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# On The Quadruple Sequences Spaces $m(\mathbb{M},\varphi,\triangle_{j}^{i},p)_{\mathbb{F}}^{4}$ of Fuzzy Numbers Identified by Triple Young Functions

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***Abstract***— *In this paper, we examine the triple young functions defined by quadruple sequences spaces $\boldsymbol{m}(\mathbb{M},\boldsymbol{\varphi},\triangle_{j}^{i},\boldsymbol{p})_{\mathbb{F}}^{4}$ of fuzzy numbers and demonstrate several properties, such as the fact that the space $\boldsymbol{m}(\mathbb{M},\boldsymbol{\varphi},\triangle_{j}^{i},\boldsymbol{p})_{\mathbb{F}}^{4}$ is a complete metric space .*

**Keywords**—Quadruple sequences spaces , fuzzy numbers , young functions , triple young functions , metric space , complete metric space .

## 1. Introduction

Sargent ([2],[3]) introduced the $m(\varphi)$ space. He investigated a few $m(\varphi)$ space-related properties. Later, it was examined from the perspective of sequence space, and several matrix classes with one member like $m(\varphi)$ were used by Rath and Tipathy [1],Tripathy and Sen ([5],[6],[7]), Tripathy and Mahanta [4], and others.

The quadruple sequences space $m(\mathbb{M},\varphi,\triangle_{j}^{i},p)_{\mathbb{F}}^{4}$ , $0 \prec p \prec \infty$ of fuzzy numbers has been introduced in this study. Section two contains the definitions and introductions required for our work. We examine some of the features of the space $m(\mathbb{M},\varphi,\triangle_{j}^{i},p)_{\mathbb{F}}^{4}$ for both $0 \prec p \prec 1$ and $1 \preccurlyeq p \prec \infty$ in the third section .

Assuming that $\mathfrak{Q} = (\mathfrak{Q}_{\mathfrak{n}\mathfrak{t}\mathfrak{p}\mathfrak{q}})$ is a quadruple sequence, $\mathbb{P}(\mathfrak{Q})$ denotes the set of all permutations of the element of $(\mathfrak{Q}_{\mathfrak{n}\mathfrak{t}\mathfrak{p}\mathfrak{q}})$ , i.e. $\mathbb{P}(\mathfrak{Q}) = \{(\mathfrak{Q}_{\pi(\mathfrak{n}\mathfrak{t}\mathfrak{p}\mathfrak{q})}) : \pi$ is a permutation on $\mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers .

Assume that $\mathfrak{Y}_{\mathfrak{s}\mathfrak{r}\mathfrak{a}\mathfrak{b}}$ is the class of all subsets of $\mathbb{N}$ that do not contain more than a certain number of each of the components $\mathfrak{s}, \mathfrak{r}, \mathfrak{a}$, and $\mathfrak{b}$. All through $(\varphi_{\mathfrak{n}\mathfrak{t}\mathfrak{p}\mathfrak{q}})$ is a positive quadruple numbers are arranged in a non-decreasing quadruple sequence such that $\mathfrak{t}\varphi_{(\mathfrak{n}+1)(\mathfrak{t}+1)(\mathfrak{p}+1)(\mathfrak{q}+1)} \preccurlyeq (\mathfrak{n}+1)(\mathfrak{t}+1)(\mathfrak{p}+1)(\mathfrak{q}+1)\varphi_{\mathfrak{n}\mathfrak{t}\mathfrak{p}\mathfrak{q}}$ , $\forall\, \mathfrak{n},\mathfrak{t},\mathfrak{p},\mathfrak{q} \in \mathbb{N}$ .

## 2. Definitions and Preliminaries

$\Omega : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing, and convex with $\Omega(0) = 0, \Omega(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\Omega(\mathfrak{A}) \to \infty$ as $\mathfrak{A} \to \infty$ implies that $\Omega$ is an Orlicz function .

$\mathcal{H}: [0,\infty) \to [0,\infty) \ni \mathcal{H}(\mathfrak{A}) = \frac{\Omega(\mathfrak{A})}{\mathfrak{A}}$ , $\mathfrak{A} \succ 0$ and $\mathcal{H}(0) = 0, \mathcal{H}(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\mathcal{H}(\mathfrak{A}) \to 0$ as $\mathfrak{A} \to \infty$ tends to $\mathcal{H}$ is a young function .

A triple young function is a function : $[0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty) \times [0,\infty) \ni \mathbb{M}(\mathfrak{A},\mathfrak{S},\mathfrak{R}) = (\mathbb{M}_1(\mathfrak{A}),\mathbb{M}_2(\mathfrak{S}),\mathbb{M}_2(\mathfrak{R}))$ , where $\mathbb{M}_1 : [0,\infty) \to [0,\infty) \ni \mathbb{M}_1(\mathfrak{A}) = \frac{\Omega_1(\mathfrak{A})}{\mathfrak{A}}, \mathfrak{A} \succ 0$ and $\mathbb{M}_2 : [0,\infty) \to [0,\infty) \ni \mathbb{M}_2(\mathfrak{S}) = \frac{\Omega_2(\mathfrak{S})}{\mathfrak{S}}, \mathfrak{S} \succ 0$ and $\mathbb{M}_3 : [0,\infty) \to [0,\infty) \ni \mathbb{M}_3(\mathfrak{R}) = \frac{\Omega_3(\mathfrak{R})}{\mathfrak{R}}, \mathfrak{R} \succ 0$ . These functions are non-decreasing , continuous , even , convex , and satisfy the following condition:

i) $\mathbb{M}_1(0) = 0, \mathbb{M}_2(0) = 0$ , $\mathbb{M}_3(0) = 0 \Longrightarrow \mathbb{M}(0,0,0) = (\mathbb{M}_1(0),\mathbb{M}_2(0),\mathbb{M}_2(0)) = (0,0,0)$

ii) $\mathbb{M}_1(\mathfrak{A}) \succ 0, \mathbb{M}_2(\mathfrak{S}) \succ 0, \mathbb{M}_3(\mathfrak{R}) \succ 0 \Longrightarrow \mathbb{M}(\mathfrak{A},\mathfrak{S},\mathfrak{R}) = (\mathbb{M}_1(\mathfrak{A}),\mathbb{M}_2(\mathfrak{S}),\mathbb{M}_2(\mathfrak{R})) \succ (0,0,0)$, for $\mathfrak{A} \succ 0, \mathfrak{S} \succ 0, \mathfrak{R} \succ 0$ we mean by $(\mathfrak{A},\mathfrak{S},\mathfrak{R}) \succ (0,0,0)$ implies that $\mathbb{M}_1(\mathfrak{A}) \succ 0, \mathbb{M}_2(\mathfrak{S}) \succ 0, \mathbb{M}_3(\mathfrak{R}) \succ 0$.

iii) $\mathbb{M}_1(\mathfrak{A}) \to 0, \mathbb{M}_2(\mathfrak{S}) \to 0, \mathbb{M}_3(\mathfrak{R}) \to 0$ as $\mathfrak{A} \to \infty, \mathfrak{S} \to \infty, \mathfrak{R} \to \infty$ then $\mathbb{M}(\mathfrak{A},\mathfrak{S},\mathfrak{R}) = (\mathbb{M}_1(\mathfrak{A}),\mathbb{M}_2(\mathfrak{S}),\mathbb{M}_2(\mathfrak{R})) \to (0,0,0)$ as $(\mathfrak{A},\mathfrak{S},\mathfrak{R}) \to (\infty,\infty,\infty)$,we mean by $\mathbb{M}(\mathfrak{A},\mathfrak{S},\mathfrak{R}) \to (0,0,0)$ as $\mathbb{M}_1(\mathfrak{A}) \to 0, \mathbb{M}_2(\mathfrak{S}) \to 0, \mathbb{M}_3(\mathfrak{R}) \to 0$ .

It is satisfies the following conditions :

1. $\mathbb{F}$ is a convex if for each $\mathbb{F}(\mathbb{r}_2) \succcurlyeq \mathbb{F}(\mathbb{r}_1) \wedge \mathbb{F}(\mathbb{r}_3) = \min\{\mathbb{F}(\mathbb{r}_1),\mathbb{F}(\mathbb{r}_3)\}$ , $\forall \mathbb{r}_1 \prec \mathbb{r}_2 \prec \mathbb{r}_3$ , $\forall\, \mathbb{r}_1,\mathbb{r}_2,\mathbb{r}_3 \in \mathbb{R}$ .
2. $\mathbb{F}$ is normal if there is a $\mathbb{r}_0 \in \mathbb{R}$ and $\mathbb{F}(\mathbb{r}_0) = 1$.
3. $\mathbb{F}$ is upper-semi-continuous $\forall\, \mathbb{a} \in \mathbb{I}$ , $\forall\, \varepsilon \succ 0$ and $\mathbb{F}^{-1}([0,\mathbb{a}+\varepsilon))$ is open in the usual topology of $\mathbb{R}$
4. $\mathbb{F}$ is a non-negative fuzzy number $\forall\, \mathbb{r} \prec 0$ implies $\mathbb{F}(\mathbb{r}) = 0$ leads to $\mathbb{F} : \mathbb{R} \to [0,1]$ is a

fuzzy real number .

The set of all non-negative fuzzy numbers of $\mathbb{R}(\mathbb{I})$ denoted by $\mathbb{R}^*(\mathbb{I})$ . Let $\mathbb{R}(\mathbb{I})$ denote the set of all fuzzy numbers which are upper-semi continuous , normal .

In this study, we introduce and define the space $m(\mathbb{M},\varphi,\triangle_j^i,p)_{\mathbb{F}}^4$ as follows :

$$m(\mathbb{M},\varphi,\triangle_j^i,p)_{\mathbb{F}}^4=\Bigg\{\mathfrak{Q}_{\mathfrak{ntpq}}=\left((\mathfrak{Q}_1)_{\mathfrak{ntpq}},(\mathfrak{Q}_2)_{\mathfrak{ntpq}},(\mathfrak{Q}_3)_{\mathfrak{ntpq}}\right):\sup_{\mathfrak{s,r,a,b}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}}\frac{1}{\varphi_{\mathfrak{srab}}}\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left\{\left(\mathbb{M}_1\left(\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}},\ \bar{0}\right)}{\rho}\right)\right)\curlyvee\mathbb{M}_2\left(\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}},\ \bar{0}\right)}{\rho}\right)\right)\curlyvee\mathbb{M}_3\left(\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}},\ \bar{0}\right)}{\rho}\right)\right)\right)^{p}\right\}\prec\infty,\text{for some }\rho\succ 0\Bigg\},\text{ for }0\prec p\prec\infty,\text{ where }\mathbb{M}=(\mathbb{M}_1,\mathbb{M}_2,\mathbb{M}_3\ ).$$

### 3. MAIN RESULTS

**<u>Theorem 3.1</u>** :

$m(\mathbb{M},\varphi,\triangle_j^i,p)_{\mathbb{F}}^4$ is a complete metric space under the metric,

$$\mathcal{G}(\mathfrak{Q},\mathfrak{S})=\sum_{i=1}^{\mathfrak{ij}}\sum_{j=1}^{\mathfrak{ij}}\sum_{k=1}^{\mathfrak{ij}}\sum_{\ell=1}^{\mathfrak{ij}}\bar{d}\left(((\mathfrak{Q}_1)_{ijk\ell},(\mathfrak{S}_1)_{ijk\ell}),((\mathfrak{Q}_2)_{ijk\ell},(\mathfrak{S}_2)_{ijk\ell}),((\mathfrak{Q}_3)_{ijk\ell},(\mathfrak{S}_3)_{ijk\ell})\right)+\inf\left[(\rho,\rho,\rho)\succ(0,0,0):\sup_{\mathfrak{s,r,a,b}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}}\frac{1}{\varphi_{\mathfrak{srab}}}\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left\{\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}},\triangle_j^i(\mathfrak{S}_1)_{\mathfrak{ntpq}}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}},\triangle_j^i(\mathfrak{S}_2)_{\mathfrak{ntpq}}\right)}{\rho}\right)\curlyvee\mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}},\triangle_j^i(\mathfrak{S}_3)_{\mathfrak{ntpq}}\right)}{\rho}\right)\right)^{p}\right\}\preccurlyeq(1,1,1)\right],$$

$\forall\ \mathfrak{Q},\mathfrak{S}\in m(\mathbb{M},\varphi,\triangle_j^i,p)_{\mathbb{F}}^4$ ,where $\mathfrak{Q}=(\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3),\mathfrak{S}=(\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3),\mathfrak{i}\geqslant 1,\mathfrak{j}\geqslant 1$ and $0\prec p\prec 1$, where $\mathbb{M}=(\mathbb{M}_1,\mathbb{M}_2,\mathbb{M}_3)$

**<u>Proof</u>**

Let $(\mathfrak{Q}^{(g\mathfrak{f}ec)})$ be a Cauchy quadruple sequence in $m(\mathbb{M},\varphi,\triangle_j^i,p)_{\mathbb{F}}^4\ni\mathfrak{Q}^{(g\mathfrak{f}ec)}=(\mathfrak{Q}_{\mathfrak{ntpq}}^{(g\mathfrak{f}ec)})_{\mathfrak{n,t,p,q}=1}^{\infty}$.

$\forall\ \varepsilon\succ 0$ be given . $\exists$ a fixed point $\mathfrak{x}_0\succ 0$, choose $r\succ 0\ni\left(\mathbb{M}_1\left(\frac{r\mathfrak{x}_0}{2}\right)\curlyvee\mathbb{M}_2\left(\frac{r\mathfrak{x}_0}{2}\right)\curlyvee\mathbb{M}_3\left(\frac{r\mathfrak{x}_0}{2}\right)\right)\succcurlyeq(1,1,1)$. Then $\exists$ a positive integer $\mathfrak{n}_0=\ \mathfrak{n}_0(\varepsilon)\ni\mathcal{G}\left(((\mathfrak{Q}_1)^{(g\mathfrak{f}ec)},(\mathfrak{Q}_1)^{(uvwx)}),((\mathfrak{Q}_2)^{(g\mathfrak{f}ec)},(\mathfrak{Q}_2)^{(uvwx)}),((\mathfrak{Q}_3)^{(g\mathfrak{f}ec)},(\mathfrak{Q}_3)^{(uvwx)})\right)\prec\left(\frac{\varepsilon}{r\mathfrak{x}_0},\frac{\varepsilon}{r\mathfrak{x}_0},\frac{\varepsilon}{r\mathfrak{x}_0}\right),\forall\ g,\mathfrak{f},e,c,u,v,w,x\geqslant\mathfrak{n}_0$ .

By the definition of , we arrive that

$$\sum_{i=1}^{\mathfrak{ij}}\sum_{j=1}^{\mathfrak{ij}}\sum_{k=1}^{\mathfrak{ij}}\sum_{\ell=1}^{\mathfrak{ij}}\bar{d}\left(\left((\mathfrak{Q}_1)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_1)_{ijk\ell}^{(uvwx)}\right),\left((\mathfrak{Q}_2)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_2)_{ijk\ell}^{(uvwx)}\right),\left((\mathfrak{Q}_3)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_3)_{ijk\ell}^{(uvwx)}\right)\right)+\inf\left[(\rho,\rho,\rho)\succ(0,0,0):\sup_{\mathfrak{s,r,a,b}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}}\frac{1}{\varphi_{\mathfrak{srab}}}\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left\{\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(g\mathfrak{f}ec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(g\mathfrak{f}ec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\rho}\right)\curlyvee\mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(g\mathfrak{f}ec)},\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\rho}\right)\right)^{p}\right\}\preccurlyeq(1,1,1)\right]\prec(\varepsilon,\varepsilon,\varepsilon),\forall\ g,\mathfrak{f},e,c,u,v,w,x\geqslant\mathfrak{n}_0.\quad\ldots(3\text{-}1)$$

Which implies that,

$$\sum_{i=1}^{\mathfrak{ij}}\sum_{j=1}^{\mathfrak{ij}}\sum_{k=1}^{\mathfrak{ij}}\sum_{\ell=1}^{\mathfrak{ij}}\bar{d}\left(\left((\mathfrak{Q}_1)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_1)_{ijk\ell}^{(uvwx)}\right),\left((\mathfrak{Q}_2)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_2)_{ijk\ell}^{(uvwx)}\right),\left((\mathfrak{Q}_3)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_3)_{ijk\ell}^{(uvwx)}\right)\right)\prec(\varepsilon,\varepsilon,\varepsilon),\forall\ g,\mathfrak{f},e,c,u,v,w,x\geqslant\mathfrak{n}_0.$$

$$\Rightarrow\bar{d}\left(\left((\mathfrak{Q}_1)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_1)_{ijk\ell}^{(uvwx)}\right),\left((\mathfrak{Q}_2)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_2)_{ijk\ell}^{(uvwx)}\right),\left((\mathfrak{Q}_3)_{ijk\ell}^{(g\mathfrak{f}ec)},(\mathfrak{Q}_3)_{ijk\ell}^{(uvwx)}\right)\right)\prec(\varepsilon,\varepsilon),\forall\ g,\mathfrak{f},e,c,u,v,w,x\geqslant\mathfrak{n}_0\ ,\forall\ i,j,k,\ell=1,2,3,\ldots.\mathfrak{ij}.$$

Therefore $\left((\mathfrak{Q}_1)_{ijk\ell}^{(g\mathfrak{f}ec)}\right),\left((\mathfrak{Q}_2)_{ijk\ell}^{(g\mathfrak{f}ec)}\right),\left((\mathfrak{Q}_3)_{ijk\ell}^{(g\mathfrak{f}ec)}\right)$ are Cauchy quadruple sequences in $\mathbb{R}(\mathbb{I})$, so is convergent in $\mathbb{R}(\mathbb{I})$ by the completeness property of $\mathbb{R}(\mathbb{I}),\forall\ i,j,k,\ell=1,2,3,\ldots.\mathfrak{ij}$ .

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Let $\lim_{g,f,e,c\to\infty}(\mathfrak{Q}_1)_{ijk\ell}^{(gfec)} = (\mathfrak{Q}_1)_{ijk\ell}$ and $\lim_{\ell,k\to\infty}(\mathfrak{Q}_2)_{ijk\ell}^{(gfec)} = (\mathfrak{Q}_2)_{ijk\ell}$ and $\lim_{\ell,k\to\infty}(\mathfrak{Q}_3)_{ijk\ell}^{(gfec)} = (\mathfrak{Q}_3)_{ijk\ell}, \forall i,j,k,\ell = 1,2,3,\dots$ ij. (3-2).

Also,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left\{\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\rho}\right)\curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\rho}\right)\right)^{p}\right\} \preccurlyeq (1,1), \forall g,f,e,c,u,v,w,x \geqslant \mathfrak{n}_0 \;\dots(3\text{-}3)$$

For $\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} = 1$ and $\sigma$ varying over $\mathfrak{Y}_{\mathfrak{srab}}$, we obtain,

$$\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\mathcal{G}((\mathfrak{Q}_1)^{((gfec))},(\mathfrak{Q}_1)^{((uvwx)})}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\mathcal{G}((\mathfrak{Q}_2)^{((gfec))},(\mathfrak{Q}_2)^{((uvwx)})}\right)\curlyvee\mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\mathcal{G}((\mathfrak{Q}_3)^{((gfec))},(\mathfrak{Q}_3)^{((uvwx)})}\right)\right)^{p} \preccurlyeq \varphi_{1111}, \forall g,f,e,c,u,v,w,x \geqslant \mathfrak{n}_0. \Longrightarrow$$

$$\left[\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\mathcal{G}((\mathfrak{Q}_1)^{((gfec))},(\mathfrak{Q}_1)^{((uvwx)})}\right)\curlyvee\mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\mathcal{G}((\mathfrak{Q}_2)^{((gfec))},(\mathfrak{Q}_2)^{((uvwx)})}\right)\curlyvee\mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(uvwx)}\right)}{\mathcal{G}((\mathfrak{Q}_3)^{((gfec))},(\mathfrak{Q}_3)^{((uvwx)})}\right)\right] \preccurlyeq \varphi_{1111}^{\frac{1}{p}} \preccurlyeq \left(\mathbb{M}_1\left(\frac{r\mathfrak{x}_0}{2}\right)\curlyvee\mathbb{M}_2\left(\frac{r\mathfrak{x}_0}{2}\right)\curlyvee\mathbb{M}_3\left(\frac{r\mathfrak{x}_0}{2}\right)\right), \forall g,f,e,c,u,v,w,x \geqslant \mathfrak{n}_0.$$

By the continuity of $\mathbb{M}$, we get

$$\bar{d}\left(\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right),\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right),\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_{13})_{\mathfrak{ntpq}}^{(uvwx)}\right)\right) \preccurlyeq \left(\frac{r\mathfrak{x}_0}{2},\frac{r\mathfrak{x}_0}{2},\frac{r\mathfrak{x}_0}{2}\right).\mathcal{G}\left(\left((\mathfrak{Q}_1)^{(gfec)},(\mathfrak{Q}_1)^{(uvwx)}\right),\left((\mathfrak{Q}_2)^{(gfec)},(\mathfrak{Q}_2)^{(uvwx)}\right),\left((\mathfrak{Q}_3)^{(gfec)},(\mathfrak{Q}_3)^{(uvwx)}\right)\right),$$
$$\forall g,f,e,c,u,v,w,x \geqslant \mathfrak{n}_0.$$

$$\Longrightarrow \bar{d}\left(\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right),\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right),\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(uvwx)}\right)\right) \preccurlyeq \left(\frac{r\mathfrak{x}_0}{2},\frac{r\mathfrak{x}_0}{2},\frac{r\mathfrak{x}_0}{2}\right).\left(\frac{\varepsilon}{r\mathfrak{x}_0},\frac{\varepsilon}{r\mathfrak{x}_0},\frac{\varepsilon}{r\mathfrak{x}_0}\right) = \left(\frac{\varepsilon}{2},\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right), \forall g,f,e,c,u,v,w,x \geqslant \mathfrak{n}_0.$$

$$\Longrightarrow \bar{d}\left(\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(uvwx)}\right),\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(uvwx)}\right),\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(uvwx)}\right)\right) \preccurlyeq \left(\frac{\varepsilon}{2},\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right), \forall g,f,e,c,u,v,w,x \geqslant \mathfrak{n}_0.$$

Consequently $\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)}\right),\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)}\right),\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)}\right)$ are Cauchy quadruple sequences in $\mathbb{R}(\mathbb{I})$, so is convergent in $\mathbb{R}(\mathbb{I})$ by the completeness property of $\mathbb{R}(\mathbb{I})$.

Let $\lim_{gfec}\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)} = (\mathfrak{S}_1)_{\mathfrak{ntpq}}$ and $\lim_{gfec}\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)} = (\mathfrak{S}_2)_{\mathfrak{ntpq}}$ and $\lim_{gfec}\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)} = (\mathfrak{S}_3)_{\mathfrak{ntpq}}$ in $\mathbb{R}(\mathbb{I}), \forall \mathfrak{n},\mathfrak{t},\mathfrak{p},\mathfrak{q} \in \mathbb{N}$.

We have to prove that,

$\lim_{gfec}(\mathfrak{Q}_1)^{(gfec)} = \mathfrak{Q}_1$ and $\lim_{gfec}(\mathfrak{Q}_2)^{(gfec)} = \mathfrak{Q}_2$ and $\lim_{gfec}(\mathfrak{Q}_3)^{(gfec)} = \mathfrak{Q}_3$, $\forall \mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in m(\mathbb{M},\varphi,\triangle_j^i,p)_{\mathbb{F}}^4$.

$\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}} = \sum_{\mathfrak{r}=0}^{i}(-1)\binom{j}{\mathfrak{r}}(\mathfrak{Q}_1)_{(\mathfrak{n}+\mathfrak{r}j)(\mathfrak{t}+\mathfrak{r}j)(\mathfrak{p}+\mathfrak{r}j)(\mathfrak{q}+\mathfrak{r}j)}$ and $\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}} = \sum_{\mathfrak{r}=0}^{i}(-1)\binom{j}{\mathfrak{r}}(\mathfrak{Q}_2)_{(\mathfrak{n}+\mathfrak{r}j)(\mathfrak{t}+\mathfrak{r}j)(\mathfrak{p}+\mathfrak{r}j)(\mathfrak{q}+\mathfrak{r}j)}$ and $\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}} = \sum_{\mathfrak{r}=0}^{i}(-1)\binom{j}{\mathfrak{r}}(\mathfrak{Q}_3)_{(\mathfrak{n}+\mathfrak{r}j)(\mathfrak{t}+\mathfrak{r}j)(\mathfrak{p}+\mathfrak{r}j)(\mathfrak{q}+\mathfrak{r}j)}$. (**)

and

Let $\lim_{g,f,e,c\to\infty}(\mathfrak{Q}_1)_{ijk\ell}^{(gfec)} = (\mathfrak{Q}_1)_{ijk\ell}$ and $\lim_{\ell,k\to\infty}(\mathfrak{Q}_2)_{ijk\ell}^{(gfec)} = (\mathfrak{Q}_2)_{ijk\ell}$ and $\lim_{\ell,k\to\infty}(\mathfrak{Q}_3)_{ijk\ell}^{(gfec)} = (\mathfrak{Q}_3)_{ijk\ell}, \forall i,j,k,\ell = 1,2,3,\dots$ ij. (3-2).

For $\mathfrak{n},\mathfrak{t},\mathfrak{p},\mathfrak{q} = 1$, from (**) and (3-2),we have

$\lim_{g,f,e,c\to\infty}(\mathfrak{Q}_1)_{ij+1}^{(gfec)} = (\mathfrak{Q}_1)_{ij+1}$ and $\lim_{g,f,e,c\to\infty}(\mathfrak{Q}_2)_{ij+1}^{(gfec)} = (\mathfrak{Q}_2)_{ij+1}$ and

$\lim_{g,f,e,c\to\infty}(\mathfrak{Q}_3)_{ij+1}^{(gfec)} = (\mathfrak{Q}_3)_{ij+1}, \forall i \geqslant 1, j \geqslant 1$.

This mean that,

$\lim_{\ell,k\to\infty}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)} = (\mathfrak{Q}_1)_{\mathfrak{ntpq}}$ and $\lim_{\ell,k\to\infty}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)} = (\mathfrak{Q}_2)_{\mathfrak{ntpq}}$ and

$\lim_{\ell,k\to\infty}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)} = (\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \forall \mathfrak{n},\mathfrak{t},\mathfrak{p},\mathfrak{q} \in \mathbb{N}$

Also,

$\lim_{g,f,e,c\to\infty}\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)} = \triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}$ and $\lim_{g,f,e,c\to\infty}\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)} = \triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}$, and $\lim_{g,f,e,c\to\infty} \triangle_{\mathrm{j}}^{\mathrm{i}} (\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)} = \triangle_{\mathrm{j}}^{\mathrm{i}} (\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \forall\ \mathfrak{n}, \mathfrak{t}, \mathfrak{p}, \mathfrak{q} \in \mathbb{N}$.

By the continuity of $\mathbb{M}$, from (3-3), we get

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b}\succcurlyeq 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}\right)}{\rho}\right)\right)^{p}\right] \preccurlyeq (1,1,1), \text{ for some } \rho \succ 0,\ \forall\ g,f,e,c \succcurlyeq \mathfrak{n}_0.$$

Now on taking the infimum of $p$'s and using (3-1) ,we get

$$\inf\left[(\rho,\rho,\rho) \succ (0,0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b}\succcurlyeq 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}\right)}{\rho}\right)\right)^{p}\right] \preccurlyeq (1,1,1)\right] \prec (\varepsilon,\varepsilon,\varepsilon),\ \forall\ g,f,e,c \succcurlyeq \mathfrak{n}_0.$$

Moreover , we get,

$$\sum_{i=1}^{\mathrm{ij}}\sum_{j=1}^{\mathrm{ij}}\sum_{k=1}^{\mathrm{ij}}\sum_{\ell=1}^{\mathrm{ij}} \bar{d}\left(\left((\mathfrak{Q}_1)_{ijk\ell}^{(gfec)}, (\mathfrak{Q}_1)_{ijk\ell}\right), \left((\mathfrak{Q}_2)_{ijk\ell}^{(gfec)}, (\mathfrak{Q}_2)_{ijk\ell}\right), \left((\mathfrak{Q}_2)_{ijk\ell}^{(gfec)}, (\mathfrak{Q}_2)_{ijk\ell}\right)\right) + \inf\left[(\rho,\rho,\rho) \succ (0,0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b}\succcurlyeq 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\left(\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}^{(gfec)},\ \triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}}\right)}{\rho}\right)\right)^{p}\right] \preccurlyeq (1,1,1)\right] \prec (\varepsilon,\varepsilon,\varepsilon) + (\varepsilon,\varepsilon,\varepsilon) = (2\varepsilon,2\varepsilon,2\varepsilon),\ \forall\ \ell,k \succcurlyeq \mathfrak{n}_0.$$

Which leads that,

$$\mathcal{G}\left(\left((\mathfrak{Q}_1)^{(gfec)},\mathfrak{Q}_1\right), \left((\mathfrak{Q}_2)^{(gfec)},\mathfrak{Q}_2\right), \left((\mathfrak{Q}_3)^{(gfec)},\mathfrak{Q}_3\right)\right) \prec (2\varepsilon,2\varepsilon,2\varepsilon), \forall\ g,f,e,c \succcurlyeq \mathfrak{n}_0.$$

i.e. $\lim_{\ell k}(\mathfrak{Q}_1)^{(gfec)} = \mathfrak{Q}_1$ and $\lim_{\ell k}(\mathfrak{Q}_2)^{(gfec)} = \mathfrak{Q}_2$ and $\lim_{\ell k}(\mathfrak{Q}_3)^{(gfec)} = \mathfrak{Q}_3$

Now, it is to show that $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in m(\mathbb{M},\varphi,\Delta_m^n,p)_{\mathbb{F}}^4$.

We know that,

$$\mathcal{G}\left((\mathfrak{Q}_1,\bar{\theta}), (\mathfrak{Q}_2,\bar{\theta}), (\mathfrak{Q}_3,\bar{\theta})\right) \preccurlyeq \mathcal{G}\left(\left(\mathfrak{Q}_1,(\mathfrak{Q}_1)^{(nmba)}\right), \left(\mathfrak{Q}_2,(\mathfrak{Q}_2)^{(nmba)}\right), \left(\mathfrak{Q}_3,(\mathfrak{Q}_3)^{(nmba)}\right)\right) +$$
$$\mathcal{G}\left(\left((\mathfrak{Q}_1)^{(nmba)},\bar{\theta}\right), \left((\mathfrak{Q}_2)^{(nmba)},\bar{\theta}\right), \left((\mathfrak{Q}_3)^{(nmba)},\bar{\theta}\right)\right) \prec (\varepsilon,\varepsilon,\varepsilon) + (\mathbb{M}_1,\mathbb{M}_2,\mathbb{M}_3), \forall\ n,m,b,a \succcurlyeq \mathfrak{n}_0(\varepsilon).$$

i.e. $\mathcal{G}\left((\mathfrak{Q}_1,\bar{\theta}), (\mathfrak{Q}_2,\bar{\theta}), (\mathfrak{Q}_3,\bar{\theta})\right)$ is finite.

Therefore $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in m(\mathbb{M},\varphi,\triangle_{\mathrm{j}}^{\mathrm{i}},p)_{\mathbb{F}}^4$.

Thus,

$m(\mathbb{M},\varphi,\triangle_{\mathrm{j}}^{\mathrm{i}},p)_{\mathbb{F}}^4$ is complete metric space .

**Proposition 3.2 :**

$m(\mathbb{M},\varphi,\triangle_{\mathrm{j}}^{\mathrm{i}})_{\mathbb{F}}^4 \subseteq m(\mathbb{M},\varphi,\triangle_{\mathrm{j}}^{\mathrm{i}},p)_{\mathbb{F}}^4$ , for $1 \preccurlyeq p \prec \infty$ , where $\mathbb{M} = (\mathbb{M}_1,\mathbb{M}_2,\mathbb{M}_3)$.

**Proof** :

Let $= (\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3) \in m(\mathbb{M},\varphi,\triangle_{\mathrm{j}}^{\mathrm{i}})_{\mathbb{F}}^4$.

Then we have,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b}\succcurlyeq 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right)\right] = \mathbb{K}\ (\prec \infty), \text{ for some } \rho \succ 0.$$

$\forall\ \mathfrak{s}, \mathfrak{r}, \mathfrak{a}, \mathfrak{b}$ and $\sigma \in \mathfrak{Y}_{\mathfrak{srab}}$ , we have

$$\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right)\right] \preccurlyeq \mathbb{K}\varphi_{\mathfrak{srab}}, \text{ for some } \rho \succ 0.$$

$$\Rightarrow \left(\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right)\right]^{p}\right)^{\frac{1}{p}} \preccurlyeq \mathbb{K}\varphi_{\mathfrak{srab}}.$$

$$\Rightarrow \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b}\succcurlyeq 1,\sigma\in\mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}}\left(\sum_{\mathfrak{n}\in\sigma}\sum_{\mathfrak{t}\in\sigma}\sum_{\mathfrak{p}\in\sigma}\sum_{\mathfrak{q}\in\sigma}\left[\mathbb{M}_1\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_1)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_2)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{\bar{d}\left(\triangle_{\mathrm{j}}^{\mathrm{i}}(\mathfrak{Q}_3)_{\mathfrak{ntpq}},\bar{0}\right)}{\rho}\right)\right]^{p}\right)^{\frac{1}{p}} \preccurlyeq \mathbb{K} \prec \infty.$$

Therefore $\mathfrak{Q} = (\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3) \in m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4$, for $1 \preccurlyeq p \prec \infty$.

**Proposition 3.3 :**

$$m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 \subseteq m(\mathbb{M}, \Psi, \triangle_j^i, p)_{\mathbb{F}}^4 \Leftrightarrow \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} \left(\frac{\varphi_{\mathfrak{srab}}}{\Psi_{\mathfrak{srab}}}\right) \prec \infty, \text{ for } 0 \prec p \prec \infty .$$

**Proof** :

Let $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} \left(\frac{\varphi_{\mathfrak{srab}}}{\Psi_{\mathfrak{srab}}}\right) = \mathbb{K} \prec \infty$ .To prove that $(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 \subseteq m(\mathbb{M}, \Psi, \triangle_j^i, p)_{\mathbb{F}}^4$ .

Assume that $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} \left(\frac{\varphi_{\mathfrak{srab}}}{\Psi_{\mathfrak{srab}}}\right) = \mathbb{K} \prec \infty$ , we have $\varphi_{\mathfrak{srab}} \preccurlyeq \mathbb{K}\, \Psi_{\mathfrak{srab}}$ .

Now, if $\left(\mathfrak{Q}_{\mathfrak{ntpq}}\right) = \left((\mathfrak{Q}_1)_{\mathfrak{ntpq}}, (\mathfrak{Q}_2)_{\mathfrak{ntpq}},, (\mathfrak{Q}_2)_{\mathfrak{ntpq}}\right) \in m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4$ , then ,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1, \sigma \in \mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma} \sum_{\mathfrak{t}\in\sigma} \sum_{\mathfrak{p}\in\sigma} \sum_{\mathfrak{q}\in\sigma} \left[\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right)\right]^{p} \prec \infty.$$

$$\Rightarrow \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1, \sigma \in \mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\mathbb{K}\, \Psi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma} \sum_{\mathfrak{t}\in\sigma} \sum_{\mathfrak{p}\in\sigma} \sum_{\mathfrak{q}\in\sigma} \left[\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right)\right]^{p} \prec \infty.$$

$\Rightarrow \left(\mathfrak{Q}_{\mathfrak{ntpq}}\right) \in m(\mathbb{M}, \Psi, \triangle_j^i, p)_{\mathbb{F}}^4$ .

Hence $(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 \subseteq m(\mathbb{M}, \Psi, \triangle_j^i, p)_{\mathbb{F}}^4$ .

Conversely, suppose that $m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 \subseteq m(\mathbb{M}, \Psi, \triangle_j^i, p)_{\mathbb{F}}^4$ .To prove that $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} \left(\frac{\varphi_{\mathfrak{srab}}}{\Psi_{\mathfrak{srab}}}\right) = \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} (\mathfrak{J}_{\mathfrak{srab}}) \prec \infty$.

Suppose $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} (\mathfrak{J}_{\mathfrak{srab}}) = \infty$. Then there exists a double subsequence $\left(\mathfrak{J}_{\mathfrak{srab}_{ijk\ell}}\right)$ of $(\mathfrak{J}_{\mathfrak{srab}})$ such that $\lim_{ijk\ell \to \infty} \left(\mathfrak{J}_{\mathfrak{sr}_{ijk\ell}}\right) = \infty$.

Then for $\left(\mathfrak{Q}_{\mathfrak{ntpq}}\right) \in m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4$, we have,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1, \sigma \in \mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\Psi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma} \sum_{\mathfrak{t}\in\sigma} \sum_{\mathfrak{p}\in\sigma} \sum_{\mathfrak{q}\in\sigma} \left[\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right)\right]^{p} \succcurlyeq$$

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1, \sigma \in \mathfrak{Y}_{\mathfrak{srab}}} \frac{\mathfrak{J}_{\mathfrak{srab}_{ijk\ell}}}{\varphi_{\mathfrak{srab}_{ijk\ell}}} \sum_{\mathfrak{n}\in\sigma} \sum_{\mathfrak{t}\in\sigma} \sum_{\mathfrak{p}\in\sigma} \sum_{\mathfrak{q}\in\sigma} \left[\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right)\right]^{p} = \infty.$$

i.e. $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1, \sigma \in \mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\Psi_{\mathfrak{srab}}} \sum_{\mathfrak{n}\in\sigma} \sum_{\mathfrak{t}\in\sigma} \sum_{\mathfrak{p}\in\sigma} \sum_{\mathfrak{q}\in\sigma} \left[\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\rho}\right)\right]^{p} = \infty$ .

Therefore $\left(\mathfrak{Q}_{\mathfrak{ntpq}}\right) \notin m(\mathbb{M}, \Psi, \triangle_j^i, p)_{\mathbb{F}}^4$ is a contradiction. Hence $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1} \left(\frac{\varphi_{\mathfrak{srab}}}{\Psi_{\mathfrak{srab}}}\right) \prec \infty$.

**Proposition 3.4 :**

$$\ell_{p}(\mathbb{M}, \triangle_j^i)_{\mathbb{F}}^4 \subseteq m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 \subseteq \ell_\infty(\mathbb{M}, \triangle_j^i)_{\mathbb{F}}^4 \text{ for } 1 \preccurlyeq p \prec \infty \text{, where } \mathbb{M} = (\mathbb{M}_1, \mathbb{M}_2).$$

**Proof** :

On taking $\mathbb{M}(x_1, x_2) = \left(x_1^{p}, x_2^{p}, x_3^{p}\right), \forall\, x_1, x_2, x_3 \in [0, \infty)$ and for $1 \preccurlyeq p \prec \infty$ and $\varphi_{\mathfrak{ntpq}} = (1,1,1), \forall\, \mathfrak{n}, \mathfrak{t}, \mathfrak{p}, \mathfrak{q} \in \mathbb{N}$, we arrive that $m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 = \ell_{p}(\mathbb{M}, \triangle_j^i)_{\mathbb{F}}^4$ .So, the first inclusion is clear.

Next, suppose that, $(\mathfrak{Q}_{\mathfrak{ntpq}}) = ((\mathfrak{Q}_1)_{\mathfrak{ntpq}}, (\mathfrak{Q}_2)_{\mathfrak{ntpq}}, (\mathfrak{Q}_3)_{\mathfrak{ntpq}}) \in m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4$

This tends that,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{a},\mathfrak{b} \geqslant 1, \sigma \in \mathfrak{Y}_{\mathfrak{srab}}} \frac{1}{\varphi_{\mathfrak{srab}}} \left[\sum_{\mathfrak{a}\in\sigma} \sum_{\mathfrak{b}\in\sigma} \left(\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right)\right)^{p}\right]^{\frac{1}{p}} = \mathbb{K}(\prec \infty).$$

$$\forall\, \mathfrak{s}, \mathfrak{r}, \mathfrak{a}, \mathfrak{b} = 1, \left(\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right)\right) \preccurlyeq \mathbb{K}\varphi_{1111} .$$

Which indicates that ,

$$\sup_{\mathfrak{ntpq} \geqslant 1} \left[\mathbb{M}_1\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_1)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right) \curlyvee \mathbb{M}_2\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_2)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right) \curlyvee \mathbb{M}_3\left(\frac{d\left(\triangle_j^i(\mathfrak{Q}_3)_{\mathfrak{ntpq}}, \bar{0}\right)}{\mathfrak{p}}\right)\right] \prec \infty .$$

Therefore $\left(\mathfrak{Q}_{\mathfrak{ntpq}}\right) \in \ell_\infty(\mathbb{M}, \triangle_j^i)_{\mathbb{F}}^4$ .

Thus,

$$\ell_{p}(\mathbb{M}, \triangle_j^i)_{\mathbb{F}}^4 \subseteq m(\mathbb{M}, \varphi, \triangle_j^i, p)_{\mathbb{F}}^4 \subseteq \ell_\infty(\mathbb{M}, \triangle_j^i)_{\mathbb{F}}^4$$

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