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# Solving First Order of Applied Differential Equations Using Runge Kutta Methods

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***Abstract***--*This work is mainly analytic and comparative among various numerical methods of Runge Kutta methods for solving differential equations of first order. Also, the results of solution with relative error are presented in tables that good approvals appeared in the application of the studied methods.*

**Keywords--** Analytical solution; Approximate solution; Runge Kutta methods.

## 1. Introduction:

Ordinary differential equations are very essential for solving some physical and biological phenomena, which are used to solve some problems in all different sciences fields [5]. There are a lot of applications for first order ordinary differential equations, for example, population growth to know the extent of diseases among the population, Nuclear physics problem, problem pharmacokinetics, the problem of steady heat transfer and a problem in Newton's law of cooling [6,13,14]. Analytically, there are several ways to solve ordinary differential equations of first order [1,3]:

$$\acute{y} = f(x, y), \qquad y(0) = y_0 \qquad \ldots(1)$$

In addition, numerical methods are extensively used to find the approximate solution of differential equations, therefore, there are several methods such as the Taylor' method, Picard's method, Euler method and Hyun (modified Uller) …etc [10, 2,4 , 9].

A lot of people have been working on developing these methods and using them to solve some important applications, among these important applications are the ones mentioned above when we talked about ordinary differential equations of first order.

Runge Kutta is one of the effective approximate methods for solving ordinary differential equations[7, 11], this method is an elementary value problem that can be defined as a number of equations consisting of number of first order equations which are subject to initial conditions involving two types of problems: a single step problem [8] and a multistep problem [12].

## 2-Derivation of Runge-Kutta Methods

Runge-Kutta Methods are considered of the most widely used and common methods, however, and many researchers are still looking to develop the

$$y_{n+1} = y_n + \sum_{i=1}^{v} W_i K_i \qquad \cdots (2)$$

where $K_i$ is defined as:-

$K_i = \hbar . f\left(t_n + c_i \hbar, y_n + \sum_{j=1}^{i-1} a_{ij} \; K_j\right)$ , $c_1 = 0,$ $\quad i = 1,2, \ldots, v$

Or in the form:

$K_1 = \hbar \cdot f(t_n, y_n)$

$K_2 = \hbar \cdot f(t_n + c_2 \hbar, y_n + a_{21} k_1)$

$K_3 = \hbar \cdot f(t_n + c_3 \hbar, y_n + a_{31} k_1 + a_{32} k_2)$

$K_4 = \hbar \cdot f(t_n + c_4 \hbar, y_n + a_{41} k_1 + a_{42} k_2 + a_{43} \; k_3)$ .

⋮

$y_{n+1} = y_n + \sum_{i=1}^{v} W_i K_i.$

where $W_i$ , $a_{2j}, a_{3j}, a_{4j}, \cdots, a_{v(v-1)}$ , $c_2, c_3, c_4, \cdots, c_v$ Optional constant.

To calculate the values of the constants, we spread $y_{n+1}$ according to the powers of h, so that it matches the Tayler series to solve differential equation f terms.

**2-1 Runge-Kutta of second order**

The constitution that arises from (1) is may be one of the simplest Runge-Kutta constitutions:

$y_{n+1} = y_n + \sum_{i=1}^{2} W_i K_i = y_n + W_1 K_1 + W_2 K_2$ ,

where

$K_1 = \hbar \cdot f(t_n, y_n)$ , $\cdots (3)$

$K_2 = \hbar \cdot f(t_n + c_2 \hbar, y_n + a_{21} K_1)$ .

And $w_1, w_2, a_{21}, c_2$ undefind coefficients that must be specified to match $y_{n+1}$ in (3) to Tayler's series of the highest order possible. We assume that y(t) is a solution of problem (1),which is continuous and differentiable as many times as necessary on a domain[a,b] so that it can be spread y(t) about $t_n$ ,and substitute each t in $t_{n+1}$ to produce:

$$y(t_{n+1}) = y(t_n) + \hbar\, y'(t_n) + \frac{\hbar^2}{2!} y''(t_n) + \frac{\hbar^3}{3!} y^{(3)}(t_n) + \cdots$$

In other words,

$$y(t_{n+1}) = y(t_n) + \hbar\, f(t_n, y_n) + \frac{\hbar^2}{2}\left(f_t + f_y . f\right)_n y''(t_n)$$

$$+ \frac{\hbar^3}{6}(f_t + 2f_{ty} . f + f_{yy} . f^2 + f_y . f_t + f_y^2 . f)_n \quad \dots (4)$$

All derivatives of the function are calculated in$(t_n, y_n)$.

$$\frac{K_2}{\hbar} = f(t_n, c_2\hbar, y_n + a_{21}K_1) = f(t_n, y_n) + c_2\hbar f_t +$$

$$a_{21}K_1 f_y + \tfrac{c_2^2 \hbar^2}{2} f_{tt} + \tfrac{a_{21}^2 K_1^2}{2} f_{yy} + c_2\, \hbar a_{21} K_1 f_{ty} \quad \dots (5)$$

And compensate for the value of $K_1 = \hbar\, f(t_n, y_n)$ , $K_2$ from the (5) in (2) and arrange them according to the powers $\hbar$, we find that :

$$y_{n+1} = y_n + (w_1 + w_2)\hbar f + w_2 \hbar^2 \left(c_2 f_t + a_{21} f f_y\right) +$$

$$a_{21}\hbar^2 \left(\tfrac{c_2^2}{2} f_{tt} + c_2 a_{21} f_{ty} f + \tfrac{a_{21}^2}{2} f^2 f_{yy}\right). \quad \cdots (6)$$

From the relation (4) and (6) we get:

$$(w_1 + w_2)\, \hbar = \hbar \quad \Longrightarrow \quad (w_1 + w_2) = 1$$

$$w_2 \hbar^2 c_2 = a_{21} \hbar^2 w_2 = \frac{\hbar^2}{2}$$

where the above equation has the solution :-

$$w_1 = w_2 = \frac{1}{2}$$

$$c_2 = a_{21} = 1$$

therefore, there are four unknown coefficients from which the Runge-kutta method results from the second order, and it is written in the following:

$$y_{n+1} = y_n + \frac{1}{2}[K_1 + K_2]$$

$$K_1 = \hbar \cdot f(t_n, y_n)$$

$$K_2 = \hbar \cdot f(t_n + \hbar, y_n + K_1) \qquad \cdots (7)$$

### 2-2 Runge-Kutta method of third order

In similar way, a third order Runge-kutta method can be derived to solve the ordinary differential equation ,which has the form:-

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

$$K_1 = \hbar \cdot f(t_n, y_n)$$

$$K_2 = \hbar \cdot f\left(t_n + \frac{\hbar}{2}, y_n + \frac{k_1}{2}\right)$$

$$K_3 = \hbar \cdot f(t_n + \hbar, y_n + 2k_2 - k_1) \qquad \cdots (8)$$

and $t_n = t_0 + i\hbar$

### 2-3 Runge-Kutta method of fourth order.

We write the generalized fourth order Runge-kutta formula as follows:

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = \hbar \cdot f(t_n, y_n)$$

$$K_2 = \hbar \cdot f\left(t_n + \frac{\hbar}{2}, y + \frac{k_1}{2}\right)$$

$$K_3 = \hbar \cdot f\left(t_n + \frac{\hbar}{2}, y_n + \frac{k_2}{2}\right)$$

$$K_4 = \hbar \cdot f(t_n + \hbar, y_n + k_3) \qquad \cdots (9)$$

## 3 Remark

The numerical error is the difference between the exact and approximate solution. There are different errors which dependent in this work.

### 3-1 Definition: Absolute Error

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The absolute error is the difference between the exact value and the approximate value $\xi_{\Upsilon} = |\Upsilon_{exact} - \Upsilon_{approximate}$ $\cdots (10)$

### 3-2 Definition: Relative Error

The relative error is the ratio of the absolute error to the exact value.

$$\xi_r = \frac{|\xi_{\Upsilon}|}{\Upsilon_{exact}} \times 100\% \qquad \cdots (11)$$

## 4 Applications:-

In this section , some applied differential equations are solved by Runge-Kutta method:

**Example(1): ( Newton's law of cooling)**

A hot Nescafe of 115 Fahrenheit kept in a room temperature of 350 Fahrenheit. The temperature is varying at a pace of 200 Fahrenheit ea minute. How long will it take for Nescafe to cool to 400 degrees F.? Suppose that Nescafe applies Newton's law of cooling.

After represent the above problem as a equation of first order**,**

$$\frac{d\Upsilon}{dt} = -0 \cdot 25(\Upsilon - 35) \qquad \cdots (12)$$

$$\Upsilon(0) = 115 \quad . \mathfrak{h} = 0 \cdot 1 \quad , \quad 0 < t < \frac{1}{2}$$

The exact solution is $\Upsilon(t) = 35 - 80e^{-0 \cdot 25(t)}$

We solve this equation using Runge-Kutta for second, third and four order as in Table(1)

Table(1) The solution of equation (12) by second, third and fourth order

| $t_n$ | Exact solution | RK-2nd order | Error | RK-3rd order | Error | RK-4th order | Error |
|---|---|---|---|---|---|---|---|
| 0.1 | 113.024793 | 113.025000 | 1.831456573-E5 | 113.02479166 | 1185580583-E7 | 113.02479296 | 3539046517-E9 |
| 0.2 | 111.098354 | 111.098757 | 3.627416478-E5 | 111.09835143 | 2313265595-E7 | 111.09835397 | 2700310033-E9 |
| 0.3 | 109.2194789 | 109.2200697 | 5409291511-E5 | 109.21947520 | 3387674101-E7 | 109.219478924 | 2197410228-E9 |
| 0.4 | 107.3869934 | 107.3877617 | 7154497725-E5 | 107.3869886 | 4469815057-E7 | 107.3869934669 | 6229804735-E9 |
| 0.5 | 105.5997522 | 105.6006888 | 8869338995-E5 | 105.5997463 | 5587134323-E7 | 105.5997522360 | 3409098909-E9 |

**Example( 2): (Nuclear physics problem)**

The equation

$$\frac{d\mathcal{P}}{dt} + \mathcal{J}\mathcal{P} = 0 \quad , \quad \mathcal{P}(0) = \mathcal{P}_0 \qquad \cdots (13)$$

is the quantity of atoms in a sample of a radioactive isotope that are motionless un decayed at time t, *somewhere* $\mathcal{J}$ is the decay constant.

$$Let \quad \mathcal{P}_0 = 1 \quad \Longrightarrow \quad \mathcal{P}(0) = 1 \quad , \; Let \quad \mathcal{J} = -1$$

The Exact solution is $\mathcal{P} = e^{t}$ .
Second, third, and fourth order of Runge-Kutta methods are used to solve this equation as shown below in Table (2)

Table(2) The solution of equation (13) by second, third and fourth order

| $t_n$ | Exact solution | RK-2nd order | Error | RK-3rd order | Error | RK-4th order | Error |
|---|---|---|---|---|---|---|---|
| 0.1 | 1.105170918 | 1.105170918 | 1.546530018-E3 | 1.105166666 | 3.847368702-E5 | 1.105170833 | 7.691118054-E7 |
| 0.2 | 1.221402758 | 1.221402758 | 3.092820919-E3 | 1.221393361 | 7.693612888-E5 | 1.221402570 | 1.539213816-E6 |
| 0.3 | 1.349858808 | 1.349858808 | 4.638877757-E3 | 1.349843229 | 1.154120706-E4 | 1.349858497 | 2.303944666-E6 |
| 0.4 | 1.491824698 | 1.491824698 | 6.184694497-E3 | 1.491321742 | 3.37141489-E3 | 1.491824240 | 3.07006581-E6 |
| 0.5 | 1.648721271 | 1.648721271 | 7.730269648-E3 | 1.648689559 | 1.923430028-E4 | 1.648720638 | 3.839339075-E6 |

**Example (3): (Problem of Pharmacokinetics)**

The problem of pharmacokinetics can be considered as equation:

$$\frac{d\mathcal{K}(t)}{dt} + a\mathcal{K}(t) = \frac{\mathfrak{P}}{vOL} \qquad , t > 0 \qquad \cdots (14)$$

$with\ \mathcal{K}(0)=0$ ,

Here $\mathcal{K}(t)$: drug concentration in the blood at any time ,

$a$: elimination velocity constant

$\mathfrak{B}$: infusion rate (in mg/min) ,

$vOL$: volume in which drug is distributed

The exact solution is $e^{t} - 1$

The result obtained using Runge-kutta formula of the second, third and fourth order are calculated in Table(3):

Table(3) The solution of equation (14) by second, third and fourth order

| $t_n$ | Exact solution | RK-2nd order | Error | RK-3rd order | Error | RK-4th order | Error |
|---|---|---|---|---|---|---|---|
| 0.1 | 0.1051709181 | 0.1050000000 | 1.62514603-E2 | 0.1051666666 | 4.042467325-E4 | 0.1051708333 | 8.063065487-E6 |
| 0.2 | 0.2214027582 | 0.2210250000 | 1.706203676-E2 | 0.2213933611 | 4.244346401-E4 | 0.2214025708 | 8.464212529-E6 |
| 0.3 | 0.3498588076 | 0.3492326250 | 1.789815167-E2 | 0.3498432295 | 4.452681957-E4 | 0.3498458499 | 7.0694096-E4 |
| 0.4 | 0.4918246976 | 0.4909020506 | 1.875967198-E2 | 0.4918017425 | 4.667333729-E4 | 0.4918242400 | 9.304128122-E6 |
| 0.5 | 0.6487212707 | 0.6474467659 | 1.964641607-E2 | 0.6486895591 | 4.888324375-E24 | 0.6487206385 | 9.74532559-E6 |

**Example(4):(Problem A Steady Heat Transform)**

Consider the stable heat transform problem.

$$\frac{d\mathrm{M}(\mathfrak{t})}{d\mathfrak{t}} + \zeta\mathrm{M}(\mathfrak{t}) = \mathfrak{t} \qquad \text{, with } \mathrm{M}(0)=0 . \qquad \cdots (15)$$

Where $\zeta$ represents the “thermal diffusivity”, and $\mathrm{M}(t)$ is “the temperature.

The exact solution $\mathfrak{t} - 1 + e^{-\mathfrak{t}}$, when $\zeta = 1$.

In order to solve this equation for the second, third, and fourth orders, we use the Runge-Kutta methods such as in Table(4).

Table(4) The solution of equation (15) by second, third and fourth order

| $t_n$ | Exact solution | RK-2nd order | Error | RK-3rd order | Error | RK-4th order | Error |
|---|---|---|---|---|---|---|---|

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.1 | 0.00483741 | 0.00500000 | 3.361096124-E1 | 0.00483333 | 8.434265444-E3 | 0.00483750 | 1.86049973-E4 |
| 0.2 | 0.01873075 | 0.01902500 | 1.570946171-E1 | 0.01872336 | 3.945383927-E3 | 0.01873090 | 8.008221774-E5 |
| 0.3 | 0.04081822 | 0.041217625 | 9.784968575-E2 | 0.04080818 | 2.459685895-E3 | 0.04081842 | 4.8997727-E5 |
| 0.4 | 0.07032004 | 0.07080125 | 6.843141727-E2 | 0.07030794 | 1.720704368-E3 | 0.07032028 | 3.412967342-E5 |
| 0.5 | 0.10653065 | 0.10707576 | 5.116931137-E2 | 0.10651696 | 1.285076173-E3 | 0.10653093 | 2.628344651-E5 |

**Example (5): (Problem of Population Growth)**

A town's population growth is proportionate to its present population. if after three years, the population doubles, and after five years, it reaches 10,000. Calculate the town's original population.

Mathematically, the aforementioned application (problem) can be detailed as:

$$\frac{dŊ}{d\tau} = 0\cdot 231Ŋ \quad , \quad Ŋ(0) = 3151 \quad \cdots (16)$$

t is the number of people who live in the city at any given moment, $t$ is denoted by Ɗ$(t)$ and $\varsigma$ =0.231 is the proportionality constant. The original population of the town at the time $\tau$ =0 is given by $N_0$.

The exact solution is $3151\ e^{0.231(\ \tau)}$

The Runge_Kutta method of the second, third and fourth order can be used to solve this problem as in table (5)

Table(5) The solution of equation (16) by second, third and fourth order

| $t_n$ | Exact solution | RK-2nd order | Error | RK-3rd order | Error | RK-4th order | Error |
|---|---|---|---|---|---|---|---|
| 0.1 | 3224.635314 | 3224.628802 | 2.019453168-E5 | 3224.635275 | 1.20943909-E7 | 3224.635313 | 3.101125872-E9 |
| 0.2 | 3299.991401 | 3299.978074 | 4.038495372-E5 | 3299.991324 | 2.333339413-E7 | 3299.991400 | 3.030310927-E9 |
| 0.3 | 3377.108476 | 3377.088019 | 6.057585673-E5 | 3377.108357 | 3.523724537-E7 | 3377.108475 | 2.961113056-E9 |
| 0.4 | 3456.027689 | 3455.999776 | 6.057585673-E5 | 3456.027527 | 4.687462445-E7 | 3456.027688 | 2.893495336-E9 |
| 0.5 | 3536.791155 | 3536.755449 | 1.009559186-E4 | 3536.790949 | 5.824488667-E7 | 3536.791154 | 2.827421683-E9 |

Form comparison Runge-Kutta methods in tables 1, 2, 3, 4 and 5 with exact solution, we conclude the fourth order method is the most accurate for analytical solution.

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