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On The Statistically Convergent on Some Classes of Difference Double Sequence Spaces of Fuzzy Numbers Described by Double Modulus Functions

Aqeel Mohammed Hussein
University of Al-Qadisiyah , Faculty of Education, Department of Mathematics,
E-mail: aqeel.Hussein@qu.edu.iq

Abstract*: In this paper, we offer the statistically convergent on some classes of difference double sequence spaces of fuzzy numbers described by double modulus functions and so introduce the double sequence space of strongly $\wp$-Cesaro summable and discuss the properties like completeness, and other properties*.

Keywords—Statistically convergent , strongly $\wp$-Cesaro summable , convergent-free, solidity, monotony, symmetry , double sequence , fuzzy number, double modulus function.

1. Introduction

We present the $\overline{\mathbb{c}}$ and $\overline{\mathbb{c}_0}$ of statistically convergent and statistically null double sequences, respectively, by combining the spaces $\overline{\mathbb{c}}(\mathbb{M},\triangle_{m}^{n})_{\mathbb{F}}^{2}$, $\overline{\mathbb{c}_0}(\mathbb{M},\triangle_{m}^{n})_{\mathbb{F}}^{2}$ and so introduce the strongly $\wp$-Cesaro summable double sequence space $\mathbb{w}(\mathbb{M},\triangle_{m}^{n},\wp)_{\mathbb{F}}^{2}$. The statistically convergent of sequences spaces was independently introduced by Quick [3], Buck [1], and Schoenberg [9]. From a sequence space perspective, Fridy and Orhan [5], Salat [8], Connor [2], Fridy [4], Maddox [6], Rath and Tripathy [7], Tripathy ([10], [11], [12]), Tripathy and Sen [13], and others studied it and linked it to summability.

2. Definitions and Preliminaries

It is satisfies the following condition :

(a) $\mathcal{M}(\mathfrak{A}) = 0 \Leftrightarrow \mathfrak{A} = 0$.

(b) $\mathcal{M}(\mathfrak{A}_1 + \mathfrak{A}_2) \preccurlyeq \mathcal{M}(\mathfrak{A}_1) + \mathcal{M}(\mathfrak{A}_2)$,$\forall\ \mathfrak{A}_1 \succcurlyeq 0, \mathfrak{A}_2 \succcurlyeq 0$.

(c) $\mathcal{M}$ is increasing.

(d) $\mathcal{M}$ is continuous from the right at 0 implies that $\mathcal{M}: [0,\infty) \to [0,\infty)$ is a modulus function

A double modulus functions is a function : $[0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty) \ni \mathbb{M}(\mathfrak{A} , \mathfrak{S}) = \big(\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S})\big)$, where $\mathbb{M}_1 : [0,\infty) \to [0,\infty)$ and $\mathbb{M}_2 : [0,\infty) \to [0,\infty)$.

These functions are increasing, continuous from the right at 0, that hold the following conditions :

i) $\mathbb{M}_1(\mathfrak{A}) = 0 \Leftrightarrow \mathfrak{A} = 0$ and $\mathbb{M}_2(\mathfrak{S}) = 0 \Leftrightarrow \mathfrak{S} = 0$ then $\mathbb{M}(\mathfrak{A} , \mathfrak{S}) = \big(\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S})\big) = (0,0) \Leftrightarrow (\mathfrak{A} , \mathfrak{S}) = (0,0)$

ii) $\mathbb{M}_1(\mathfrak{A}_1 + \mathfrak{A}_2) \preccurlyeq \mathbb{M}_1(\mathfrak{A}_1) + \mathbb{M}_1(\mathfrak{A}_2)$ and $\mathbb{M}_2(\mathfrak{A}_1 + \mathfrak{A}_2) \preccurlyeq \mathbb{M}_2(\mathfrak{A}_1) + \mathbb{M}_2(\mathfrak{A}_2)$ then $\mathbb{M}(\mathfrak{A}_1 + \mathfrak{A}_2) = \big(\mathbb{M}_1(\mathfrak{A}_1 + \mathfrak{A}_2), \mathbb{M}_2(\mathfrak{A}_1 + \mathfrak{A}_2)\big) \preccurlyeq \big(\mathbb{M}_1(\mathfrak{A}_1) + \mathbb{M}_1(\mathfrak{A}_2), \mathbb{M}_2(\mathfrak{A}_1) + \mathbb{M}_2(\mathfrak{A}_2)\big) = \big(\mathbb{M}_1(\mathfrak{A}_1), \mathbb{M}_2(\mathfrak{A}_1)\big) + \big(\mathbb{M}_1(\mathfrak{A}_2), \mathbb{M}_2(\mathfrak{A}_2)\big) = \mathbb{M}(\mathfrak{A}_1) + \mathbb{M}(\mathfrak{A}_2) \Longrightarrow \mathbb{M}(\mathfrak{A}_1 + \mathfrak{A}_2) \preccurlyeq \mathbb{M}(\mathfrak{A}_1) + \mathbb{M}(\mathfrak{A}_2)$

It is satisfies the following conditions :

1. $\mathbb{F}$ is a convex if for each $\mathbb{F}(\mathbb{r}_2) \succcurlyeq \mathbb{F}(\mathbb{r}_1) \wedge \mathbb{F}(\mathbb{r}_3) = \min\{\ \mathbb{F}(\mathbb{r}_1), \mathbb{F}(\mathbb{r}_3)\}$, $\forall \mathbb{r}_1 \prec \mathbb{r}_2 \prec \mathbb{r}_3$,$\forall\ \mathbb{r}_1, \mathbb{r}_2, \mathbb{r}_3 \in \mathbb{R}$.
2. $\mathbb{F}$ is normal if there is a $\mathbb{r}_0 \in \mathbb{R}$ and $\mathbb{F}(\mathbb{r}_0) = 1$.
3. $\mathbb{F}$ is upper-semi-continuous $\forall\ \mathbb{a} \in \mathbb{I}$, $\forall\ \varepsilon \succ 0$ and $\mathbb{F}^{-1}$ $([0 , \mathbb{a} + \varepsilon))$ is open in the usual topology of $\mathbb{R}$
4. $\mathbb{F}$ is a non-negative fuzzy number $\forall\ \mathbb{r} \prec 0$ implies $\mathbb{F}(\mathbb{r}) = 0$ leads to $\mathbb{F} : \mathbb{R} \to [0,1]$ is a fuzzy real number .

The set of all non-negative fuzzy numbers of $\mathbb{R}(\mathbb{I})$ denoted by $\mathbb{R}^{*}(\mathbb{I})$. Let $\mathbb{R}(\mathbb{I})$ denote the set of all fuzzy numbers which are upper-semi continuous , normal .

$\forall\ \varepsilon \succ 0$, the set $\mathcal{S}(\{\mathfrak{u},\mathfrak{t} \in \mathbb{N}: |\mathfrak{W}_{\mathfrak{ut}} - \mathbb{L}| \succcurlyeq \varepsilon\}) = 0$ tends to $\mathfrak{W} = (\mathfrak{W}_{\mathfrak{ut}})$ is statistically convergent to $\mathbb{L}$. It is written as $\mathfrak{W}_{\mathfrak{ut}} \xrightarrow{stat} \mathbb{L}$ or $\underset{\mathfrak{u},\mathfrak{t}\to\infty}{\text{stat}} \lim \mathfrak{W}_{\mathfrak{ut}} = \mathbb{L}$.

Assume that $\mathbb{x} \in \mathbb{W}^2 = \{\mathfrak{W}_{\mathfrak{ut}} = ((\mathfrak{W}_1)_{\mathfrak{ut}},(\mathfrak{W}_2)_{\mathfrak{ut}}) : \mathfrak{W}_{\mathfrak{ut}} \in \mathbb{R}\}$ and Let $\wp$ be a real positive number. If there is a real number $\mathbb{L} \ni \lim_{\mathfrak{n}} \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \sum_{\mathfrak{t}=1}^{\mathfrak{n}} |\mathfrak{W}_{\mathfrak{ut}} - \mathbb{L}|^{\wp} = 0$, $\mathbb{x}$ is said to be a strongly $\wp$-Cesaro Summable sequence. In this case, we say that $\mathbb{x}$ is strongly $\mathfrak{p}$- Cesaro Summable to $\mathbb{L}$.

In this study, we offer and define these spaces as follows:

$$\overline{\mathbb{c}}(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^2 = \Big\{((\mathfrak{W}_1)_{\mathfrak{ut}},(\mathfrak{W}_2)_{\mathfrak{ut}}) \in \mathbb{W}_{\mathbb{F}}^2 : \text{stat} - \lim\Big[\mathbb{M}_1\Big(\frac{\bar{d}(\triangle_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{W}_1)_{\mathfrak{ut}}, \mathbb{L}_1)}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\bar{d}(\triangle_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{W}_2)_{\mathfrak{ut}}, \mathbb{L}_2)}{\rho}\Big)\Big] = 0, \text{for some } \rho \succ 0, \mathbb{L} \in \mathbb{R}^2(\mathbb{I})\Big\}$$, where $(\mathfrak{W}_{\mathfrak{ut}}) = ((\mathfrak{W}_1)_{\mathfrak{ut}},(\mathfrak{W}_2)_{\mathfrak{ut}})$, $\mathbb{L} = (\mathbb{L}_1, \mathbb{L}_2)$.

$$\overline{\mathbb{c}_0}(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^2 = \Big\{((\mathfrak{W}_1)_{\mathfrak{ut}},(\mathfrak{W}_2)_{\mathfrak{ut}}) \in \mathbb{W}_{\mathbb{F}}^2 : \text{stat} - \lim\Big[\mathbb{M}_1\Big(\frac{\bar{d}(\triangle_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{W}_1)_{\mathfrak{ut}}, \bar{0})}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\bar{d}(\triangle_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{W}_2)_{\mathfrak{ut}}, \bar{0})}{\rho}\Big)\Big] = 0, \text{for some } \rho \succ 0\Big\}.$$

$$\mathbb{w}(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}},\wp)_{\mathbb{F}}^2 = \Big\{((\mathfrak{W}_1)_{\mathfrak{ut}},(\mathfrak{W}_2)_{\mathfrak{ut}}) \in \mathbb{W}_{\mathbb{F}}^2 : \lim_{\mathfrak{n}\to\infty} \frac{1}{\mathfrak{n}} \sum_{\mathfrak{a}=1}^{\mathfrak{n}} \sum_{\mathfrak{b}=1}^{\mathfrak{n}} \Big[\mathbb{M}_1\Big(\frac{\bar{d}(\triangle_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{W}_1)_{\mathfrak{ut}}, \mathbb{L}_1)}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\bar{d}(\triangle_{\mathfrak{b}}^{\mathfrak{a}}(\mathfrak{W}_2)_{\mathfrak{ut}}, \mathbb{L}_2)}{\rho}\Big)\Big]^{\wp} = 0, \text{for some } \rho \succ 0, \mathbb{L} \in \mathbb{R}^2(\mathbb{I})\Big\}.$$

We also define

$$(\mathbb{m})_{\mathbb{F}}^2(\mathbb{M}) = \overline{\mathbb{c}}(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^2 \cap \ell_\infty(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^2.$$

$$(\mathbb{m}_0)_{\mathbb{F}}^2(\mathbb{M}) = \overline{\mathbb{c}_0}(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^2 \cap \ell_\infty(\mathbb{M},\triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^2.$$

3. Main Results

Theorem 3.1:

$(\mathbb{m})_{\mathbb{F}}^2(\mathbb{M})$ and $(\mathbb{m}_0)_{\mathbb{F}}^2(\mathbb{M})$ are complete metric space by the metric

$$\mathbb{g}(\mathfrak{W},\mathfrak{V}) = \sum_{\mathbb{i}=1}^{\mathfrak{nm}} \sum_{\mathbb{j}=1}^{\mathfrak{nm}} \bar{\mathbb{d}}\Big(((\mathfrak{W}_1)_{\mathbb{ij}},(\mathfrak{V}_1)_{\mathbb{ij}}),((\mathfrak{W}_2)_{\mathbb{ij}},(\mathfrak{V}_2)_{\mathbb{ij}})\Big) + \inf\Big\{(\rho,\rho) \succ (0,0) : \sup_{\mathfrak{rs}}\Big(\mathbb{M}_1\Big(\frac{\bar{\mathbb{d}}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}, \triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{V}_1)_{\mathfrak{rs}})}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\bar{\mathbb{d}}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}, \triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{V}_2)_{\mathfrak{rs}})}{\rho}\Big)\Big) \preccurlyeq (1,1)\Big\}.$$

Proof:

Assume $\mathfrak{W}^{(\ell k)} = \big(\mathfrak{W}_{\mathfrak{rs}}^{(\ell k)}\big)_{\mathfrak{r},\mathfrak{s}=1}^{\infty}$ be a Cauchy double sequence in $(\mathbb{m})_{\mathbb{F}}^2(\mathbb{M})$.

$\forall\ \varepsilon \succ 0$. Choose $r \succ 0 \ni \Big(\mathbb{M}_1\Big(\frac{rx_0}{2}\Big) \curlyvee \mathbb{M}_2\Big(\frac{rx_0}{2}\Big)\Big) \succcurlyeq (1,1), \forall\ \ell,k,u,v \succcurlyeq \mathfrak{n}_0$, $\exists$ a positive integer $\mathfrak{n}_0 = \mathfrak{n}_0(\varepsilon) \ni$ $\mathcal{G}\Big(((\mathfrak{W}_1)^{(\ell k)},(\mathfrak{W}_1)^{(uv)}),((\mathfrak{W}_2)^{(\ell k)},(\mathfrak{W}_2)^{(uv)})\Big) \prec \Big(\frac{\varepsilon}{rx_0},\frac{\varepsilon}{rx_0}\Big)$.

As,

$$\sum_{\mathbb{i}=1}^{\mathfrak{nm}} \sum_{\mathbb{j}=1}^{\mathfrak{nm}} \bar{\mathbb{d}}\Big(((\mathfrak{W}_1)_{\mathbb{ij}}^{(\ell k)},(\mathfrak{W}_1)_{\mathbb{ij}}^{(uv)}),((\mathfrak{W}_2)_{\mathbb{ij}}^{(\ell k)},(\mathfrak{W}_2)_{\mathbb{ij}}^{(uv)})\Big) + \inf\Big[(\rho,\rho) \succ (0,0) : \sup_{\mathfrak{rs}}\Big\{\mathbb{M}_1\Big(\frac{\bar{\mathbb{d}}\big(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}, \Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{ab}}^{(uv)}\big)}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\bar{\mathbb{d}}\big(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)}, \Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\big)}{\rho}\Big)\Big\} \preccurlyeq (1,1)\Big] \prec (\varepsilon,\varepsilon), \forall\ \ell,k,u,v \succcurlyeq \mathfrak{n}_0.$$

(3-1)

This leads that,

$$\sum_{\mathbb{i}=1}^{\mathfrak{nm}} \sum_{\mathbb{j}=1}^{\mathfrak{nm}} \bar{\mathbb{d}}\Big(((\mathfrak{W}_1)_{\mathbb{ij}}^{(\ell k)},(\mathfrak{W}_1)_{\mathbb{ij}}^{(uv)}),((\mathfrak{W}_2)_{\mathbb{ij}}^{(\ell k)},(\mathfrak{W}_2)_{\mathbb{ij}}^{(uv)})\Big) \prec (\varepsilon,\varepsilon), \forall\ \ell,k,u,v \succcurlyeq \mathfrak{n}_0.$$

$$\Rightarrow \bar{\mathbb{d}}\Big(((\mathfrak{W}_1)_{\mathbb{ij}}^{(\ell k)},(\mathfrak{W}_1)_{\mathbb{ij}}^{(uv)}),((\mathfrak{W}_2)_{\mathbb{ij}}^{(\ell k)},(\mathfrak{X}_2)_{\mathbb{ij}}^{(uv)})\Big) \prec (\varepsilon,\varepsilon), \forall\ \ell,k,u,v \succcurlyeq \mathfrak{n}_0, \forall\ \mathbb{i},\mathbb{j} = 1,2,3,\ldots,\mathfrak{nm}.$$

Therefore $\big((\mathfrak{W}_1)_{\mathbb{ij}}^{(\ell k)}\big),\big((\mathfrak{W}_2)_{\mathbb{ij}}^{(\ell k)}\big)$ are Cauchy double sequences in $\mathbb{R}^n(\mathbb{I})$ by the completeness property of $\mathbb{R}^n(\mathbb{I})$, so is convergent in $\mathbb{R}^n(\mathbb{I})$, $\forall\ \mathbb{i},\mathbb{j} = 1,2,3,\ldots,\mathfrak{nm}$.

Let $\lim_{\ell,k\to\infty}(\mathfrak{W}_1)_{\mathbb{ij}}^{(\ell k)} = (\mathfrak{W}_1)_{\mathbb{ij}}$ and $\lim_{\ell,k\to\infty}(\mathfrak{W}_2)_{\mathbb{ij}}^{(\ell k)} = (\mathfrak{W}_2)_{\mathbb{ij}}$ $\forall\ \mathbb{i},\mathbb{j} = 1,2,3,\ldots,\mathfrak{nm}$.

(3-2)

Moreover, $\sup_{\mathfrak{rs}}\left\{\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(uv)}\right)}{\rho}\right)\curlyvee\right.$ $\left.\mathbb{M}_2\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\right)}{\rho}\right)\right\}\preccurlyeq(1,1),\ \forall\ \ell,k,u,v\succcurlyeq\mathfrak{n}_0$

(3-3)

$$\Longrightarrow\left\{\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(uv)}\right)}{\mathcal{G}(\mathfrak{W}^{(\ell k)},\mathfrak{W}^{(uv)})}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\right)}{\mathcal{G}(\mathfrak{W}^{(\ell k)},\mathfrak{W}^{(uv)})}\right)\right\}\preccurlyeq(1,1)\preccurlyeq\left(\mathbb{M}_1\left(\frac{r\mathfrak{x}_0}{2}\right)\curlyvee\mathbb{M}_2\left(\frac{r\mathfrak{x}_0}{2}\right)\right),\forall\ \ell,k,u,v\succcurlyeq\mathfrak{n}_0.$$

By the continuity of $\mathbb{M}$, we get

$$\bar{\mathbb{d}}\left(\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(uv)}\right),\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\right)\right)\preccurlyeq\left(\frac{r\mathfrak{x}_0}{2},\frac{r\mathfrak{x}_0}{2}\right).\mathcal{G}\left(\left((\mathfrak{W}_1)^{(\ell k)},(\mathfrak{W}_1)^{(uv)}\right),\left((\mathfrak{W}_2)^{(\ell k)},(\mathfrak{W}_2)^{(uv)}\right)\right),\forall\ \ell,k,u,v\succcurlyeq\mathfrak{n}_0.$$

Then

$$\bar{\mathbb{d}}\left(\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(uv)}\right),\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\right)\right)\preccurlyeq\left(\frac{r\mathfrak{x}_0}{2},\frac{r\mathfrak{x}_0}{2}\right).\left(\frac{\varepsilon}{r\mathfrak{x}_0},\frac{\varepsilon}{r\mathfrak{x}_0}\right)=\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)\ ,\forall\ \ell,k,u,v\succcurlyeq\mathfrak{n}_0\ .$$

Then

$$\bar{\mathbb{d}}\left(\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(uv)}\right),\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\right)\right)\preccurlyeq\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)\ ,\forall\ \ell,k,u,v\succcurlyeq\mathfrak{n}_0\ .$$

Therefore $\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}\right),\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)}\right)$ are Cauchy double sequence in $\mathbb{R}^n(\mathbb{I})$, so is convergent in $\mathbb{R}^n(\mathbb{I})$ by the completeness property of $\mathbb{R}^n(\mathbb{I})$.

Assume $\lim_{\ell k}\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}=(\mathfrak{V}_1)_{\mathfrak{rs}}$ and $\lim_{\ell k}\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)}=(\mathfrak{V}_2)_{\mathfrak{rs}}$ in $\mathbb{R}^n(\mathbb{I})$, $\forall\ \mathfrak{a},\mathfrak{b}\in\mathbb{N}$.

We must prove that,

$\lim_{\ell k}(\mathfrak{W}_1)^{(\ell k)}=\mathfrak{W}_1$ and $\lim_{\ell k}(\mathfrak{W}_2)^{(\ell k)}=\mathfrak{W}_2,\forall\ \mathfrak{W}_1,\mathfrak{W}_2\in(\mathfrak{m})_{\mathbb{F}}^2\,(\mathbb{M})$.

$\triangle_{\mathfrak{m}}^{\mathfrak{n}}\,(\mathfrak{W}_1)_{\mathfrak{rs}}=\sum_{\mathfrak{i}=0}^{\mathfrak{n}}(-1)\binom{\mathfrak{n}}{\mathfrak{i}}(\mathfrak{W}_1)_{(\mathfrak{n}+\mathfrak{im})(\mathfrak{s}+\mathfrak{im})}$ and $\triangle_{\mathfrak{m}}^{\mathfrak{n}}\,(\mathfrak{W}_2)_{\mathfrak{rs}}=\sum_{\mathfrak{i}=0}^{\mathfrak{n}}(-1)\binom{\mathfrak{n}}{\mathfrak{i}}(\mathfrak{W}_2)_{(\mathfrak{n}+\mathfrak{im})(\mathfrak{s}+\mathfrak{im})}$

(**)

And

$$\sum_{\mathfrak{i}=1}^{\mathfrak{nm}}\sum_{\mathfrak{j}=1}^{\mathfrak{nm}}\mathbb{d}_\infty\left(\left((\mathfrak{W}_1)_{\mathfrak{ij}}^{(\ell k)},(\mathfrak{W}_1)_{\mathfrak{ij}}^{(uv)}\right),\left((\mathfrak{W}_2)_{\mathfrak{ij}}^{(\ell k)},(\mathfrak{W}_2)_{\mathfrak{ij}}^{(uv)}\right)\right)+\inf\left[(\rho,\rho)\succ(0,0):\sup_{\mathfrak{rs}}\left\{\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(uv)}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(uv)}\right)}{\rho}\right)\right\}\preccurlyeq(1,1)\right]\prec(\varepsilon,\varepsilon)\ ,\forall\ \ell,k,u,v\succcurlyeq\mathfrak{n}_0\ .$$

(3-1)

For $\mathfrak{r},\mathfrak{s}=1$, from (**) and (3-1), we get,

$\lim_{\ell,k\to\infty}(\mathfrak{W}_1)_{\mathfrak{nm}+1}^{(\ell k)}=(\mathfrak{W}_1)_{\mathfrak{nm}+1}$ and $\lim_{\ell,k\to\infty}(\mathfrak{W}_2)_{\mathfrak{nm}+1}^{(\ell k)}=(\mathfrak{W}_2)_{\mathfrak{nm}+1},\forall\ \mathfrak{n}\succcurlyeq1,\mathfrak{m}\succcurlyeq1.$

Proceeding in this way of induction, we arrive that,

$\lim_{\ell,k\to\infty}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}=(\mathfrak{W}_1)_{\mathfrak{rs}}$ and $\lim_{\ell,k\to\infty}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)}=(\mathfrak{W}_2)_{\mathfrak{rs}}\ ,\forall\ \mathfrak{r},\mathfrak{s}\in\mathbb{N}$

Moreover,

$\lim_{\ell,k\to\infty}\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}=\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}$ and $\lim_{\ell,k\to\infty}\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}=\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}\ ,\forall\ \mathfrak{r},\mathfrak{s}\in\mathbb{N}$. By the continuity of $\mathbb{M}$ and taking $u,v\to\infty$ and fixing (ℓk) , it follows that (3-3)

$$\sup_{\mathfrak{rs}}\left\{\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}\right)}{\rho}\right)\right\}\preccurlyeq(1,1),\text{ for some }\rho\succ0\ .$$

Now, on taking the infimum number $\wp$'s ,we obtain,

$$\inf\left\{(\rho,\rho)\succ(0,0):\sup_{\mathfrak{rs}}\left\{\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}\right)}{\rho}\right)\right\}\preccurlyeq(1,1)\right\}\prec(\varepsilon,\varepsilon)\ ,\forall\ \ell,k\succcurlyeq\mathfrak{n}_0$$

(by (3-1))

consequently,

$$\sum_{\mathfrak{i}=1}^{\mathfrak{nm}}\sum_{\mathfrak{j}=1}^{\mathfrak{nm}}\mathbb{d}_\infty\left(\left((\mathfrak{W}_1)_{\mathfrak{ij}}^{(\ell k)},(\mathfrak{W}_1)_{\mathfrak{ij}}\right),\left((\mathfrak{W}_1)_{\mathfrak{ij}}^{(\ell k)},(\mathfrak{W}_1)_{\mathfrak{ij}}\right)\right)+\inf\left\{(\rho,\rho)\succ(0,0):\sup_{\mathfrak{rs}}\left\{\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{\mathbb{d}}\left(\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)},\,\Delta_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}\right)}{\rho}\right)\right\}\preccurlyeq(1,1)\right\}\prec(\varepsilon,\varepsilon)+(\varepsilon,\varepsilon)=(2\varepsilon,2\varepsilon)\ ,\forall\ \ell,k\succcurlyeq\mathfrak{n}_0$$

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Which indicates that,

$$\mathcal{G}\Big(\big((\mathfrak{W}_1)^{(\ell k)},(\mathfrak{W}_1)\big),\big((\mathfrak{W}_2)^{(\ell k)},(\mathfrak{W}_2)\big)\Big) \prec$$

$(2\varepsilon, 2\varepsilon), \forall\, \ell, k \geqslant \mathfrak{n}_0$.

i.e. $\lim_{\ell k}(\mathfrak{W}_1)^{(\ell k)} = (\mathfrak{W}_1)$ and $\lim_{\ell k}(\mathfrak{W}_2)^{(\ell k)} = (\mathfrak{W}_2)$.

Now it's to show that $(\mathfrak{W}_1), (\mathfrak{W}_2) \in (\mathfrak{m})^2_{\mathbb{F}}\,(\mathbb{M})$.

Suppose $(\mathfrak{W}_1)^{(\ell k)}, (\mathfrak{W}_2)^{(\ell k)} \in (\mathfrak{m})^2_{\mathbb{F}}\,(\mathbb{M})$. Then,

$\forall\, \ell, k, \exists\, \mathbb{L}_{\ell k} = ((\mathbb{L}_1)_{\ell k}, (\mathbb{L}_2)_{\ell k}) \ni$

$$\underset{\mathfrak{rs}\to\infty}{\text{stat}} - \lim\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\ell k)}, (\mathbb{L}_1)_{\ell k}\right)}{\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\ell k)}, (\mathbb{L}_2)_{\ell k}\right)}{\rho}\right)\right) = (0,0)$$

, for some $\rho \succ 0$, $\mathbb{L}_{\ell k} \in \mathbb{R}^2(\mathbb{I})$, $\forall\, \ell, k \in \mathbb{N}$.

We have to show that,

i) $(\mathbb{L}_{\ell k})$ converges to $\mathbb{L} = (\mathbb{L}_1, \mathbb{L}_2)$, $\forall \ell, k \to \infty$.

ii) $$\underset{\mathfrak{rs}\to\infty}{\text{stat}} - \lim\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) = (0,0)$$

, for some $\rho \succ 0$, $\mathbb{L} = (\mathbb{L}_1, \mathbb{L}_2) \in \mathbb{R}^2(\mathbb{I})$.

Since $\big((\mathfrak{W}_1)^{(\ell k)}\big), \big((\mathfrak{W}_2)^{(\ell k)}\big)$ are convergent double sequence of elements from $(\mathfrak{m})^2_{\mathbb{F}}\,(\mathbb{M})$.

So for given $\varepsilon \succ 0, \exists\, \mathfrak{n}_0 \in \mathbb{N} \ni$

$$\mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\ell k)}, (\mathfrak{W}_1)^{(uv)}\big), \big((\mathfrak{W}_1)^{(\ell k)}, (\mathfrak{W}_1)^{(uv)}\big)\Big) \prec \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right)$$

.

Again, for given $\varepsilon \succ 0$, we have,

$$\mathcal{S}(\mathbb{A}_{\ell k}) = \mathcal{S}\Big(\Big\{\mathfrak{r}, \mathfrak{s} \in \mathbb{N} : \mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\ell k)}, (\mathbb{L}_1)_{\ell k}\big), \big((\mathfrak{W}_2)^{(\ell k)}, (\mathbb{L}_2)_{\ell k}\big)\Big) \prec \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right)\Big\}\Big) = (1,1).$$

And

$$\mathcal{S}(\mathbb{A}_{uv}) = \mathcal{S}\Big(\Big\{\mathfrak{r}, \mathfrak{s} \in \mathbb{N} : \mathcal{g}\Big(\big((\mathfrak{W}_1)^{(uv)}, (\mathbb{L}_1)_{uv}\big), \big((\mathfrak{W}_2)^{(uv)}, (\mathbb{L}_2)_{uv}\big)\Big) \prec \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right)\Big\}\Big) = (1,1)$$

Let $\mathbb{A} = \mathbb{A}_{\ell k} \cap \mathbb{A}_{uv}$, then $\mathcal{S}(\mathbb{A}) = 1$.We choose $\mathfrak{r}, \mathfrak{s} \in \mathbb{A}$. Then , $\forall\, \ell, k, u, v \geqslant \mathfrak{n}_0$, we have,

$$\mathcal{g}\big(((\mathbb{L}_1)_{\ell k}, (\mathbb{L}_1)_{uv}), ((\mathbb{L}_2)_{\ell k}, (\mathbb{L}_2)_{uv})\big) \preccurlyeq \mathcal{g}\Big(\big((\mathbb{L}_1)_{\ell k}, (\mathfrak{W}_1)^{(\ell k)}\big), \big((\mathbb{L}_2)_{\ell k}, (\mathfrak{W}_2)^{(\ell k)}\big)\Big) + \mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\ell k)}, (\mathfrak{W}_1)^{(uv)}\big), \big((\mathfrak{W}_2)^{(\ell k)}, (\mathfrak{W}_2)^{(uv)}\big)\Big) + \mathcal{g}\Big(\big((\mathfrak{W}_1)^{(uv)}, (\mathbb{L}_1)_{uv}\big), \big((\mathfrak{W}_1)^{(uv)}, (\mathbb{L}_1)_{uv}\big)\Big) \prec \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right) + \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right) + \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right) = \left(\frac{3\varepsilon}{3}, \frac{3\varepsilon}{3}\right) = (\varepsilon, \varepsilon)\,.$$

Since the double sequence $((\mathbb{L}_1)_{\ell k})$, $((\mathbb{L}_2)_{\ell k})$ are Cauchy convergence condition, it must be convergent to a fuzzy numbers $\mathbb{L}_1, \mathbb{L}_2$.

Consequently,

$\lim_{\ell, k\to\infty} (\mathbb{L}_1)_{\ell k} = \mathbb{L}_1$ and $\lim_{\ell, k\to\infty} (\mathbb{L}_1)_{\ell k} = \mathbb{L}_1$. Let $\mathfrak{z} \succ 0$.

Its prove that,

$$\mathcal{S}(\mathbb{F}) = \mathcal{S}\big(\{\mathfrak{r}, \mathfrak{s} \in \mathbb{N} : \mathcal{g}\big(((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathbb{L}_1)_{\ell k}), ((\mathfrak{W}_2)_{\mathfrak{rs}}, (\mathbb{L}_2)_{\ell k})\big) \prec (\mathfrak{z}, \mathfrak{z})\}\big) = (1,1).$$

Since $(\mathfrak{W}_1)^{(\mathfrak{ab})} \to \mathfrak{W}_1$ and $(\mathfrak{W}_2)^{(\mathfrak{ab})} \to \mathfrak{W}_2$, there is $\mathfrak{u}, \mathfrak{t} \in \mathbb{N} \ni$

$$\mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\mathfrak{ut})}, \mathfrak{W}_1\big), \big((\mathfrak{W}_2)^{(\mathfrak{ut})}, \mathfrak{W}_2\big)\Big) \prec \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right).$$

The numbers $\mathfrak{u}, \mathfrak{t}$ can be chosen in such a way that together with $\mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\mathfrak{ut})}, \mathfrak{W}_1\big), \big((\mathfrak{W}_2)^{(\mathfrak{ut})}, \mathfrak{W}_2\big)\Big) \prec \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right)$, we have $\mathcal{g}\big(((\mathbb{L}_1)_{\mathfrak{ut}}, \mathbb{L}_1), ((\mathbb{L}_1)_{\mathfrak{ut}}, \mathbb{L}_1)\big) \prec \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right)$.

Since

$$\underset{\mathfrak{rs}\to\infty}{\text{stat}} - \lim\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}^{(\mathfrak{ut})}, (\mathbb{L}_1)_{\mathfrak{ut}}\right)}{\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}\left(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}^{(\mathfrak{ut})}, (\mathbb{L}_2)_{\mathfrak{ut}}\right)}{\rho}\right)\right) = (0,0)$$

, we have a subset $\mathbb{B}$ of $\mathbb{N}$ such that $\mathcal{S}(\mathbb{B}) = 1$,where $\mathbb{B} = \Big\{\mathfrak{r}, \mathfrak{s} \in \mathbb{N} : \mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\mathfrak{ut})}, (\mathbb{L}_1)_{\mathfrak{ut}}\big), \big((\mathfrak{W}_2)^{(\mathfrak{ut})}, (\mathbb{L}_2)_{\mathfrak{ut}}\big)\Big) \prec \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right)\Big\}$.

Therefore,

$\forall\, \mathfrak{r}, \mathfrak{s} \in \mathbb{B}$, we have

$$\mathcal{g}\big((\mathfrak{W}_1, \mathbb{L}_1), (\mathfrak{W}_2, \mathbb{L}_2)\big) \preccurlyeq \mathcal{g}\Big(\big(\mathfrak{W}_1, (\mathfrak{W}_1)^{(\mathfrak{ut})}\big), \big(\mathfrak{W}_2, (\mathfrak{W}_2)^{(\mathfrak{ut})}\big)\Big) + \mathcal{g}\Big(\big((\mathfrak{W}_1)^{(\mathfrak{ut})}, (\mathbb{L}_1)_{\mathfrak{ut}}\big), \big((\mathfrak{W}_1)^{(\mathfrak{ut})}, (\mathbb{L}_1)_{\mathfrak{ut}}\big)\Big) + \mathcal{g}\big(((\mathbb{L}_1)_{\mathfrak{ut}}, \mathbb{L}_1), ((\mathbb{L}_1)_{\mathfrak{ut}}, \mathbb{L}_1)\big) \prec \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right) + \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right) + \left(\frac{\mathfrak{z}}{3}, \frac{\mathfrak{z}}{3}\right) = (\mathfrak{z}, \mathfrak{z}, \mathfrak{z})\,.$$

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Thus ,

$(\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M})$ is complete metric space .

Other case is similar .

Theorem 3.2 :

Let $\mathbb{M}, \mathbb{M}_1$, and $\mathbb{M}_2$ be three double modulus functions which satisfy $\triangle_2$- condition then :

i) $(\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_1) \subseteq (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M} \circ \mathbb{M}_1)$,

ii) $(\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_1) \cap (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_2) \subseteq (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_1 + \mathbb{M}_2)$, where $\mathbb{M} = (\mathbb{M}_3, \mathbb{M}_4), \mathbb{M}_1 = (\mathbb{M}_5, \mathbb{M}_6), \mathbb{M}_2 = (\mathbb{M}_7, \mathbb{M}_8)$

Proof:

i) Let $(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_1)$. For $\varepsilon \succ 0, \exists\, \mathfrak{y} \succ 0 \ni (\varepsilon,\varepsilon) = \mathbb{M}_3(\mathfrak{y}) \curlyvee \mathbb{M}_4(\mathfrak{y})$. Then there's a set $\mathbb{A} \subseteq \mathbb{N}$, with

$$\mathcal{S}(\mathbb{A}) = 1 \ni \left(\mathbb{M}_5\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_6\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) \prec$$

$(\mathfrak{y},\mathfrak{y})$, $\forall\, \mathfrak{r},\mathfrak{s} \in \mathbb{A}$, for some $\rho \succ 0, \forall\, \mathbb{L} = (\mathbb{L}_1, \mathbb{L}_2) \in \mathbb{R}^2(\mathbb{I})$.

Assume $\mathfrak{V}_{\mathfrak{rs}} = \left(\mathbb{M}_5\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_6\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right)$,for some $\rho \succ 0$. Since $\mathbb{M} = (\mathbb{M}_3, \mathbb{M}_4)$ is increasing and continuous, we get

$$(\mathbb{M}_3, \mathbb{M}_4)(\mathfrak{V}_{\mathfrak{rs}}) = (\mathbb{M}_3, \mathbb{M}_4)\left(\mathbb{M}_5\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_6\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) \prec \left(\mathbb{M}_3\left(\mathbb{M}_5\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right)\right) \curlyvee \mathbb{M}_4\left(\mathbb{M}_6\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right)\right)\right) \prec \left(\mathbb{M}_3(\mathfrak{y}) \curlyvee \mathbb{M}_4(\mathfrak{y})\right) = (\varepsilon,\varepsilon),$$

for some $\rho \succ 0$.

Which implies that,

$(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M} \circ \mathbb{M}_1)$.

ii) Suppose $(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_1) \cap (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_2)$.

Then there is a set $\mathbb{A} \subseteq \mathbb{N}$, with $\mathcal{S}(\mathbb{A}) = 1 \ni$

$$\left(\mathbb{M}_5\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_6\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) \prec$$

$(\varepsilon,\varepsilon), \forall\, \mathfrak{r},\mathfrak{s} \in \mathbb{A}$ and for some $\rho \succ 0$.

And

$$\left(\mathbb{M}_7\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_8\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) \prec$$

$(\varepsilon,\varepsilon), \forall\, \mathfrak{r},\mathfrak{s} \in \mathbb{A}$ and for some $\rho \succ 0$.

The remainder of the proof is come of the equality,

$$((\mathbb{M}_5, \mathbb{M}_6) + (\mathbb{M}_7, \mathbb{M}_8))\left(\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) = \left(\left(\mathbb{M}_5\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_6\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right) + \left(\mathbb{M}_7\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_1)_{\mathfrak{rs}}, \mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_8\left(\frac{\bar{\mathbb{d}}(\triangle_3^2(\mathfrak{W}_2)_{\mathfrak{rs}}, \mathbb{L}_2)}{\rho}\right)\right)\right) \prec (\varepsilon,\varepsilon) + (\varepsilon,\varepsilon) = (2\varepsilon, 2\varepsilon)$$

, for some $\succ 0$.

This leads that,

$(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in (\mathbb{m}\mathbb{m})_{\mathbb{F}}^{2}(\mathbb{M}_1 + \mathbb{M}_2)$.

Theorem 3.3 :

i) $\mathbb{w}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}}, \wp)_{\mathbb{F}}^{2} \subseteq \bar{\mathbb{c}}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^{2}$,

ii) For $= (\mathfrak{W}_{\mathfrak{rs}}) \in \bar{\mathbb{c}}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^{2}$, $(\mathfrak{W}_{\mathfrak{rs}})$ is strongly $\wp$-Cesaro summable to $\mathfrak{W}_{00}$, if it is bounded .

Proof :

i) Let $(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in \mathbb{w}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}}, \wp)_{\mathbb{F}}^{2}$. $\forall\, \varepsilon \succ 0$ and $\wp \in \mathbb{R}$, $0 \prec \wp \prec \infty$ and using the continuity of $\mathbb{M}$, we get

$$\sum_{\mathfrak{r}=1}^{\mathfrak{n}} \sum_{\mathfrak{s}=1}^{\mathfrak{n}} \left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{00})}{\rho}\right)\right]^{\wp} \succcurlyeq (\varepsilon^{\wp}, \varepsilon^{\wp}) \cdot \left|\left\{\mathfrak{r}, \mathfrak{s} \preccurlyeq \mathfrak{n} : \left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle_{\mathfrak{m}}^{\mathfrak{n}}(\mathfrak{W}_2)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{00})}{\rho}\right)\right] \succcurlyeq (\varepsilon,\varepsilon)\right\}\right|$$

, for some $\rho \succ 0$.

Therefore $(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}})$ is statistically convergent to $(\mathfrak{W}_{00}) = ((\mathfrak{W}_1)_{00}, (\mathfrak{W}_2)_{00})$. Consequently $(\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in \bar{\mathbb{c}}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^{2}$.

Thus ,

$$\mathbb{w}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}}, \wp)_{\mathbb{F}}^{2} \subseteq \bar{\mathbb{c}}(\mathbb{M}, \triangle_{\mathfrak{m}}^{\mathfrak{n}})_{\mathbb{F}}^{2}.$$

ii) $\forall\, \varepsilon \succ 0$ and suppose $\mathbb{K} = \bar{d}\left(((\mathfrak{W}_1)_{\mathfrak{rs}}, \bar{0}), ((\mathfrak{W}_2)_{\mathfrak{rs}}, \bar{0})\right) + \bar{d}\left(((\mathfrak{W}_1)_{00}, \bar{0}), ((\mathfrak{W}_2)_{00}, \bar{0})\right)$. Since $\mathfrak{W} = (\mathfrak{W}_{\mathfrak{rs}}) =$

$((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in (\mathfrak{m})^2_{\mathbb{F}}(\mathbb{M})$ is bounded and statistically convergent to $(\mathfrak{W}_{00}) = ((\mathfrak{W}_1)_{00}, (\mathfrak{W}_2)_{00})$, ∃ a positive number

$$\mathcal{N}(\varepsilon) \ni \frac{1}{\mathfrak{n}}\left|\left\{\mathfrak{r},\mathfrak{s} \preccurlyeq \mathfrak{n} : \left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},(\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},(\mathfrak{W}_2)_{00})}{\rho}\right)\right]^{\wp} \succcurlyeq \left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)\right\}\right| \prec \frac{\varepsilon}{\mathbb{K}^{\wp}}, \forall\, \mathfrak{n} \succcurlyeq \mathcal{N}(\varepsilon).$$

Now , the set

$$\mathbb{J}_{\mathfrak{n}} = \left\{\mathfrak{r},\mathfrak{s} \preccurlyeq \mathfrak{n} : \left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},(\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},(\mathfrak{W}_2)_{00})}{\rho}\right)\right]^{\wp} \succcurlyeq \left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)\right\}.$$

Then $\forall\, \mathfrak{n} \succcurlyeq \mathcal{N}(\varepsilon)$, we have

$$\frac{1}{\mathfrak{n}}\sum_{\mathfrak{r}=1}^{\mathfrak{n}}\sum_{\mathfrak{s}=1}^{\mathfrak{n}}\left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},(\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},(\mathfrak{W}_2)_{00})}{\rho}\right)\right]^{\wp} =$$

$$\frac{1}{\mathfrak{n}}\left(\sum_{\mathfrak{r}\in\mathbb{J}_{\mathfrak{n}}}\sum_{\mathfrak{s}\in\mathbb{J}_{\mathfrak{n}}}\left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},(\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},(\mathfrak{W}_2)_{00})}{\rho}\right)\right]^{\wp} + \right.$$

$$\left.\sum_{\mathfrak{r}\notin\mathbb{J}_{\mathfrak{n}},\mathfrak{r}\preccurlyeq\mathfrak{n}}\sum_{\mathfrak{s}\notin\mathbb{J}_{\mathfrak{n}},\mathfrak{s}\preccurlyeq\mathfrak{n}}\left[\mathbb{M}_1\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},(\mathfrak{W}_1)_{00})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},(\mathfrak{W}_2)_{00})}{\rho}\right)\right]^{\wp}\right) \prec \varepsilon, \text{ for some } \rho \succ 0.$$

<u>**Proposition 3.4**</u> **:**

$\mathbb{Z}(\mathbb{M},\triangle^{\mathfrak{i}}_{\mathfrak{m}})^2_{\mathbb{F}} \subset \mathbb{Z}(\mathbb{M},\triangle^{\mathfrak{n}}_{\mathfrak{m}})^2_{\mathbb{F}}, \forall\, 0 \preccurlyeq \mathfrak{i} \prec \mathfrak{n}, \forall\, \mathbb{Z} = \bar{c}$ and $\bar{c}_0$.

<u>**Proof**</u>

Let $= (\mathfrak{W}_{\mathfrak{rs}}) = ((\mathfrak{W}_1)_{\mathfrak{rs}}, (\mathfrak{W}_2)_{\mathfrak{rs}}) \in \bar{c}(\mathbb{M},\triangle^{\mathfrak{n}-1}_{\mathfrak{m}})^2_{\mathbb{F}}$.

Then , we have

$$\underset{\mathfrak{rs}\to\infty}{\mathrm{stat}-\lim}\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},\mathbb{L}_1)}{\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},\mathbb{L}_2)}{\rho}\right)\right) = (0,0), \text{ for some } \rho \succ 0 \text{ and } \mathbb{L}_1, \mathbb{L}_2 \in \mathbb{R}(\mathbb{I}).$$

Now , we have

$$\underset{\mathfrak{rs}\to\infty}{\mathrm{stat}-\lim}\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},\bar{0})}{2\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}}_{\mathfrak{m}}(\mathfrak{W}_2)_{\mathfrak{rs}},\bar{0})}{2\rho}\right)\right) =$$

$$\underset{\mathfrak{rs}\to\infty}{\mathrm{stat}-\lim}\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}} - \triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{(\mathfrak{r}+1)(\mathfrak{s}+1)},\bar{0})}{2\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}} - \triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{(\mathfrak{r}+1)(\mathfrak{s}+1)},\bar{0})}{2\rho}\right)\right) \preccurlyeq \underset{\mathfrak{rs}\to\infty}{\mathrm{stat}-\lim}\frac{1}{2}\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1}{\rho}\right.\right.$$

$$\left.\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{\mathfrak{rs}},\bar{0})}{\rho}\right)\right) +$$

$$\underset{\mathfrak{rs}\to\infty}{\mathrm{stat}-\lim}\frac{1}{2}\left(\mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{(\mathfrak{r}+1)(\mathfrak{s}+1)},\bar{0})}{\rho}\right) \curlyvee \mathbb{M}_1\left(\frac{\bar{\mathbb{d}}(\triangle^{\mathfrak{n}-1}_{\mathfrak{m}}(\mathfrak{W}_1)_{(\mathfrak{r}+1)(\mathfrak{s}+1)},\bar{0})}{\rho}\right)\right) = (0,0).$$

Proceeding in this way by induction , we have $\mathbb{Z}(\mathbb{M},\triangle^{\mathfrak{i}}_{\mathfrak{m}})^2_{\mathbb{F}} \subset \mathbb{Z}(\mathbb{M},\triangle^{\mathfrak{n}}_{\mathfrak{m}})^2_{\mathbb{F}}, \forall\, 0 \preccurlyeq \mathfrak{i} \prec \mathfrak{n}, \forall\, \mathbb{Z} = \bar{c}$ and $\bar{c}_0$.

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