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Solution of Fractional Differential Equation by using Mahgoub Transform

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Abstract *In this paper, we used Mahgoub transform for solving linear fractional differential equations and some lemmas and properties about Mahgoub transform are proved. Illustrative examples are given in order to demonstrate the effectiveness of Mahgoub transform for solving linear fractional differential equations.*

Keywords: Linear fractional differential equations; Mahgoub transform; Inverse Mahgoub Transform.

1. Introduction

In the literature there are numerous integral transforms that are widely used in physics, astronomy, as well as engineering. In order to solve the differential equations, the integral transforms were extensively used and thus there are several works on the theory and application of integral transforms such as the (Laplace transform [1], Fourier transform [1], Hankel transform [1], Mellin transform [1], Z-transform [1], Wavelet transform [1], Mahgoub transform [2], Kamal transform [3], Elzaki transform [4], Aboodh transform [5], Mohand transform [6], Sumudu transform [7], Hermite transform [1] etc.). These transformation it will allow us to transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown function by using the Inverse Transform.

In 2016, Mahgoub [8] defined a new integral transform "Mahgoub transform'' Mahgoub and Alshikh [9] applied Mahgoub transform for solving partial differential equations. Fadhil [10] gave the convolution for Kamal and Mahgoub transforms. Taha etc. al. [11] gave the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms. For modeling biofluids flow in fractured biomaterials, Zadeh [12] gave an integro-partial differential equation.

In this work, we apply Mahgoub transform method to derive fundamental system of solutions to linear homogeneous equation of the following form:

$$a\frac{d^{2\alpha}y}{dx^{2\alpha}}+b\frac{d^{\alpha}y}{dx^{\alpha}}+cy=0\,, \qquad 0<\alpha\leq 1 \tag{1}$$

subject to the initial conditions:

$$y(0)=c_1\,,\qquad y'(0)=c_2$$

Where a, b, c, c_1 and c_2 are arbitrary constants.

This paper has been organized as follows: In Section 2, we begin by some basic definitions related to fractional calculus theory. In Section 3, the Mahgoub transform and invers Mahgoub transform of fractional integrals and derivatives have been discussed. In Section 4, the proposed method is applied to several examples to provided to illustrate the efficiency of this method.

2. Fractional calculus

In this section, we mention the some basic definitions of the fractional calculus.

Definition 1 The Riemann-Liouvlle fractional integral operator (I^{α}) of order $\alpha \geq 0$, of a function $f(x)\in c_{\mu}$,$\mu\geq -1$ is define as:

$$I^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}f(\tau)\,d\tau \qquad (\alpha>0)$$

$$I^0 f(x)=f(x)$$

Where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator (I^{α}), which we will need here, are as follows.

(1) $I^{\alpha}[af(x)+b\,g(x)]=aI^{\alpha}f(x)+bI^{\alpha}\,g(x)$,

(2) $I^{\alpha}I^{\beta}f(x)=I^{\alpha+\beta}f(x)=I^{\beta}I^{\alpha}f(x)$,

(3) $I^{\alpha}x^{v}=\frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)}x^{v+\alpha}$.

Definition 2 The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D^{\alpha} f(x) = I^{n-\alpha} D^{n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau$$

For $\alpha > 0, x > 0, \alpha, x \in \mathbb{R}, n \in \mathbb{N}, n-1 < \alpha < n.$ for the Caputo derivative we have $D^{\alpha} c = 0$, c is constant and

$$D^{\alpha} x^{v} = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} x^{v-\alpha}.$$

Definition 3 The Mittag-Leffler function of one parameter α is denoted by $E_{\alpha}(z)$ and is defined as:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)}, \qquad \alpha > 0$$

The Mittag-Leffler function with two parameters α and β is denoted by $E_{\alpha,\beta}(z)$ and is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+\beta)}, \qquad \alpha, \beta > 0$$

Where, C is the set of complex numbers. For $\beta = 1$, we get,$E_{\alpha,1}(z) = E_{\alpha}(z)$ which is the direct generalization of exponential series.

3. Mahgoub Transform

The Mahgoub transform of the functin for all is defined as: [9]

$$\mathbf{M}[f(x)] = v \int_{0}^{\infty} f(x)\, e^{-vx} dx = F(v), \qquad x \geq 0$$

Where $\mathbf{M}$ the operator is called the Mahgoub transform operator. The Mahgoub transform of the functin $f(x)$ for $x \geq 0$ exist if $f(x)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Mahgoub transforms of the functin $f(x)$.
Now, if $\mathbf{M}[f(x)] = F(v)$, then $f(x)$ is called the inverse Mahgoub transform of $F(v)$. In symbol,

$$f(x) = \mathbf{M}^{-1}\, [F(v)]$$

Where $\mathbf{M}^{-1}$is called the inverse Mahgoub transform operator.

Some properties of Mahgoub Transform which are applied in this paper, are as follows: [13,14-17]
For $\mathbf{M}[f\ (x)] = F(v)$ and $\mathbf{M}[g\ (x)] = G(v)$, we have

1. Linearity property.

$$\mathbf{M}[af(x) + bg(x)] = aF(v) + bG(v),$$

where a, b are arbitrary constants.

2. Mahgoub transform for Power function.

$$\mathbf{M}[x^{n}] = \frac{\Gamma(n+1)}{v^{n}} \quad ,$$

3. Mahgoub transform of derivatives.

$$\mathbf{M}\left[f^{(n)}(x)\right] = v^{n} F(v) - v^{n} f(0) - v^{n-1} f'(0) - \cdots - v f^{(n-1)}(0),$$

4. Multiplication by x.

$$\mathbf{M}[xf(x)] = \left(-\frac{d}{dv} + \frac{1}{v}\right) F(v)$$

5. Mahgoub transform of integrals.

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$$\mathbf{M}\left[\int_0^x f(t)\,dt\right] = \frac{F(v)}{v},$$

6. Convolution for the Mahgoub Transform.

$$\mathbf{M}\left[\int_0^x f(x-t)\,g(t)dt\right] = \frac{1}{v}F(v)G(v).$$

Mahgoub transform for some elementary functions given by follows [18, 19]

$f(x)$	$\mathbf{M}[f(x)] = F(v)$
c $\equiv$ constant	c
x^n	$\frac{\Gamma(n+1)}{v^n}$
e^{ax}	$\frac{v}{v-a}$
$\sin at$	$\frac{av}{v^2+a^2}$
$\cos at$	$\frac{v^2}{v^2+a^2}$

4. Main Results

Lemma 1 The following properties are satisfied to Mahgoub transform for $m-1<\alpha\le m, \alpha>0$, $m\in\mathbb{N}$

i. $\mathbf{M}[I^\alpha f(x)] = \frac{F(v)}{v^\alpha}$,

ii. $\mathbf{M}[D^\alpha f(x)] = \frac{v^m F(v) - v^m f(0) - v^{m-1} f'(0) - \cdots - v f^{(m-1)}(0)}{v^{m-\alpha}}$,

Proof.

i. The Mahgoub transform of Riemann-Liouville fractional given by:

$$\mathbf{M}[I^\alpha f(x)] = \mathbf{M}\left[\frac{1}{\Gamma(\alpha)}\int_a^t (x-t)^{\alpha-1} f(t)\,dt\right] = \frac{1}{\Gamma(\alpha)} vF(v)G(v),$$

Where

$$G(v) = \mathbf{M}[x^{\alpha-1}] = \frac{\Gamma(\alpha)}{v^{\alpha-1}}.$$

ii. The Mahgoub transform of Caputo fractional derivative of order $\alpha > 0$ is:

$$\mathbf{M}[D^\alpha f(x)] = \mathbf{M}\left[I^{m-\alpha} f^{(m)}(x)\right] = \frac{\mathbf{M}\left[f^{(m)}(x)\right]}{v^{m-\alpha}}.$$

Lemma 2 For α, β *and* $\gamma > 0, a \in R$, we have the following inverse Mahgoub transform formula

i. $\mathbf{M}^{-1}\left[\frac{v^{\alpha-\beta}}{v^\alpha+a}\right] = x^\beta E_{\alpha,\beta+1}(-ax^\alpha)$,

ii. $\mathbf{M}^{-1}\left[\frac{v^\gamma}{v^\beta+v^\alpha+a}\right] = \sum_{n=0}^\infty (-a)^n \sum_{k=0}^\infty \binom{n+k}{k} (-1)^k \frac{t^{k(\beta-\alpha)+\beta(n+1)-\gamma}}{\Gamma(k(\beta-\alpha)+\beta(n+1)-\gamma+1)}$

Proof.

i. $\frac{v^{\alpha-\beta}}{v^\alpha+a}$, by using the series expansion can be rewritten as:

$$\frac{v^{\alpha-\beta}}{v^\alpha+a} = \frac{1}{v^\beta}\frac{1}{1+\frac{a}{v^\alpha}} = \frac{1}{v^\beta}\sum_{n=0}^\infty \left(\frac{-a}{v^\alpha}\right)^n = \sum_{n=0}^\infty \frac{(-a)^n}{v^{n\alpha+\beta}} .$$

The inverse Mahgoub transform of above function is

$$\sum_{n=0}^\infty \frac{(-a)^n x^{n\alpha+\beta}}{\Gamma(n\alpha+\beta+1)} = x^\beta \sum_{n=0}^\infty \frac{(-ax^\alpha)^n}{\Gamma(n\alpha+\beta+1)} = x^\beta E_{\alpha,\beta+1}(-ax^\alpha) .$$

ii. $\frac{v^{\gamma}}{v^{\beta}+v^{\alpha}+a}$, by using the series expansion can be rewritten as:

$$\frac{v^{\gamma}}{v^{\beta}+v^{\alpha}+a}=\frac{v^{\gamma}}{v^{\beta}+v^{\alpha}}\left(\frac{1}{1+\frac{a}{v^{\beta}+v^{\alpha}}}\right)=\frac{v^{\gamma}}{v^{\beta}+v^{\alpha}}\sum_{n=0}^{\infty}\left(\frac{-a}{v^{\beta}+v^{\alpha}}\right)^{n}=\sum_{n=0}^{\infty}\frac{(-a)^{n}v^{\gamma}}{(v^{\beta}+v^{\alpha})^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{(-a)^{n}v^{\gamma}}{v^{\beta(n+1)}}\frac{1}{(1+v^{\alpha-\beta})^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{(-a)^{n}v^{\gamma}}{v^{\beta(n+1)}}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-v^{\alpha-\beta}\right)^{k}=\sum_{n=0}^{\infty}(-a)^{n}\sum_{k=0}^{\infty}\binom{n+k}{k}(-1)^{k}v^{k(\alpha-\beta)-\beta(n+1)+\gamma}$$

Now, the inverse Mahgoub transform of above function is given by

$$\mathbf{M}^{-1}\left[\frac{v^{\gamma}}{v^{\beta}+v^{\alpha}+a}\right]=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-a)^{n}(-1)^{k}\binom{n+k}{k}\frac{t^{k(\beta-\alpha)+\beta(n+1)-\gamma}}{\Gamma(k(\beta-\alpha)+\beta(n+1)-\gamma+1)}$$

5. Application

Example 4.1. Consider the following fractional differential equation [20]:

$$\frac{d^{\alpha}y}{dx^{\alpha}}=Ay\,,\qquad 0<\alpha\leq 1 \tag{6}$$

$$y(0)=1$$

Taking Mahgoub transform of equation (6), we have

$$\frac{v\,\mathbf{M}[y(x)]-vy(0)}{v^{1-\alpha}}=A\mathbf{M}[y(x)], \tag{7}$$

Substitute the initial condition, and solve equation (7), with respect to $\mathbf{M}[y(x)]$, we have

$$\mathbf{M}[y(x)]=\frac{v^{\alpha}}{v^{\alpha}-A},$$

Applying inverse Mahgoub transform,

$$y(x)=\mathbf{M^{-1}}\left[\frac{v^{\alpha}}{v^{\alpha}-A}\right],$$

Use lemma (2.i), we get the solution of equation (6)

$$y(x)=E_{\alpha,1}(Ax^{\alpha})=E_{\alpha}(Ax^{\alpha})$$

Example 4.2. Consider the fractional differential equation [21]:

$$\frac{d^{2\alpha}y}{dx^{2\alpha}}-y=0\,,\qquad 0<\alpha\leq 1 \tag{8}$$

$$y(0)=1\,,\qquad y'(0)=1$$

The second initial condition is for $\alpha>\frac{1}{2}$ only.

In two cases of α:

Case I: If $0<\alpha\leq\frac{1}{2}$, then equation (8), after taking Mahgoub transform and Substitute the initial condition, gives

$$\mathbf{M}[y(x)]=\frac{v^{2\alpha}}{v^{2\alpha}-1},$$

Now, applying inverse Mahgoub transform, and used lemma (2.i), we have

$$y(x) = E_{2\alpha}(x^{2\alpha}), \qquad 0 < \alpha \leq \frac{1}{2} \tag{9}$$

Which is the exact solution of equation (8).

Case II: If $\frac{1}{2} < \alpha \leq 1$, then applying Mahgoub transform for equation (8), we have

$$\frac{v^2\, \mathbf{M}[y(x)] - v^2 y(0) - v y'(0)}{v^{2-2\alpha}} - \mathbf{M}[y(x)] = 0\,, \tag{10}$$

Substitute the initial condition, and solve equation (10), with respect to $\mathbf{M}[y(x)]$, we have

$$\mathbf{M}[y(x)] = \frac{v^{2\alpha}}{v^{2\alpha}-1} + \frac{v^{2\alpha-1}}{v^{2\alpha}-1}\,,$$

Applying inverse Mahgoub transform, and used lemma (2.i), we have

$$y(x) = E_{2\alpha}(x^{2\alpha}) + x\, E_{2\alpha,2}(x^{2\alpha}), \qquad \frac{1}{2} < \alpha \leq 1 \tag{11}$$

Example 4.3. Consider the fractional differential equation [20]:

$$\frac{d^{2\alpha}y}{dx^{2\alpha}} + \frac{d^{\alpha}y}{dx^{\alpha}} - 2y = 0\,, \qquad 0 < \alpha \leq 1 \tag{12}$$

$$y(0) = 1\,, \qquad y'(0) = 1$$

The second initial condition is for $\alpha > \frac{1}{2}$ only.

Case I: If $0 < \alpha \leq \frac{1}{2}$, then equation (12), after taking Mahgoub transform and Substitute the initial condition, gives

$$\frac{v\mathbf{M}[y(x)] - v y(0)}{v^{1-2\alpha}} - \frac{v\mathbf{M}[y(x)] - v y(0)}{v^{1-\alpha}} - 2\mathbf{M}[y(x)] = 0\,,$$

$$\mathbf{M}[y(x)] = \frac{v^{2\alpha} + v^{\alpha}}{v^{2\alpha} + v^{\alpha} - 2} = 1 + \frac{2}{3}\left(\frac{1}{v^{\alpha}-1} - \frac{1}{v^{\alpha}+2}\right),$$

Applying inverse Mahgoub transform, and used lemma (2.i), we have

$$y(x) = 1 + \frac{2}{3} x^{\alpha} E_{\alpha,\alpha+1}(x^{\alpha}) - \frac{2}{3} x^{\alpha} E_{\alpha,\alpha+1}(-2x^{\alpha})$$

Case II: If $\frac{1}{2} < \alpha \leq 1$, then applying Mahgoub transform for equation (12), we have

$$\frac{v^2\, \mathbf{M}[y(x)] - v^2 y(0) - v y'(0)}{v^{2-2\alpha}} - \frac{v\mathbf{M}[y(x)] - v y(0)}{v^{1-\alpha}} - 2\mathbf{M}[y(x)] = 0\,, \tag{13}$$

Substitute the initial condition, and solve equation (13), with respect to $\mathbf{M}[y(x)]$, we have

$$\mathbf{M}[y(x)] = \frac{v^{2\alpha} + v^{2\alpha-1} + v^{\alpha}}{v^{2\alpha} + v^{\alpha} - 2}\,,$$

Or,

$$\mathbf{M}[y(x)] = \frac{v^{2\alpha}}{v^{2\alpha} + v^{\alpha} - 2} + \frac{v^{2\alpha-1}}{v^{2\alpha} + v^{\alpha} - 2} + \frac{v^{\alpha}}{v^{2\alpha} + v^{\alpha} - 2}\,,$$

Applying inverse Mahgoub transform, and used lemma (2.ii), we have

$$y(x)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2)^n(-1)^k\binom{n+k}{k}\frac{t^{\alpha(k+2n)}}{\Gamma(\alpha(k+2n)+1)}+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2)^n(-1)^k\binom{n+k}{k}\frac{t^{\alpha(k+2n)+1}}{\Gamma(\alpha(k+2n)+2)}$$
$$+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2)^n(-1)^k\binom{n+k}{k}\frac{t^{\alpha(k+2n)+\alpha}}{\Gamma(\alpha(k+2n+1)+1)}.$$

5 Conclusion

Mahgoub and Laplace transforms are very useful integral transforms for solving many advanced problems of engineering and sciences. In this paper Mahgoub transform have successfully find the exact solution of linear fractional differential equations. The fractional derivatives are described in the Caputo sense. From the rustle of the proposed example we can show Mahgoub transform is a powerful and efficient techniques for obtaining exact solution of linear fractional differential equations.

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