.7([2-4]). $A\ nonempty\ subset\ I\ of\ an\ AB-algebra$ (X; *,ℵ) is called **an AB-ideal of X** if it satisfies the following conditions: for any m, y,x ∈ X,

(I_1) ℵ ∈ I ,
(I_2) (m * y)*x ∈ I and y ∈ I imply m*x ∈ I.

Proposition 2.9 ([2-4]). $Every\ AB-ideal\ of\ AB-algebra\ is\ an\ AB-subalgebra.$

Proposition 2.8 ([2-4]). Let {I_i | i∈Λ} be a family of AB-ideals of AB-algebra (X; *, ℵ). The intersection of $any\ set\ of\ AB-ideals\ of\ X\ is\ also\ an$ AB-ideal.

Definition 2.9 ([13,14]). Let (X ; *,ℵ) and (Y; *`,ℵ `) $be\ nonempty\ sets. The\ mapping$ f:(X; *,ℵ) → (Y; *`,ℵ `) is called **a homomorphism** if it satisfies:

f (m*y) =f (m) *`f (y), for all m , y ∈ X. The set {m∈X | f (m) = ℵ '}is *called* ***the kernel of f*** *denoted* by ker f.

Theorem 2.10 ([2-4]). Let f:(X; *,ℵ) → (Y; *`,ℵ `)$be\ a\ homomorphism\ of\ an\ AB-algebra\ X\ into\ an\ AB$ −algebra Y, then :

A. f (ℵ) = ℵ '.
B. f is *injective* if and only if ker f ={ℵ }.
C. m ≤ y impliesf (m) ≤ f (y).

Theorem 2.11 ([2-4]). Let f:(X; *,ℵ) → (Y; *`,ℵ `) $be\ a\ homomorphism\ of\ an\ AB-algebra\ X\ into\ an\ AB$ −algebra Y, then:

(F_1) If S is an AB-subalgebra of X, $then\ f\ (S)\ is\ an\ AB-subalgebra\ of$ Y.
(F_2) If I is an AB-ideal of X, then f (I) i$s\ an\ AB-ideal\ of\ Y\ , where\ f\ is\ onto.$
(F_3) $If\ H\ \ is\ an\ AB-subalgebra\ of$ Y, thenf^{-1} (H) is an AB-subalgebra of X .
(F_4) If J is an AB-ideal of Y, then f^{-1} (J) $s\ an\ AB-ideal\ of\ X$.
(F_5) ker f is an AB-ideal of X.
(F_6) Im(f) $is\ an\ AB-subalgebra\ of\ Y.$

Definition 2.12([30]). Let (X; *,ℵ) $be\ a\ nonempty\ set, a\ fuzzy\ subset\ \ \mu$ of X is a function μ: X → [ℵ ,1].

Definition 2.13 ([29]). $Let\ X\ be\ a\ nonempty\ set\ and\ \mu\ \ be\ a\ fuzzy\ subset\ of$ (X; *,ℵ), for t ∈ [ℵ ,1] , the set $L(\mu,t) = \mu_t$ ={ m ∈ X | μ(m) ≥ t} *is called* ***a level subset of μ*** .

Definition 2.14([5]). Let $(X;*,$ℵ) $be\ an\ AB-algebra, a\ fuzzy\ subset\ \mu\ of\ X\ is\ called$ ***a fuzzy AB − subalgebra*** **of *X*** if for all m, y ∈ X, μ (m*y) ≥ min {μ (m), μ (y)} .

Definition 2.15([5]). Let $(X;*,$ℵ) be an AB-algebra, $a\ fuzzy\ subset\ \mu\ of\ X\ is\ called$ ***a fuzzy AB − ideal of X*** if it $satisfies\ the\ following\ conditions, for\ all\ m, y, z\ \in\ X,$

(FAB_1) μ (ℵ) ≥ μ (m) ,
(FAB_2) μ (m*x) ≥ min {μ ((m ∗ y)*x), μ (y)} .

Proposition 2.17([5]).

1- $The\ intersection\ of\ any\ set\ of\ fuzzy\ AB-ideals\ of$ AB-algebra is also fuzzy AB-ideal.
2- The union of any set of fuzzy AB-ideals of AB-algebra is also fuz$zy\ AB-ideal\ where\ is\ chain.$

Proposition 2.18([5]). E$very\ fuzzy\ AB-ideal\ of\ AB-algebra\ is\ a\ fuzz$y AB-subalgebra.

Proposition 2.19([5]).

1- Let μ be $a\ fuzzy\ subset\ of\ AB-algebra\ (X;*,$ℵ) . If $\mu\ is\ a\ fuzzy\ AB-subalgebra\ of\ X\ i$f and only if for every t ∈ [ℵ ,1], μ_t is an AB-subalgebra of X .
2- Let μ be a fuzzy AB-ideal of AB-algebra $(X;*,$ℵ), μ is a fuzzy AB − ideal of X if and only if for every t ∈ [ℵ ,1], μ_t is an AB-ideal of X .

Lemma 2.20([5]). Let μ be a fuzzy AB-ideal of AB-algebra X and if m ≤ y , then μ(m) ≥ μ(y) , for all m, y∈ X.

Definition 2.21 ([33]). Let $f:(X;*,$ℵ) → $(Y;*`,$ℵ `)be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by: $f(\mu)(y) =$

$$\begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if\ \ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ ℵ & otherwise \end{cases}$$

is said to be **the image of *μ* under *f*.**

Similarly if β is a fuzzy subset of Y , then the fuzzy subset $\mu\ =\ (\beta \circ f)$ of X (i.e the fuzzy subset defined by μ (m) = β (f (m)), for all

$x \in X$) is called **the pre-image of β under f**.

Definition 2.22 ([29]). A fuzzy subset μ of a set X has sup property if for any subset T of X, there exist $t_0 \in T$ such that $\mu(t_{\aleph}) = sup\{\mu(t) | t \in T\}$.

Proposition 2.23 ([5]). Let $f: (X; *, \aleph) \to (Y; *`, \aleph`)$ be a homomorphism between AB − algebras X and Y respectively.

1− For every fuzzy AB − subalgebra β of Y, $f^{-1}(\beta)$ is a fuzzy AB − subalgebra of X.

2- For every fuzzy AB − subalgebra μ of X, $f(\mu)$ is a fuzzy AB − subalgebra of Y.

3- For every fuzzy AB − ideal β of Y, $f^{-1}(\beta)$ is a fuzzy AB − ideal of X.

4- For every fuzzy AB − ideal μ of X with sup property, $f(\mu)$ is a fuzzy AB − ideal of Y, where f is onto.

Now, we will recall the concept of interval-valued fuzzy subsets.

Remark 2.24[1,8]. An interval number is $\tilde{a} = [a^-, a^+]$, where $\aleph \leq a^- \leq a^+ \leq 1$. Let I be a closed unit interval, (i.e., I = [ℵ, 1]).

Let D[ℵ, 1] denote the family of all closed subintervals of I = [ℵ, 1], that is, D[ℵ, 1] = { $\tilde{a} = [a^-, a^+] | a^- \leq a^+$, for $a^-, a^+ \in$ I}.

Now, we define what is known as refined minimum (briefly, rmin) of two element in D[ℵ, 1].

Definition 2.25[1,7]. We also define the symbols (≽), (≼), (=), rmin and rmax in case of two elements in D[ℵ, 1]. Consider two interval numbers (elements numbers)

$\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+]$ in D[ℵ, 1] : Then

(1) $\tilde{a} \succcurlyeq \tilde{b}$ if and only if, $a^- \geq b^-$ and $a^+ \geq b^+$,

(2) $\tilde{a} \preccurlyeq \tilde{b}$ if and only if, $a^- \leq b^-$ and $a^+ \leq b^+$,

(3) $\tilde{a} = \tilde{b}$ if and only if, $a^- = b^-$ and $a^+ = b^+$,

(4) rmin $\{\tilde{a}, \tilde{b}\}$ = [min $\{a^-, b^-\}$, min $\{a^+, b^+\}$],

(5) rmax $\{\tilde{a}, \tilde{b}\}$ = [max $\{a^-, b^-\}$, max $\{a^+, b^+\}$],

Remark2.26[1,7]. It is obvious that (D[ℵ, 1], ≼, ∨, ∧) is a complete lattice with $\tilde{\aleph}$ = [ℵ, ℵ] as its least element and $\tilde{1}$ = [1, 1] a sits greatest element. Let $\tilde{a}_i \in$ D[ℵ, 1] where i∈Λ. We define $\text{rinf}_{i\in\Lambda}\tilde{a}$ = [r $\inf_{i\in\Lambda} a^-$, $\text{rinf}_{i\in\Lambda} a^+$], $\text{rsup}_{i\in\Lambda}\tilde{a}$ = [$\text{rsup}_{i\in\Lambda} a^-$, $\text{rsup}_{i\in\Lambda} a^+$].

Definition 2.27[1,7]. An **interval − valued fuzzy subset $\tilde{\mu}_A$ on X** is defined as $\tilde{\mu}_A$ = {< m, [μ_A^-(m), μ_A^+(m)] >| m∈ X}. Where μ_A^-(m) $\leq \mu_A^+$(m), for all m ∈ X. Then the ordinary fuzzy subsets $\mu_A^-: X \to$ [ℵ, 1] and $\mu_A^+: X \to$ [ℵ, 1] are called a **lower fuzzy subset and an upper fuzzy subset** of $\tilde{\mu}_A$ respectively. Let $\tilde{\mu}_A$ (m) = [μ_A^-(m), μ_A^+(m)],

$\tilde{\mu}_A: X \to$ D[ℵ, 1], then A = {< m, $\tilde{\mu}_A$ (m) >| m∈ X }.

Definition 2.28([1,7]). Let $(X; *, \aleph)$ be a nonempty set. A cubic set Ω in a structure

Ω = {< m, $\tilde{\mu}_\Omega$ (m), λ_Ω (m)>| m∈ }, which is briefly denoted by Ω =< $\tilde{\mu}_\Omega$, λ_Ω >, where $\tilde{\mu}_\Omega : X \to$ D[ℵ, 1], $\tilde{\mu}_\Omega$ is an interval − valued fuzzy subset of X and

$\lambda_\Omega: X \to$ [ℵ, 1], λ_Ω is a fuzzy subset of X.

Definition 2.29([1,7]). For a family $\Omega_i = \{\langle m, \tilde{\mu}_{\Omega i}(m)\rangle | m \in X\}$ on fuzzy subsets of X, where i∈Λ and Λ is index set, we define the join (∨) and meet (∧) operations as follows:

$$\vee_{i\in\Lambda} \Omega_i = \left(\vee_{i\in\Lambda} \tilde{\mu}_{\Omega i}\right)(m) = \sup\{\tilde{\mu}_{\Omega i}(m) | i \in \Lambda\},$$

$$\wedge_{i\in\Lambda} \Omega_i = \left(\wedge_{i\in\Lambda} \tilde{\mu}_{\Omega i}\right)(m) = \inf\{\tilde{\mu}_{\Omega i}(m) | i \in \Lambda\},$$

3. ζ -Dot Cubic AB-ideals of AB-algebra

In this section, we shall define the notion of ζ − dot cubic of AB − ideals, and we study some of the relations, theorems, propositions and examples on ζ -dot cubic of AB-ideals of AB-algebra.

Definition 3.1.

Let $(X; *, \aleph)$ be an AB-algebra. A cubic set Ω =< $\tilde{\mu}_\Omega$ (m), λ_Ω(m) > of X is called **cubic AB-ideal of X** if, for all $m, y, z \in X$:

(1) $\tilde{\mu}_\Omega(\aleph) \succcurlyeq \tilde{\mu}_\Omega(x)$ and $\lambda_\Omega(\aleph) \leq \lambda_\Omega(m)$},

(2) $\tilde{\mu}_\Omega(m * z) \succcurlyeq rmin\{\tilde{\mu}_\Omega((m * y) * z), \tilde{\mu}_\Omega(y)\}$ and
$\lambda_\Omega(m * z) \leq max\{\lambda_\Omega((m * y) * z), \lambda_\Omega(y)\}$.

Definition 3.2.

Let $(X; *, \aleph)$ be an AB-algebra. A cubic set Ω^ζ =< $\tilde{\mu}_\Omega^\zeta$ (m), λ_Ω^ζ(m) > of is called **ζ − dot cubic AB − ideal of X** if it satisfies the following conditions: for all $x, y, z \in X$:

(1) $\tilde{\mu}_\Omega^\zeta(\aleph) \succcurlyeq \tilde{\mu}_\Omega^\zeta(m)$ and $\lambda_\Omega(\aleph) \leq \lambda_\Omega(m)$},

(2) $\tilde{\mu}_\Omega^\zeta(m * z) \succcurlyeq rmin\{\tilde{\mu}_\Omega^\zeta((m * y) * z), \tilde{\mu}_\Omega^\zeta(y)\}$ and

$\lambda_{\Omega}^{\zeta}(m*z) \leq max\{\lambda_{\Omega}^{\zeta}((m*y)*z), \lambda_{\Omega}^{\zeta}(y)\}$.

Example 3.3. Let $X = \{\aleph, 1,2,3\}$ in which the operation as in example $*$ be define by the following table:

$*$	ℵ	1	2	3
ℵ	ℵ	ℵ	ℵ	ℵ
1	1	ℵ	ℵ	ℵ
2	2	2	ℵ	ℵ
3	3	3	3	ℵ

Then $(X;*,\aleph)$ is an AB − algebra. Define a cubic set $\Omega = < \tilde{\mu}_{\Omega}(m), \lambda_{\Omega}(m) >$ of X is fuzzy subset $\mu: X \rightarrow [\aleph, 1]$ by:

$$\tilde{\mu}_{\Omega}(m) = \begin{cases} [0.3,0.9] & if x = \{\aleph, 2\} \\ [0.1,0.6] & otherwise \end{cases} \quad \text{and} \quad \lambda_{\Omega} = \begin{cases} 0.1 & if x = \{\aleph, 2\} \\ 0.6 & otherwise \end{cases}.$$

Define a cubic set $\Omega^{\zeta} = < \tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m) >$ of X and $\zeta = 0.4$ as follows:

$$\tilde{\mu}_{\Omega}^{\zeta}(m) = \begin{cases} [0.12,0.32] & if x = \{\aleph, 2\} \\ [0.04,0.24] & otherwise \end{cases} \quad \text{and} \quad \lambda_{\Omega}^{\zeta} = \begin{cases} 0.04 & if x = \{\aleph, 2\} \\ 0.24 & otherwise \end{cases}.$$

The ζ −dot cubic set $\Omega^{\zeta} = < \tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m) >$ is a ζ -dot cubic AB-ideal of X.

Theorem 3.4.

If $(X;*,\aleph)$ be an AB − algebra and $\Omega = < \tilde{\mu}_{\Omega}(m), \lambda_{\Omega}(m) >$ is a cubic AB − ideal of X , then $\Omega^{\zeta} = < \tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m) >$ is ζ −dot cubic AB − ideal of X, where $\zeta \in (\aleph, 1]$.

Proof :

Assume that Ω is a ζ -dot cubic AB − ideal of X and let $\zeta \in (\aleph, 1]$. Then for all $x, y, z \in X$.

$\tilde{\mu}_{\Omega}^{\zeta}(\aleph) = \tilde{\mu}_{\Omega}(\aleph).\zeta \geqslant \tilde{\mu}_{\Omega}(m).\zeta = \tilde{\mu}_{\Omega}^{\zeta}(m)$ and so $\tilde{\mu}_{\Omega}^{\zeta}(\aleph) \geqslant \tilde{\mu}_{\Omega}^{\zeta}(m)$.

$\lambda_{\Omega}^{\zeta}(\aleph) = \lambda_{\Omega}(\aleph).\zeta \leq \lambda_{\Omega}(m).\zeta = \lambda_{\Omega}^{\zeta}(m)$ and so $\lambda_{\Omega}^{\zeta}(\aleph) \leq \lambda_{\Omega}^{\zeta}(m)$.

$$\begin{aligned} \tilde{\mu}_{\Omega}^{\zeta}(m*x) &= \tilde{\mu}_{\Omega}(m*x).\zeta \\ &\geqslant min\{\tilde{\mu}_{\Omega}((m*y)*x), \tilde{\mu}_{\Omega}(y)\}.\zeta \\ &= min\{\tilde{\mu}_{\Omega}((m*y)*x).\zeta, \tilde{\mu}_{\Omega}(y).\zeta\} \\ &= min\{\tilde{\mu}_{\Omega}^{\zeta}((m*y)*x), \tilde{\mu}_{\Omega}^{\zeta}(y)\}. \quad \text{And} \end{aligned}$$

$$\begin{aligned} \lambda_{\Omega}^{\zeta}(m*x) &= \lambda_{\Omega}(m*x) \\ &\leq max\{\lambda_{\Omega}((m*y)*x), \lambda_{\Omega}(y)\}.\zeta \\ &= max\{\lambda_{\Omega}((m*y)*x).\zeta, \lambda_{\Omega}(y).\zeta\} \\ &= max\{\lambda_{\Omega}^{\zeta}((m*y)*x), \lambda_{\Omega}^{\zeta}(y)\}. \end{aligned}$$

Hence $\Omega^{\zeta} = < \tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m) >$ is a ζ -dot cubic AB − ideal of X . □

Proposition 3.5.

Let $(X;*,\aleph)$ be an AB − algebra and $\Omega = < \tilde{\mu}_{\Omega}(m), \lambda_{\Omega}(m) >$ is a cubic subset of X such that $\Omega^{\zeta} = < \tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m) >$ is ζ -dot cubic AB − ideal of X , for some $\zeta \in [\aleph, 1]$, then $\Omega = < \tilde{\mu}_{\Omega}(m), \lambda_{\Omega}(m) >$ is a cubic AB − ideal of X .

Proof.

Assume that $\Omega^{\zeta} = < \tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m) >$ is a ζ -dot cubic AB − subalgebra of X for some $\zeta \in (\aleph, 1]$. Let $m, y, z \in X$, then

$\tilde{\mu}_{\Omega}^{\zeta}(\aleph) = \tilde{\mu}_{\Omega}(\aleph).\zeta \geqslant \tilde{\mu}_{\Omega}(m).\zeta = \tilde{\mu}_{\Omega}^{\zeta}(m)$ and so $\tilde{\mu}_{\Omega}^{\zeta}(\aleph) \geqslant \tilde{\mu}_{\Omega}^{\zeta}(m)$.

$\lambda_{\Omega}^{\zeta}(\aleph) = \lambda_{\Omega}(\aleph).\zeta \leq \lambda_{\Omega}(m).\zeta = \lambda_{\Omega}^{\zeta}(m)$ and so $\lambda_{\Omega}^{\zeta}(\aleph) \leq \lambda_{\Omega}^{\zeta}(m)$.

$$\begin{aligned} \tilde{\mu}_{\Omega}(m*z)\cdot\zeta &= \tilde{\mu}_{\Omega}^{\zeta}(m*z) \\ &\geqslant min\{\tilde{\mu}_{\Omega}^{\zeta}((m*y)*x), \tilde{\mu}_{\Omega}^{\zeta}(y)\} \\ &= min\{\tilde{\mu}_{\Omega}((m*y)*x)\cdot\zeta, \tilde{\mu}_{\Omega}(y)\cdot\zeta\} \\ &= min\{\tilde{\mu}_{\Omega}((m*y)*x), \tilde{\mu}_{\Omega}(y)\}\cdot\zeta. \end{aligned}$$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 3, March - 2023, Pages: 7-16

$\tilde{\mu}_{\Omega}(m * z) \geq min\{\tilde{\mu}_{\Omega}((m * y) * x), \tilde{\mu}_{\Omega}(y)\}$ and so

$\lambda_{\Omega}(m * z) \cdot \zeta = \lambda^{\zeta}_{\Omega}(m * z)$

$\leq max\{\lambda^{\zeta}_{\Omega}((m * y) * x), \lambda^{\zeta}_{\Omega}(y)\}$

$= max\{\lambda_{\Omega}((m * y) * x) \cdot \zeta , \lambda_{\Omega}(y) \cdot \zeta\}$

$= max\{\lambda_{\Omega}((m * y) * x), \mu(y)\} \cdot \zeta$.

$\lambda_{\Omega}(m * z) \leq max\{\lambda_{\Omega}((m * y) * x), \lambda_{\Omega}(y)\}$

Hence $\Omega =< \tilde{\mu}_{\Omega}(m), \lambda_{\Omega}(m) >$ is a cubic AB − ideal of X. □

Proposition 3.6.

Let $(X ;*, ℵ)$ be an AB − algebra and $\Omega =< \tilde{\mu}_{\Omega}(m), \lambda_{\Omega}(m) >$ is a cubic subset of X such that $\Omega^{\zeta} =< \tilde{\mu}^{\zeta}_{\Omega}(m), \lambda^{\zeta}_{\Omega}(m) >$ is ζ -dot cubic AB − ideal of X , for some $\zeta \in [ℵ , 1]$, then then the cubic Ω of X is a fuzzy S − extension of the ζ -dot cubic Ω^{ζ} of X.

Proof:

Since $\tilde{\mu}_{\Omega}(m) \geqslant \mu(m). \zeta = \tilde{\mu}^{\zeta}_{\Omega}(x)$, and $\lambda_{\Omega}(m) \geq \lambda_{\Omega}(m). \zeta = \lambda^{\zeta}_{\Omega}(m)$ then $\Omega(m)$ is a fuzzy S − extension of $\Omega^{\zeta}(m)$, for all m∈ X and since Ω is a fuzzy AB − ideal of X, then Ω^{ζ} of μ is a ζ -dot cubic AB- ideal, by Proposition (4.4) . □

Proposition 3.6.

Let $(X ;*, ℵ)$ be an AB − algebra. A ζ -dot cubic subset $\Omega^{\zeta} =< \tilde{\mu}^{\zeta}_{\Omega}(m), \lambda^{\zeta}_{\Omega}(m) >$ of . If Ω^{ζ} is a ζ − dot cubic AB- ideal of X, then for all $\zeta \in (ℵ ,1]$, $\tilde{t} \in D[ℵ , 1]$ and $s \in [ℵ , 1]$, with $t \leq \zeta$, then the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is an AB- ideal of X.

Proof.

Assume that $\Omega^{\zeta} =< \tilde{\mu}^{\zeta}_{\Omega}(m), \lambda^{\zeta}_{\Omega}(m) >$ is a ζ -dot cubic AB − ideal of X and let $\tilde{t} \in D[ℵ , 1]$ and $s \in [ℵ , 1]$, be such that $\widetilde{U}(\Omega; \tilde{t}, s) \neq \emptyset$.

Let $m , y, z \in X$ such that $(m * y) * x , y \in \widetilde{U}(\Omega; \tilde{t}, s)$, then

$\tilde{\mu}^{\zeta}_{\Omega}((m * y) * x) \geqslant \tilde{t}$, $\tilde{\mu}^{\zeta}_{\Omega}(y) \geqslant \tilde{t}$ and $\lambda^{\zeta}_{\Omega}((m * y) * x) \leq s, \lambda^{\zeta}_{\Omega}(y) \leq s$.

Since Ω^{ζ} is a ζ −dot cubic AB − ideal of X, we get

$\tilde{\mu}^{\zeta}_{\Omega}(m * z) \geqslant rmin\{\tilde{\mu}^{\zeta}_{\Omega}((m * y) * x), \tilde{\mu}^{\zeta}_{\Omega}(y)\} \geqslant \tilde{t}$ and

$\lambda^{\zeta}_{\Omega}(m * x) \leq max\{\lambda^{\zeta}_{\Omega}((m * y) * x), \lambda^{\zeta}_{\Omega}(y)\} \leq s$.

Hence the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is an AB- ideal of X. □

Proposition 3.7.

Let $(X ;*, ℵ)$ be an AB − algebra. A ζ − dot cubic subset $\Omega^{\zeta} =< \tilde{\mu}^{\zeta}_{\Omega}(m), \lambda^{\zeta}_{\Omega}(m) >$ of . If the set $\widetilde{U}(\Omega; \tilde{t}, s)$ is an AB − ideal of X, for all $\zeta \in (ℵ ,1]$, $\tilde{t} \in D[ℵ , 1]$ and $s \in [ℵ , 1]$, with $t \leq \zeta$, then Ω^{ζ} is a ζ -dot cubic AB − ideal of X.

Proof.

Suppose that $\widetilde{U}(\Omega; \tilde{t}, s)$ is an AB − ideal of X and let $m , y, z \in X$ be such that

$\tilde{\mu}^{\zeta}_{\Omega}(m*x) \prec rmin\{\tilde{\mu}^{\zeta}_{\Omega}((m * y) * x), \tilde{\mu}^{\zeta}_{\Omega}(y)\}$ and $\lambda^{\zeta}_{\Omega}(m*x) > max\{\lambda^{\zeta}_{\Omega}((m * y) * x), \lambda^{\zeta}_{\Omega}(y)\}$.

Consider $\tilde{\zeta} = 1/2 \{ \tilde{\mu}^{\zeta}_{\Omega}(m*x) + rmin\{\tilde{\mu}^{\zeta}_{\Omega}((m * y) * x), \tilde{\mu}^{\zeta}_{\Omega}(y)\} \}$ and

$\zeta = 1/2 \{ \lambda^{\zeta}_{\Omega}(m*x) + max\{\lambda^{\zeta}_{\Omega}((m * y) * x), \lambda^{\zeta}_{\Omega}(y)\}\}$.

We have $\tilde{\zeta} \in D[ℵ , 1]$ and $\zeta \in (ℵ , 1]$, and

$\tilde{\mu}^{\zeta}_{\Omega}(m*x) \prec \tilde{\zeta} \prec rmin\{\tilde{\mu}^{\zeta}_{\Omega}((m * y) * x), \tilde{\mu}^{\zeta}_{\Omega}(y)\}$, and

$\lambda^{\zeta}_{\Omega}(m*x) > \zeta > max\{\lambda^{\zeta}_{\Omega}((m * y) * x), \lambda^{\zeta}_{\Omega}(y)\}$.

It follows that $(x * y) * z, y \in \widetilde{U}(\Omega; \tilde{t}, s)$, and $(m * x) \notin \widetilde{U}(\Omega; \tilde{t}, s)$. *This is a contradiction and therefore* Ω^{ζ} is a ζ -dot cubic AB- ideal of X . □

Theorem 3.8. Let $(X ;*, ℵ)$ *be an AB − algebra*. A ζ -dot cubic subset $\Omega^{\zeta} =< \tilde{\mu}^{\zeta}_{\Omega}(m), \lambda^{\zeta}_{\Omega}(m) >$ of X is a ζ -dot cubic AB − ideal of X if and only if, $\mu^{-\zeta}_{\Omega}$ and $\mu^{+\zeta}_{\Omega}$ are fuzzy AB-ideals of X and λ^{ζ}_{Ω} *are anti − fuzzy* AB-ideal of X.

Proof.

Let $\mu^{-\zeta}_{\Omega}, \mu^{+\zeta}_{\Omega}$ and λ^{ζ}_{Ω} be fuzzy AB − ideals of X and $m , y, z \in X$, then

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 3, March - 2023, Pages: 7-16

$\mu^{-\zeta}_{\Omega}$ (m ∗ x) ≥ min{$\mu^{-\zeta}_{\Omega}$ ((m ∗ y) ∗ x),$\mu^{-\zeta}_{\Omega}$(y)},
$\mu^{+\zeta}_{\Omega}$ (m ∗x)≥ min{$\mu^{+\zeta}_{\Omega}$ ((m ∗ y) ∗ x),$\mu^{+\zeta}_{\Omega}$ (y)} and
λ^{ζ}_{Ω}(m ∗ x) ≤ max{λ^{ζ}_{Ω} ((m ∗ y) ∗ x), λ^{ζ}_{Ω} (y)}. Now,
$\tilde{\mu}^{\zeta}_{\Omega}$ (m ∗ x) = [$\mu^{-\zeta}_{\Omega}$(m ∗ x), $\mu^{+\zeta}_{\Omega}$(m ∗ x)]
≽ [min{$\mu^{-\zeta}_{\Omega}$((m ∗ y) ∗ x),$\mu^{-\zeta}_{\Omega}$(y)}, min{$\mu^{+\zeta}_{\Omega}$((m ∗ y) ∗ x),$\mu^{+\zeta}_{\Omega}$ (y)}]
= rmin{[$\mu^{-\zeta}_{\Omega}$((m ∗ y) ∗ x), $\mu^{+\zeta}_{\Omega}$ ((m ∗ y) ∗ x)], [$\mu^{-\zeta}_{\Omega}$ (y), $\mu^{+\zeta}_{\Omega}$(y)]}
= rmin{$\tilde{\mu}^{\zeta}_{\Omega}$ (m),$\tilde{\mu}^{\zeta}_{\Omega}$ (y)}, therefore Ω is a ζ -dot cubic AB-ideal of X.

Conversely, assume that Ω^{ζ} is a ζ -dot cubic AB-*ideal of* X, *for any* $x , y \in X$,
[$\mu^{-\zeta}_{\Omega}$ (m ∗ x), $\mu^{+\zeta}_{\Omega}$ (m ∗ x)]=$\tilde{\mu}^{\zeta}_{\Omega}$(m ∗ x) ≽ rmin{$\tilde{\mu}^{\zeta}_{\Omega}$ ((m ∗ y) ∗ x),$\tilde{\mu}^{\zeta}_{\Omega}$ (y)}
= rmin{[$\mu^{-\zeta}_{\Omega}$ ((m ∗ y) ∗ x), $\mu^{+\zeta}_{\Omega}$ ((m ∗ y) ∗ x)], [$\mu^{-\zeta}_{\Omega}$ (y), $\mu^{+\zeta}_{\Omega}$(y)]}
= [min{$\mu^{-\zeta}_{\Omega}$ ((m ∗ y) ∗ x), μ^{-}_{Ω}((m ∗ y) ∗ x)}, min{$\mu^{+\zeta}_{\Omega}$(y), $\mu^{+\zeta}_{\Omega}$ (y)}].
Thus
$\mu^{-\zeta}_{\Omega}$ (m ∗ x) ≥ min {$\mu^{-\zeta}_{\Omega}$ ((m ∗ y) ∗ x), $\mu^{-\zeta}_{\Omega}$((m ∗ y) ∗ x)},
$\mu^{+\zeta}_{\Omega}$ (m ∗ y) ≥ min{ $\mu^{+\zeta}_{\Omega}$(m), $\mu^{+\zeta}_{\Omega}$ (m)} and
λ^{ζ}_{Ω} (m ∗ x)≤ max{λ^{ζ}_{Ω}((m ∗ y) ∗ x),λ^{ζ}_{Ω}(y)} ,
Therefore, $\mu^{-\zeta}_{\Omega}$ and $\mu^{+\zeta}_{\Omega}$ are fuzzy AB − ideals of X and λ^{ζ}_{Ω} are anti − fuzzy AB − ideal of X. □

Proposition 3.9.
Let (X ;∗, ℵ) be an AB − algebra and Ω =< $\tilde{\mu}_{\Omega}$ (m), λ_{Ω}(m) > is a cubic AB − ideal of X and $\zeta_1, \zeta_2 \in$ (ℵ ,1]. If $\zeta_2 \geq \zeta_1$, then the ζ -d*ot cubic AB − ideal* $\Omega^{\zeta 2}$ is a fuzzy *S − extension of th*e $\Omega^{\zeta 1}$ ζ -dot cubic AB-ideal of X.
Proof:
For every m∈ X and $\zeta_1, \zeta_2 \in$ (ℵ ,1] and $\zeta_2 \geq \zeta_1$,we have
$\tilde{\mu}^{\zeta 2}_{\Omega}$ $(m) = \tilde{\mu}_{\Omega}(m).\zeta_2 \succcurlyeq \tilde{\mu}_{\Omega}(m).\zeta_1 = \tilde{\mu}^{\zeta 1}_{\Omega}(m)$, and
$\lambda^{\zeta 2}_{\Omega}(m) = \lambda_{\Omega}(m).\zeta_2 \geq \lambda_{\Omega}(m).\zeta_1 = \lambda^{\zeta 1}_{\Omega}(m)$, then
$\tilde{\mu}^{\zeta 2}_{\Omega}$ (m) ≽ $\tilde{\mu}^{\zeta 1}_{\Omega}$ (m), and $\lambda^{\zeta 2}_{\Omega}$ (m) ≥ $\lambda^{\zeta 1}_{\Omega}$ (m), therefore $\Omega^{\zeta 2}$ is a fuzzy S − extension of $\Omega^{\zeta 1}$.
Since Ω is a cubic AB − ideal of X, then Ω^{ζ} is a ζ -dot cubic AB − ideal of μ, by Proposition (4.4). Hence $\Omega^{\zeta 2}$ of X is a fuzzy S − extension of the ζ -dot cubic AB-ideal $\Omega^{\zeta 1}$ of X. □

Theorem 3.10.
Every ζ − dot cubic AB − ideal of AB-algebra (X; ∗, ℵ) is a ζ -dot cubic AB − subalgebra of an AB-algebra (X ;∗, ℵ) .
Proof: Let (X ;∗, ℵ) *be an AB − algebra and* Ω =< $\tilde{\mu}_{\Omega}$ (m), λ_{Ω}(m) > *is a cubic AB − ideal of X and* Ω^{ζ} =< $\tilde{\mu}^{\zeta}_{\Omega}$ (m), λ^{ζ}_{Ω}(m) > is a ζ -dot cubic subset of .
Since Ω^{ζ} is an ζ -dot *cubic AB − ideal of X, then by Propositio*n (4.6) , *for every* ζ ∈ (ℵ ,1] , $\tilde{t}$∈ D[ℵ , 1] and s ∈ [ℵ ,1] ,
$\tilde{U}$ $(\Omega; \tilde{t}, s)$ = { m∈ X | $\tilde{\mu}_{\Omega}$ (m) ≽ $\tilde{t}, \lambda_{\Omega}$(m) ≤ s}, is AB-ideal of X .
By Proposition (2.9), for every ζ ∈ (ℵ ,1] , $\tilde{t}$∈ D[ℵ , 1] and s ∈ [ℵ ,1], $\tilde{U}$ (Ω; $\tilde{t}$, s) is AB − subgalgebra of X . □
Hence μ is a ζ -dot cubic AB − subalgebra of X by Proposition (3.12). □
Remark 3.11. The converse of proposition (4.10) is not true as the following example:
Example 3.12. Let X = {ℵ , 1, 2, 3, 4} in which (∗) is defined by the following table:

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 7 Issue 3, March - 2023, Pages: 7-16

$*$	א	1	2	3	4
א	א	א	א	א	א
1	1	א	א	א	א
2	2	א	א	א	א
3	3	2	1	א	א
4	4	3	4	3	א

Then $(X;*,\aleph)$ *is an AB − algebra. Define a cubic* set $\Omega =< \tilde{\mu}_\Omega(m), \lambda_\Omega(m) >$ of X *is fuzzy subset* $\mu: X\to [\aleph,1]$ by:

$\tilde{\mu}_\Omega(m) = \begin{cases}[0.3,0.9] & if x = \{\aleph,1,2\}\\ [0.1,0.6] & otherwise\end{cases}$ and $\lambda_\Omega = \begin{cases}0.1 & if x = \{\aleph,1,2\}\\ 0.6 & otherwise\end{cases}$.

Define a ζ − dot cubic set $\Omega^\zeta =< \tilde{\mu}^\zeta_\Omega(m), \lambda^\zeta_\Omega(m) >$ of X and $\zeta = 0.4$ as follows:

$\tilde{\mu}^\zeta_\Omega(m) = \begin{cases}[0.12,0.32] & if x = \{\aleph,1,2\}\\ [0.04,0.24] & otherwise\end{cases}$ and $\lambda^\zeta_\Omega = \begin{cases}0.04 & if x = \{\aleph,1,2\}\\ 0.24 & otherwise\end{cases}$.

The ζ *− dot cubic set* $\Omega^\zeta =< \tilde{\mu}^\zeta_\Omega(m), \tilde{\mu}^\zeta_\Omega(m) >$ *is not a* ζ *− dot cubic AB − subalgebra* of X.
Note that λ_Ω is not *an anti − fuzzy AB*-ideal of X since
$\lambda_\Omega(4*2) = \lambda_\Omega(4) = 0.24 > 0.04 = \max\{\lambda_\Omega((4*1)*2), \lambda_\Omega(1)\}$
$= \max\{\lambda_\Omega(3*2), \lambda_\Omega(1)\} = \max\{\lambda_\Omega(1), \lambda_\Omega(1)\} = \lambda_\Omega(1)$.
Hence Ω^ζ is not ζ -*dot cubic AB − ideal of* X.

4. Homomorphism of ζ -Dot Cubic AB-ideals (AB-subalgebras) of AB-algebra

In this section, we will present *some results on images* and preimages of ζ -*dot cubic AB − ideals of AB − algebras*.

Definition 4.1[3].

Let : $(X;*,\aleph) \to (Y;*',\aleph')$ *be a mapping from the set* X to a set Y. If $\Omega^\zeta =< \tilde{\mu}^\zeta_\Omega(m), \lambda^\zeta_\Omega(m) >$ is a ζ -*dot cubic subset of* X, *then the cubic subset* $\beta =<\tilde{\mu}_\beta, \lambda_\beta>$ of Y defined by:

$$f(\tilde{\mu}^\zeta_\Omega)(y) = \begin{cases} \underset{m\in f^{-1}(y)}{rsup} \tilde{\mu}^\zeta_\Omega(x) if f^{-1}(y) = \{m \in X, f(m) = y\} \neq \varphi \\ \aleph \quad otherwise\end{cases}$$
$$f(\lambda^\zeta_\Omega)(y) = \begin{cases} \underset{m\in f^{-1}(y)}{inf} \lambda^\zeta_\Omega(m) if f^{-1}(y) = \{m \in X, f(m) = y\} \neq \varphi \\ 1 \quad otherwise\end{cases}$$

is said to be **the image of Ω under f**.

Similarly if $\beta^\zeta =< \tilde{\mu}^\zeta_\beta(m), \lambda^\zeta_\beta(m) >$ is a ζ -dot cubic subset of Y , then the cubic subset $\Omega^\zeta = (\beta^\zeta \circ f)$ in X (i.e the ζ -dot cubic subset defined by
$\tilde{\mu}^\zeta_\Omega(m) = \tilde{\mu}^\zeta_\beta(f(m))$, $\lambda^\zeta_\Omega(m) = \lambda^\zeta_\beta(f(m))$ for all $m \in X$) is called **the preimage of β under *f***).

Theorem 4.2.

A homomorphic pre − image of ζ -dot cubic AB-ideal is also ζ -dot cubic AB − ideal.

Proof.

Let $f: (X;*,\aleph) \to (Y;*',\aleph')$ be homomorphism from an AB-algebra X into an AB − algebra Y.
If $\beta^\zeta =< \tilde{\mu}^\zeta_\beta(m), \lambda^\zeta_\beta(m) >$ is a ζ -dot cubic AB − ideal of Y and $\Omega^\zeta =< \tilde{\mu}^\zeta_\Omega(m), \lambda^\zeta_\Omega(m) >$the pre − image of β^ζ under f, then $\tilde{\mu}^\zeta_\Omega(m) = \tilde{\mu}^\zeta_\beta(f(m))$, $\lambda^\zeta_\Omega(m) = \lambda^\zeta_\beta(f(m))$, for all $x \in X$. Let $m \in X$, then
$(\tilde{\mu}^\zeta_\Omega)(\aleph) = \tilde{\mu}^\zeta_\beta(f(\aleph)) \succcurlyeq \tilde{\mu}_\beta(f(m)) = \tilde{\mu}^\zeta_\Omega(m)$, and $(\lambda^\zeta_\Omega)(\aleph) = \lambda^\zeta_\beta(f(\aleph)) \leq \lambda^\zeta_\beta(f(m)) = \lambda^\zeta_\Omega(m)$.

Now, let $x, y, z \in X$, then
$\tilde{\mu}_{\Omega}^{\zeta}$ (m∗x) = $\tilde{\mu}_{\beta}^{\zeta}$ (f (m∗x))⩾rmin {$\tilde{\mu}_{\beta}^{\zeta}$ (f ($(m * y) * z$),$\tilde{\mu}_{\beta}^{\zeta}$ (f (y)) }
=rmin {$\tilde{\mu}_{\Omega}^{\zeta}$(m∗ (y*x)),$\tilde{\mu}_{\Omega}^{\zeta}$ (y)}, and
λ_{Ω}^{ζ} (m∗x) = λ_{β}^{ζ} (f (m∗x))≤ max {λ_{β}^{ζ} (f ($(m * y) * z$),λ_{β}^{ζ} (f (y))}
= max {$\lambda_{\Omega}^{\zeta}((m * y) * z)$,$\lambda_{\Omega}^{\zeta}$ (y)} . □

Definition 4.3. Let $f: (X;*,\aleph) \rightarrow (Y;*',\aleph')$ be a mapping from a set X into a set Y. $\Omega^{\zeta} =<\tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m)>$ is a ζ -dot cubic subset of X **has sup and inf properties** if for any subset T of X, there exist t, s ∈ T such that

$\tilde{\mu}_{\Omega}^{\zeta}(t) = \underset{t0 \in T}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t0)$ and $\lambda_{\Omega}^{\zeta}(s) = \underset{s0 \in T}{inf}\, \lambda_{\Omega}^{\zeta}(s0)$.

Theorem 4.5.
Let $: (X;*,\aleph) \rightarrow (Y;*',\aleph')$ be an epimorphism from an AB-algebra X into an AB-algebra Y. For every ζ -dot cubic AB-ideal $\Omega^{\zeta} =<\tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m)>$ of X with **sup and inf properties**, then $f(\Omega^{\zeta})$ is a ζ -dot cubic AB-ideal of Y.

Proof.
By definition $\tilde{\mu}_{\beta}^{\zeta}(y') = f(\tilde{\mu}_{\Omega}^{\zeta})(y') = \underset{x \in f^{-1}(y')}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(x)$ and
$\lambda_{\beta}^{\zeta}(y') = f(\lambda_{\Omega}^{\zeta})(y') = \underset{x \in f^{-1}(y')}{inf}\, \lambda_{\Omega}^{\zeta}(m)$ for all y' ∈ Y and
rsup(∅) = [ℵ , ℵ] and inf (∅) = ℵ . We have prove that
$\tilde{\mu}_{\beta}^{\zeta}$ (m'∗x') ⩾rmin {$\tilde{\mu}_{\beta}^{\zeta}$ ((m' ∗y')*x') , $\tilde{\mu}_{\beta}^{\zeta}$ (y')}, and
λ_{β}^{ζ} (m'∗x')≤ max{λ_{β}^{ζ} ((m' ∗y')*x') , λ_{β}^{ζ} (y')}, for all m', y' , x' ∈ Y .
Let $f : (X;*,\aleph) \rightarrow (Y;*',\aleph')$ be epimorphism of AB-algebras,
$\Omega^{\zeta} =<\tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m)>$is a ζ -dot cubic AB-ideal of X has sup and inf properties and $\beta^{\zeta} =<\tilde{\mu}_{\beta}^{\zeta}(m), \lambda_{\beta}^{\zeta}(m)>$ the image of Ω under f.
Since $\Omega^{\zeta} =<\tilde{\mu}_{\Omega}^{\zeta}(m), \lambda_{\Omega}^{\zeta}(m)>$ is a ζ −dot cubic AB − ideal of X, we have
($\tilde{\mu}_{\Omega}^{\zeta}$)(ℵ) ⩾ $\tilde{\mu}_{\Omega}^{\zeta}$ (m), and λ_{Ω}^{ζ} (ℵ) ≤ λ_{Ω}^{ζ} (m), for all m ∈X .
Note that, ℵ $\in f^{-1}$ (ℵ ') where ℵ ,ℵ ' are the zero of X and Y, respectively.
Thus

$$\tilde{\mu}_{\beta}(\aleph') = \underset{t \in f^{-1}(\aleph')}{rsup}\, \tilde{\mu}_{\Omega}(t) = \tilde{\mu}_{\Omega}(\aleph)$$
$$\geqslant \tilde{\mu}_{\Omega}(m) = \underset{t \in f^{-1}(x')}{rsup}\, \tilde{\mu}_{\Omega}(t) = \tilde{\mu}_{\beta}(x'), \text{ and}$$
$$\lambda_{\beta}(\aleph') = \underset{t \in f^{-1}(\aleph')}{inf}\, \lambda_{\Omega}^{\zeta}(t) = \lambda_{\Omega}^{\zeta}(\aleph)$$
$$\leq \lambda_{\Omega}^{\zeta}(x) = \underset{t \in f^{-1}(x')}{inf}\, \lambda_{\Omega}^{\zeta}(t) = \lambda_{\beta}^{\zeta}(x')$$,for all m ∈X , which implies that

$\tilde{\mu}_{\beta}^{\zeta}$ (ℵ ') ⩾ $\tilde{\mu}_{\beta}^{\zeta}(x')$, and $\lambda_{\beta}^{\zeta}(\aleph') \leq \lambda_{\beta}^{\zeta}(x')$, for all m' ∈Y .
For any m' , y' , x' ∈ Y, let $x_0 \in f^{-1}$(m') , $y_0 \in f^{-1}$(y') , and $z_0 \in f^{-1}$(x')be such that

$$\lambda_{\beta}^{\zeta}(y') = f(\lambda_{\Omega}^{\zeta})(y') = \underset{x \in f^{-1}(y')}{inf}\, \lambda_{\Omega}^{\zeta}(x)$$

$\tilde{\mu}_{\Omega}^{\zeta}((m_{\aleph} * y_{\aleph}) * z_{\aleph}) = \underset{t \in f^{-1}((x' * y') * z'))}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t)$, and
$\tilde{\mu}_{\Omega}^{\zeta}(y_{\aleph}) = \underset{t \in f^{-1}(y')}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t)$. then

$$\tilde{\mu}_{\Omega}^{\zeta}(m_{\aleph} * z_{\aleph}) = \tilde{\mu}_{\beta}^{\zeta}(f(x_{\aleph} * z_{\aleph}))$$
$$= \tilde{\mu}_{\beta}^{\zeta}(m' * z')$$
$$= \underset{(m_{\aleph} * z_{\aleph}) \in f^{-1}(m' * z')}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(m_{\aleph} * z_{\aleph})$$
$$= \underset{t \in f^{-1}(m' * z')}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t).$$ Also ,

$\lambda_{\Omega}^{\zeta}((m_{\aleph} * y_{\aleph}) * z_{\aleph}) = \inf_{t\in f^{-1}((m*y')*z'))} \lambda_{\Omega}^{\zeta}(t), \lambda_{\Omega}^{\zeta}(y_{\aleph}) = \inf_{t\in f^{-1}(y')} \lambda_{\Omega}^{\zeta}(t)$

and

$\lambda_{\Omega}^{\zeta}(x_{\aleph} * z_{\aleph}) = \lambda_{\beta}^{\zeta}(f(m_{\aleph} * z_{\aleph}))$

$= \lambda_{\beta}^{\zeta}\left(f(m' * z')\right)$

$= \inf_{(m_{\aleph} * z_{\aleph})\in f^{-1}(m'*z')} \lambda_{\Omega}^{\zeta}(m_{\aleph} * z_{\aleph})$

$= \inf_{t\in f^{-1}(m'*z')} \lambda_{\Omega}^{\zeta}(t)$. Then

$\tilde{\mu}_{\beta}^{\zeta}(m' * z') = \underset{t\in f^{-1}(m'*z')}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t) = \tilde{\mu}_{\Omega}^{\zeta}(m_{\aleph} * z_{\aleph})$

$\succcurlyeq rmin\, \{\tilde{\mu}_{\Omega}^{\zeta}((x_0 * y_0) * z_0), \tilde{\mu}_{\Omega}^{\zeta}(y_0)\}$,

$= rmin\{ \underset{t\in f^{-1}(x'*(y'*z'))}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t), \underset{t\in f^{-1}(y')}{rsup}\, \tilde{\mu}_{\Omega}^{\zeta}(t)\}$

$= rmin\, \{ \tilde{\mu}_{\beta}^{\zeta}((m' * y') * z')), \tilde{\mu}_{\beta}^{\zeta}(y') \}$ and

$\lambda_{\Omega}^{\zeta}(m' * z') = \inf_{t\in f^{-1}(m'*z')} \lambda_{\Omega}^{\zeta}(t)$

$\le \max\, \{\lambda_{\Omega}^{\zeta}((m_{\aleph} * y_{\aleph}) * z_{\aleph}), \lambda_{\Omega}^{\zeta}(y_{\aleph}) \}$

$= \max\, \{ \inf_{t\in f^{-1}((m'*y')*z'))} \lambda_{\Omega}(t), \inf_{t\in f^{-1}(y')} \lambda_{\Omega}(t) \}$

Hence, $\beta^{\zeta} = < \tilde{\mu}_{\beta}^{\zeta}(m), \lambda_{\beta}^{\zeta}(m) >$ is a ζ -dot cubic $AB -$ ideal of Y . □

References

[1] A.T. Hameed and B.H. Hadi, **Anti-Fuzzy AT-Ideals on AT-algebras,** Journal Al-Qadisyah for Computer Science and Mathematics , vol.10, no.3(2018), 63-74.

[2] A.T. Hameed and B.H. Hadi, **Cubic Fuzzy AT-subalgebras and Fuzzy AT-Ideals on AT-algebra,** World Wide Journal of Multidisciplinary Research and Development, vol.4, no.4(2018), 34-44.

[3] A.T. Hameed and B.H. Hadi, **Intuitionistic Fuzzy AT-Ideals on AT-algebras,** Journal of Adv Research in Dynamical & Control Systems, vol.10, 10-Special Issue, 2018

[4] A.T. Hameed and B.N. Abbas**, AB-ideals of AB-algebras,** Applied Mathematical Sciences, vol.11, no.35 (2017), pp:1715-1723.

[5] A.T. Hameed and B.N. Abbas**, Derivation of AB-ideals and fuzzy AB-ideals of AB-algebra**, LAMBERT Academic Publishing, 2018.

[6] A.T. Hameed and B.N. Abbas**, On Some Properties of AB-algebras,** Algebra Letters, vol.7 (2017), pp:1-12.

[7] A.T. Hameed and B.N. Abbas**, Some properties of fuzzy AB-ideal of AB-algebras,** Journal of AL-Qadisiyah for Computer Science and Mathematics, vol.10, no. 1(2018), pp:1-7.

[8] A.T. Hameed and E.K. Kadhim, **Interval-valued IFAT-ideals of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2020, pp:1-5.

[9] A.T. Hameed and N.H. Malik, (2021), **(β, α)-Fuzzy Magnified Translations of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.

[10] A.T. Hameed and N.H. Malik, (2021), **Magnified translation of intuitionistic fuzzy AT-ideals on AT-algebra,** Journal of Discrete Mathematical Sciences and Cryptography, (2021), pp:1-7.

[11] A.T. Hameed and N.J. Raheem, (2020), **Hyper SA-algebra**, International Journal of Engineering and Information Systems (IJEAIS), vol.4, Issue 8, pp.127-136.

[12] A.T. Hameed and N.J. Raheem, (2021), **Interval-valued Fuzzy SA-ideals with Degree (λ,κ) of SA-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.

[13] A.T. Hameed, F. F. Kareem and S.H. Ali, **Hyper Fuzzy AT-ideals of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-15.

[14] A.T. Hameed, H.A. Faleh and A.H. Abed, (2021), **Fuzzy Ideals of AB-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-7.

[15] A.T. Hameed, I.H. Ghazi and A.H. Abed, (2020), **Fuzzy α-translation AB-ideal of AB-algebras**, Journal of Physics: Conference Series (IOP Publishing), 2020, pp:1-19.

[16] A.T. Hameed, N.J. Raheem and A.H. Abed, (2021), **Anti-fuzzy SA-ideals with Degree (λ,κ) of SA-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-16.

[17] A.T. Hameed, S.H. Ali and , R.A. Flayyih, **The Bipolar-valued of Fuzzy Ideals on AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-9.
[18] A.T. Hameed, H.A. Mohammed and A.H. Abed, **Anti-fuzzy AB-ideals of AB-algebras,** to appear .
[19] A.T. Hameed, H.A. Mohammed and A.H. Abed, **Cubic AB-subalgebras and AB-ideals of AB-algebras,** to appear .
[20] A.T. Hameed, S.M. Mostafa and A.H. Abed, **Cubic KUS-ideals of KUS-algebras** , Asian Journal of Mathematical Sciences,vol.8, no.2, pp:36 - 43, (2017).
[21] K. Is´eki and S. Yanaka, **An Introduction to Theory of BCK-algebras**, Math. Japonica, vol. 23 (1979), pp:1-20.
[22] Y.B. Jun, C.S. Kim, and M.S. Kang, **Cubic subalgebras and ideals of BCK/BCI-algebras**, Far East Journal of Math. Sciences, vol. 44 , no. 2 (2010) , 239-250.
[23] Y. Komori, **The Class of BCC-algebras isnot A Variety**, Math. Japon., vol. 29 (1984), pp:391-394.
[24] J. Meng and Y. B. Jun, **BCK-algebras**, Kyung Moon Sa Co., Korea, 1994.
[25] S.M. Mostafa, A.T. Hameed and A.H. Abed, **Fuzzy KUS-Ideals of KUS-Algebras**, Basra, Journal of Science (A),vol.34, no.2, (2016), pp:73-84.
[26] C. Prabpayak and U. Leerawat, **On Ideals and Congurences in KU-algebras**, Scientia Magna J., vol. 5, no.1 (2009), pp:54-57.
[27] C. Prabpayak and U. Leerawat, **On Isomorphisms of KU-algebras**, Scientia Magna J., vol. 5, no.3 (2009), pp:25-31.
[28] O.G. Xi, **Fuzzy BCK-algebras**, Math. Japan., vol. 36(1991), pp:935-942.
[29] L. A. Zadeh, **Fuzzy sets**, Inform. Control, vol.8 (1965), pp:338-353.