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The Scale of Compact Dissipativity on Random Dynamical Systems

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Abstract: *In this work, the structure of the Levinson center will be described on a random case. In particular, the random Levinson center coincides with the positive prolongation of the Omega-limit set. Further, we create the relationship between compact dissipative and point dissipative random dynamical systems. Also, it has been demonstrated by a n example that compactly dissipative does not imply the point dissipatively.*

Keywords: Random dynamical system (RDS), Point dissipative, Compact dissipative, Random attracter, random Levinson center (RLS).

1.Introduction: Dissipative theory provides a powerful theoretical structure for control design and analysis of dynamic systems . In particular, Many physical dynamic systems have some inputs, outputs, and state haracteristics associated to preservation, dissipation, and transfer of energy and mass , and this gives importance to the dissipative theory [8]. Willems [12] , [13] presented the foundation for developing theory of dissipativity for deterministic nonlinear smooth flows. Also, one can see Cheban [1]. The concept of dissipative for RDSs was introduced by many authors, see for example, Igor [7], Crauel and Flandoli F. **[**3**],** Wang Y., Liu Y. and Wang Z. [11], Gu A. **[**6] and others, and some types of dissipativity for RDSs were established by Yasir and kadhim [16]. Many concepts of dissipativity in random case have been studied in [10], [14], and [15].
The aim of this paper is studying the scale for Compact dissipativity and the notion of Levinson center for RDSs.

2. **Preliminaries for RDSs:**

Some essential notions associated with RDS are stated here, which are important in our work. Throughout, (X, d) any matric space and We call a subset K of X is pre compact if every sequence in K admits a subsequence converges to a point of X, and is call **relatively compact** if and only if every sequence of points in A has a cluster point in X, for more details see, **[**2].

Definition 2.1[5]: Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let the measurable function $\theta: \mathbb{T} \times \Omega \longrightarrow \Omega$ satisfy the following

$$\vartheta_0 = \mathrm{id}_\Omega, \vartheta_t \circ \vartheta_s = \vartheta_{t+s} \forall\, t, s \in \mathbb{T}\ ;\ \text{and}\ \ \vartheta_t \mathbb{P} = \mathbb{P}\ \forall\, t \in \mathbb{T}.$$

if $\theta_t A = A$, for every $t \in \mathbb{T}$, then a set $A \in \mathcal{F}$ is called ϑ $-$invariant. If for any ϑ $-$invariant set $A \in \mathcal{F}$ either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$, then ϑ is called ergodic under $\mathbb{P}$.

Definition 2.2[7]: Consider topological space X and a locally compact group $\mathbb{T}$. The random dynamical system (**RDS)** is a pair (ϑ, ψ) involving a cocycle ψ of continuous functions on X over an MDS ϑ, i.e. a mapping $\psi: \mathbb{T} \times \Omega \times X \longrightarrow X$, $(t, \omega, x) \longmapsto \psi(t, \omega, x)$ is measurable and satisfy the following for every $t \in \mathbb{T}$ and $\omega \in \Omega$:

(i) the function $x \longmapsto \psi(t, \omega, x) \equiv \psi(t, \omega)x$ is continuous , and

(ii) the function $\psi(t, \omega) := \psi(t, \omega, \cdot)$ fulfill:

$$\psi(0, \omega) = \mathrm{id}_X, \psi(t + s, \omega) = \psi(t, \vartheta_s \omega) \circ \psi(s, \omega).$$

The property (ii) called cocycle property of φ**.**

Remark: In general the function $(t,x) \longmapsto \psi(t,\vartheta_{-t}\omega)x$ is not continuous (see [7]). However, in this work, we will assume that $\psi(t,\vartheta_{-t}\omega): X \to X$ is continuous unless otherwise stated.

Definition 2.3 [7]: Consider a metric space (X,d) . The set-valued function $D: \Omega \longrightarrow 2^X$ is called a **random set** if $D(\omega) \neq \emptyset$ and for every $x \in X$ the function $\omega \longmapsto \mathrm{dist}_X(x, D(\omega))$ measurable. A random set D is called a **random closed (compact) set** when for each $\omega \in \Omega$, the set $D(\omega)$ is closed(resp. compact). A random set D is said to be **bounded** if $D(\omega)$ contained in some random ball for all $\omega \in \Omega$.

For more convenient the random set $D: \Omega \longrightarrow 2^X$ will denoted by D or $\{D(\omega)\}$.

Definition 2.4 [7]: Consider a random set $D: \Omega \longrightarrow 2^X$. We say that the set- valued function $\gamma_D^t(\omega): \Omega \longrightarrow 2^X$ defined by

$$\gamma_D^t(\omega) := \bigcup_{\tau \geq t} \psi(\tau, \theta_{-\tau}\omega) D(\vartheta_{-\tau}\omega)$$

the pull back trajectories starting from D. If $D(\omega) = \{v(\omega)\}$ is a single valued function, then $\omega \longmapsto \gamma_v(\omega) \equiv \gamma_D^0(\omega)$ is said to be the (pull back) trajectory starting from v.

Definition 2.5 [7]: Consider the RDS (ϑ, ψ).

i. A random set $A: \Omega \longrightarrow 2^X$ is called **invariant** if $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t \in \mathbb{R}$, $\omega \in \Omega$.

ii. A **random pull back attractor** of the RDS (ϑ, ψ) in the universe $\mathcal{U}$ is a proper invariant random closed set $A \in \mathcal{U}$ which is attracting in $\mathcal{U}$, i.e., for all $U \in \mathcal{U}$ we have

$$\lim_{t\to\infty} d(\psi(t,\vartheta_{-t}\omega)U(\vartheta_{-\tau}\omega), A(\omega)) = 0$$

Definition 2.6 [7]: Consider the RDS (ϑ, ψ), and let t $M: \Omega \longrightarrow 2^X$ be a random set.

i. The set-valued function

$$\omega \longmapsto \Gamma_M^+(\omega) := \bigcap \overline{\gamma_M^t(\omega)} = \bigcap_{t>0} \overline{\bigcup_{\tau \geq t} \psi(\tau, \theta_{-\tau}\omega) M(\vartheta_{-\tau}\omega)}$$

is called the **omega-limit set** of the trajectories starting from M

ii. The set-valued function

$$\omega \longmapsto \Gamma_M^-(\omega) := \bigcap \overline{\gamma_M^t(\omega)} = \bigcap_{t<0} \overline{\bigcup_{\tau \leq t} \psi(\tau, \theta_{-\tau}\omega) M(\vartheta_{-\tau}\omega)}$$

is called the **alpha-limit set**.

Definition 2.7[17]: The RDS (ϑ, ψ) is called

i. **compact dissipative** if for every compact random set A in X there is random set K in X so that

$$\lim_{t\longrightarrow+\infty} \sup\{d(\psi(t,\vartheta_{-t}\omega)x, K(\omega)) : x \in A(\vartheta_{-t}\omega)\} = 0$$

ii. **point dissipative** if for every $x \in X^\Omega$, there is random set K in X so that,

$$\lim_{t\longrightarrow+\infty} d(\psi(t,\theta_{-t}\omega)x(\vartheta_{-t}\omega), K(\omega)) = 0$$

Definition 2.8 [17]: Consider RDS (ϑ, ψ). A random set M is said to be **orbitally stable** whenever for any tempered random variable ε and any non-negative number t we have

$$d(x, M(\omega)) < \delta(\omega) \text{ implies } d(\psi(t,\theta_{-t}\omega)x, M(\omega)) < \varepsilon(\omega)$$

for some tempered random variable δ.

Definition 2.9 [17]: we will call the set $L_X(\omega)$ defined by following equality:

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$$L_X(\omega) := \Gamma_K(\omega) = \cap \{\psi(t,\theta_{-t}\omega)K(\vartheta_{-t}\omega) | t \in T, \omega \in \Omega\}$$

the **random Levinson center** of the compact dissipative RDS (ϑ, ψ).

3. Characterization of Compact Dissipativity on RDS:

Assume $\Gamma_X(\omega) := \overline{\cup\{\Gamma_x(\omega) | x \in X\}}$, $\omega \in \Omega$. Let (ϑ, ψ) be a compact dissipative RDS and $L_X(\omega)$ its random Levinson center (see, for more details [15]). It is clear that the set $\Gamma_X(\omega) \subseteq L_X(\omega)$, and $\Gamma_X(\omega)$ is a main property of a dissipative RDS. We see, from Theorem 3.5 [16] in a local compact space X, the set $\Gamma_X(\omega)$ characterize point dissipativity when it is not empty and compact. Based on the foregoing, We will examine the relationship between $\Gamma_X(\omega)$ and $L_X(\omega)$.

Let $\{\varepsilon_s : \Omega \to \mathbb{R},\ s \in \mathbb{R}^+\}$ be a family of tempered random variables (t.r.v), and define the forward prolongation and the forward limit prolongation of the random set M respectively as follows : Define two operators $D^+, J^+ : X \longrightarrow 2^X$ as follows: for every random set $M : \Omega \longrightarrow X$

$$D^+(M(\omega)) := \cap_{s>0} \overline{\cup\{\psi(t,\theta_{-t}\omega)B(M,\varepsilon_s) | t \geq 0, \omega \in \Omega\}},$$

$$J^+(M(\omega)) := \cap_{s>0} \cap_{t\geq 0} \overline{\cup\{\psi(\tau,\theta_{-\tau}\omega)B(M,\varepsilon_s) | \omega \in \Omega, \tau \geq t\}}$$

Sometimes we will write $D^+\big(M(\omega)\big) = D_M^+(\omega)$ and $J^+\big(M(\omega)\big) = J_M^+(\omega)$.

In particular, if $= \{x\}$, then we set

$$D_x^+(\omega) := D^+(\{x\}), \quad \text{and} \quad J_x^+(\omega) := J^+(\{x\}).$$

The set $D_M^+(\omega)$ is called the **first forward prolongation of a random set** M and $J_M^+(\omega)$ is called **first forward prolongational limit set of a random set** M. It clear that $D_M^+(\omega)$ and $J_M^+(\omega)$ are random sets for every random set. In fact $D_M^+(\omega)$ and $J_M^+(\omega)$ are closed invariant sets as will be seen in Proposition 3.2 below.

Note that, [9] define the prolongation and prolongational of a random set in terms of sequences but here we define these concepts as in above and will give a characterization in terms of sequences in Theorem 3.2. However, our definition is different from that one given in [9].

Notation: We will introduce the following two sets in order to give the characterization of the first forward prolongation and first forward prolongational limit set of a random set M in terms of the sequence.

$\widetilde{D}_M^+(\omega) := \{y \in X$: there exist $\{x_n\}$ and $\{t_n\}$ such $d(x_n, M(\vartheta_{-t_n}\omega)) \to 0$ and $\psi(t_n, \vartheta_{-t_n}\omega)x_n \to y\}$ and

$\tilde{J}_M^+(\omega) := \{y \in X$: there exist $\{x_n\}$ and $\{t_n\}, t_n \longrightarrow +\infty$ such $d(x_n, M(\vartheta_{-t_n}\omega)) \longrightarrow 0$ and $\psi(t_n, \vartheta_{-t_n}\omega)x_n \longrightarrow y\}$

Theorem 3.1: If (ϑ, ψ) is an RDS, then for every random set M, the following hold:

i. $D_M^+(\omega) = \widetilde{D}_M^+(\omega)$,

ii. $J_M^+(\omega) = \tilde{J}_M^+(\omega)$.

Proof: i. Let $p \in D_M^+(\omega)$, then there exist sequence $\{p_n\} \subset \cup\{\psi(t,\vartheta_{-t}\omega)B(M(\vartheta_{-t}\omega),\varepsilon_s) | t \geq 0, \omega \in \Omega\}$, $p_n \to p$,

So, $p_n \in \cup\{\psi(t,\theta_{-t}\omega)B(M(\vartheta_{-t}\omega),\varepsilon_s) | t \geq 0, \omega \in \Omega\}$, $\forall\ s > 0$ such that $p_n = \psi(t_n, \vartheta_{-t_n}\omega)x_n$. But $p_n \to p$ so $\psi(t_n, \vartheta_{-t_n}\omega)x_n \to p$. Since for every $s > 0$ we have $\{x_n\} \in B\big(M(\vartheta_{-t_n}\omega), \varepsilon_s\big)$ then

$$d(x_n, B\left(M(\vartheta_{-t_n}\omega)\right) < \varepsilon_s, \text{for all } s > 0 .$$

Thus there exist $\{x_n\}$ and $\{t_n\}$ such that

$$d(x_n, M(\theta_{-t_n}\omega)) \to 0 \text{ and } \varphi(t_n, \theta_{-t_n}\omega)x_n \to p.$$

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Then $p \in \widetilde{D}_M^+(\omega)$. By reversing the above argument we get the reverse inclusion and then the result follows.

ii. Let $y \in J_M^+(\omega)$. Suppose that $t \geq 0$ and $\{U_m : m = 1,2,\ldots\}$ is a collection of all neighborhoods of y.

$$y \in \bigcap_{s>0} \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \psi(\tau, \theta_{-\tau}\omega)\mathcal{B}(M, \varepsilon_s)}$$

Then

$$y \in \overline{\bigcup_{\tau\geq t} \psi(\tau, \theta_{-\tau}\omega)\mathcal{B}(M, \varepsilon_s)} \text{ for every } t \geq 0, \quad s > 0.$$

Thus, there is sequence $\{y_m\}$ in $\bigcup_{\tau\geq t} \psi(\tau, \vartheta_{-\tau}\omega)\mathcal{B}(M, \varepsilon_s)$ such that $y_m \longrightarrow y$.

Since $y_m \in \bigcup_{\tau\geq t} \psi(\tau, \theta_{-\tau}\omega)\mathcal{B}(M, \varepsilon_s)$ for every m. Then there is $\tau_m \geq 0$ and $x_m \in B(M, \varepsilon_s)$ such that

$$y_m = \psi(\tau_m, \vartheta_{-\tau_m}\omega)x_m \,.$$

Then,

$$\psi(\tau_m, \vartheta_{-\tau_m}\omega)x_m \longrightarrow M(\vartheta_{-t_n}\omega) \text{ and } \tau_m \longrightarrow \infty.$$

This means that $y \in \tilde{\Gamma}_M^+(\omega)$.

Conversely, let $y \in \tilde{\Gamma}_M^+(\omega)$. Then $\psi(t_n, \vartheta_{-t_n}\omega)x_n \longrightarrow y$ for some sequence $\{t_n\}$ in $\mathbb{R}^+$ and $d(x_n, M(\vartheta_{-t_n}\omega)) \longrightarrow 0$ with $t_n \longrightarrow \infty$. Thus, for any neighborhood U of y and any $t \geq 0$, there exist n_0 and n_1 such that $t_n \geq t$ for every $n > n_0$ and $\psi(t_n, \vartheta_{-t_n}\omega)x_n \in U$ for every $n > n_1$. Also, we have $t_{n_2} \geq t$ and $\psi(t_{n_2}, \vartheta_{-t_{n_2}}\omega)x_n \in U$ for every $n_2 = \max\{n_0, n_1\}$. Therefore,

$$U \cap \left(\bigcup_{\tau\geq t} \psi(\tau, \theta_{-\tau}\omega)B(M(\vartheta_{-\tau}\omega)\,, \varepsilon_s)\right) \neq \emptyset\,.$$

It follows that,

$$y \in \overline{\bigcup_{\tau\geq t} \psi(\tau, \theta_{-\tau}\omega)B(M(\vartheta_{-\tau}\omega), \varepsilon_s),} \text{ , for every } t \geq 0 \text{ and } s \geq 0$$

Thus,

$$y \in \bigcap_{s>0} \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \psi(\tau, \theta_{-\tau}\omega)B(M, \varepsilon_s)} = \Gamma_M^+(\omega)\,. \qquad \blacksquare$$

Preposition 3.2: The set D_M^+ (respectively, J_M^+) is closed and forward invariant.

Proof: By definition, D_M^+ is closed. For forward invariance, suppose that $x \in D_M^+(\omega)$ and $t > 0$. Then there exist two sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathbb{R}^+$ such that $d(x_n, M(\vartheta_{-t_n}\omega)) \longrightarrow 0$ and $\psi\big(t_n, \vartheta_{-t_n}\omega\big)x_n \longrightarrow x$.

Since $\varphi(t,\omega): X \longrightarrow X$ is continuous, then $\psi(t,\omega) \circ \psi\big(t_n, \vartheta_{-t_n}\omega\big)x_n \longrightarrow \psi(t,\omega)y$. So

$$\psi\big(t + t_n, \vartheta_{-(t+t_n)}\vartheta_t\omega\big)x_n \longrightarrow \psi(t,\omega)x.$$

Also since $d(x_n, M(\vartheta_{-t_n}\omega)) \longrightarrow 0$, then $d(x_n, M(\vartheta_{-(t+t_n)}\omega) \longrightarrow 0$. Hence $\psi(t,\omega)x \in D_M^+(\vartheta_t\omega)$. Consequently $D_M^+(\omega) \subset D_M^+(\vartheta_t\omega)$ for all $t > 0$.

Let $x \in D_M^+(\vartheta_t\omega)$ and let $t > 0$. Then there exist two sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathbb{R}^+$ such that $d(x_n, M(\vartheta_{-t_n}\vartheta_t\omega) \longrightarrow 0$ and $\big(t_n, \vartheta_{-t_n}\vartheta_t\omega\big)x_n \longrightarrow x$. So

$$d(x_n, M(\vartheta_{-t_n+t}\omega) \longrightarrow 0 \text{ and } \varphi(t,\omega)z_n \longrightarrow x \qquad (3.1)$$

where $z_n = \varphi(t_n - t, \vartheta_{-t_n+t}\omega)x_n$ .Moreover, by Theorem 3.1 (i) , $b \in D_M^+$. By (3.1) $x = \psi(t,\omega)b$. So,

$$D_M^+(\vartheta_t\omega) \subset \psi(t,\omega)D_M^+(\omega) \text{ for every } t > 0 \text{ and } \omega \in \Omega.$$

Thus $D_M^+(\omega)$ is an invariant.

Similarly, We can show that J_M^+ is closed and invariant.

Lemma 3.3: Let (ϑ,ψ) be an RDS, $\psi(t,\vartheta_{-t}\omega): X \to X$ is continuous for every $t > 0$ and $y \in \Gamma_x(\omega)$. Then $J_x^+(\omega) \subseteq J_y^+(\omega)$.

Proof: Let $y \in \Gamma_x(\omega)$ and p $\in J_x^+(\omega)$. Then there exist sequences $\{\bar{\tau_n}\} \to +\infty, \varphi\left(\bar{\tau_n}, \theta_{-\bar{\tau_n}}\omega\right)x \to y, \{\bar{t_n}\}$ and $\{x_n\}$ such that $x_n \to x$, $t_n \to +\infty$, and $\varphi\left(\bar{t_n}, \theta_{-\bar{t_n}}\omega\right)x_n \to p$. According to Theorem 3.1 we can consider that $\bar{t_n} - \bar{\tau_n} > n$ for all $n \in N$. $\forall\ k \in N$ consider the sequence $\{\varphi\left(\bar{\tau_k}, \theta_{-\bar{\tau_k}}\omega\right)x_n\}$. By hypothesis ,

$$\varphi\left(\bar{\tau}_{nk}, \theta_{-\bar{t}_{nk}}\omega\right)x_n \to \varphi\left(\bar{\tau}_{nk}, \theta_{-\bar{t}_{nk}}\omega\right)x \ \ as\ n \to +\infty,\ \ \forall\ k \in N,$$

and consequently, $\forall\ k \in N$ there exists $n_k \geq k$ such that

$$d(\varphi\left(\bar{\tau_k}, \theta_{-\bar{t}_{nk}}\omega\right)x_n, \varphi\left(\bar{\tau_k}, \theta_{-\bar{t}_{nk}}\omega\right)x) \leq k^{-1} \text{ for all } n \geq n_k.$$

As $\varphi\left(\bar{\tau_n}, \theta_{-\bar{\tau_n}}\omega\right)x \to y$, we have

$$d\left(y, \varphi\left(\bar{\tau}_{nk}, \theta_{-\bar{t}_{nk}}\omega\right)x_{nk}\right) \leq d\left(y, \varphi\left(\bar{\tau_k}, \theta_{-\bar{\tau_k}}\omega\right)x\right) + d\left(\varphi\left(\bar{\tau_k}, \theta_{-\bar{\tau_k}}\omega\right)x, \varphi\left(\bar{\tau}_{nk}, \theta_{-\bar{t}_{nk}}\omega\right)x_{nk}\right)$$

$$\leq d\left(y, \varphi\left(\bar{\tau_k}, \theta_{-\bar{\tau_k}}\omega\right)x\right) + d\left(\varphi\left(\bar{\tau_k}, \theta_{-\bar{\tau_k}}\omega\right)x, \varphi\left(\bar{\tau_k}, \theta_{-\bar{t}_{nk}}\omega\right)x\right)$$

$$+d\left(\varphi\left(\bar{\tau_k}, \theta_{-\bar{t}_{nk}}\omega\right)x, \varphi\left(\bar{\tau}_{nk}, \theta_{-\bar{t}_{nk}}\omega\right)x_{nk}\right)$$

Take the limit as $\longrightarrow \infty$, then all terms on the right hand side of the above inequality approach zero, and so we have $d\left(y, \varphi\left(\bar{\tau}_{nk}, \theta_{-\bar{t}_{nk}}\omega\right)x_{nk}\right) \longrightarrow 0$. Note that,

$$\varphi\left(\bar{t}_{nk}, \vartheta_{-\bar{t}_{nk}}\omega\right)x_{nk} = \varphi\left(\bar{t}_{nk} - \bar{\tau}_{nk} + \bar{\tau}_{nk}, \vartheta_{-\bar{t}_{nk}}\omega\right)x_{nk},$$

$$= \varphi\left(\bar{t}_{nk} - \bar{\tau}_{nk}, \vartheta_{-(\bar{t}_{nk} - \bar{\tau}_{nk})}\omega\right) \circ \varphi\left(\bar{\tau}_{nk}, \vartheta_{-\bar{t}_{nk}}\omega\right)x_{nk}$$

$\varphi\left(\bar{\tau}_{nk}, \vartheta_{-\bar{t}_{nk}}\omega\right)x_{nk} \longrightarrow y$, and $\bar{t}_{nk} - \bar{\tau}_{nk} > n_k \geq k$. From this it follows that $p \in J_y^+(\omega)$, *i.e.*, $J_x^+(\omega) \subseteq J_y^+(\omega)$. ■

Corollary 3.4: If $y \in \Gamma_x^+(\omega)$, then $J_y^+(\omega) = D_y^+(\omega)$.

Proof: Since $J_y^+(\omega) \subseteq D_y^+(\omega)$, it is sufficient to show that $D_y^+(\omega) \subseteq J_y^+(\omega)$.

Note that $D_y^+(\omega) = \gamma_y^+(\omega) \cup J_y^+(\omega)$. Since $y \in \Gamma_x(\omega)$, we have

$$\gamma_y^+(\omega) \subseteq \Gamma_x^+(\omega) \subseteq J_x^+(\omega) \subseteq J_y^+(\omega).$$

Hence

$$D_y^+(\omega) = \gamma_y^+(\omega) \cup J_y^+(\omega) \subseteq J_y^+(\omega) \cup J_y^+(\omega) = J_y^+(\omega).$$

Lemma 3.5: Consider the RDS (ϑ,ψ) . If $x_n \to x,\ y_n \to y\ as\ n \to +\infty$ and $x_n \in D^+_{y_n}(\omega)$ (respectively, $x_n \in J^+_{y_n}(\omega)$), then $x \in D^+_y(\omega)$ (respectively $x \in J^+_y(\omega)$).

Proof: Let > 0 and $\delta > 0$. Since $x_n \in D^+_y(\omega)$ ($x_n \in J^+_y(\omega)$) for all n there is $z_n \in B(y_n,\delta/2)$ and a $t_n \geq 0$ (resp., $t_n \geq n$) with

$$\mathrm{d}\big(x_n,\psi(\ t_n,\vartheta_{-\ t_n}\omega)z_n\big) < \frac{\varepsilon}{2} \qquad (3.2)$$

From $x_n \to x$ and $y_n \to y$ It follows that there is an integer n_0 such that $\forall\, n > n_0$, the inequalities

$$d(y,y_n) < \delta/2 \quad \text{and} \quad d(x_n,x) < \delta/2 \qquad (3.3)$$

hold simultaneously. From $z_n \in B(y_n,\delta/2)$ and (3.3) we obtain $d(y,z_n) < \delta$, and from (3.2) and (3.3) follows the inequality $d\big(x,\psi(\ t_n,\vartheta_{-\ t_n}\omega)z_n\big) < \varepsilon,\ i.e., x \in D^+_y(\omega)$ (respectively, $x \in J^+_y(\omega)$). ■

Lemma 3.6: If an RDS (θ,φ) and $M \subseteq X$ is a random compact set, then

$$D^+_M(\omega) = \bigcup\{D^+_x(\omega) \mid x \in M\} \ \text{ and } \quad J^+_M(\omega) = \bigcup\{\,J^+_x(\omega) \mid x \in\ M\}.$$

Proof: Since $\bigcup\{D^+_x(\omega) \mid x \in M\} \subseteq D^+_M(\omega)$, it is sufficient to show that $D^+_M(\omega)\ \subseteq \bigcup\{D^+_x(\omega) \mid x \in M\}$. Let $y\ \in D^+_M(\omega)$. Then there exist $\{x_n\}$ and $t_n \geq 0$ such that $d(x_n, M(\theta_{-\ t_n}\omega)) \to 0$ and $y = \lim_{n\to+\infty} \varphi\big(\ t_n,\theta_{-\ t_n}\omega\big)x_n$.

Since M is compact, then $\lim_{n\to+\infty} x_n \coloneqq x \in M$. So, $x_n \to\ x,\ y \in\ D^+_x(\omega)$, and $y_n \to y$ ($y_n \coloneqq \varphi\big(\ t_n,\theta_{-\ t_n}\omega\big)x_n$). According to Lemma 3.5, $y \in D^+_x(\omega)\ \subseteq \bigcup\{D^+_x(\omega) \mid x \in M\}$. In the same way, can be established the second statement. ■

Theorem 3.7:[5] If $(t,x) \longmapsto \psi(t,\theta_{-t}\omega)x$ is continuous $\forall\ t \in \mathbb{R}$ and $\omega \in \Omega$, then

$\overline{\gamma^+_M(\omega)} = \gamma^+_M(\omega) \cup \Gamma^+_M(\omega)$.

Proof:

First, note that $\gamma^+_M(\omega) \subset \overline{\gamma^+_M(\omega)}$ .Also we have $\Gamma^+_M(\omega) \subset \overline{\gamma^+_M(\omega)}$ (by Definition(2.3.1). Therefore $\overline{\gamma^+_M(\omega)} \supset \gamma^+_M(\omega) \cup \Gamma^+_M(\omega)$.

Now, let $y \in \overline{\gamma^+_M(\omega)}$. So there is $\{y_n\}$ in $\gamma^+_M(\omega)$ with $y_n \longrightarrow y$. Now, $y_n \in \bigcup_{\tau\in\mathbb{R}-K} \psi(\tau,\vartheta_{-\tau}\omega)M(\vartheta_{-\tau}\omega)$, there exists a sequence $\{\tau_n\}$ with $\tau_n \in \mathbb{R}$ for every n and $\{x_n\}$ in $M\big(\vartheta_{-\tau_n}\omega\ \big)$ such that $y_n = \psi\big(\tau_n,\vartheta_{-\tau_n}\omega\big)x_n$. We have two cases:

Case I: The net $\{\tau_n\}$ has the property that $\tau_n \longrightarrow \infty$, in which case $y \in \Gamma^+_M(\omega)$.

Case II: There is a subnet $\{\tau_{n_k}\}$ in $\mathbb{R}$ such that

$$\tau_{n_k} \longrightarrow \tau \in \overline{\mathbb{R}^+} = \mathbb{R}^+ \ \text{(as } \mathbb{R}^+ \text{ is closed).}$$

But then $\psi\left(\tau_{n_k},\vartheta_{-\tau_{n_k}}\omega\right)x \longrightarrow \psi(\tau,\theta_{-\tau}\omega)x \in \gamma^+_M(\omega)$ (since $(t,x) \longmapsto \psi(t,\vartheta_{-t}\omega)x$ is continuous). Since $\psi\left(\tau_{n_k},\vartheta_{-\tau_{n_k}}\omega\right)x \longrightarrow y$, then from the uniqueness of the limit we have $\psi(\tau,\vartheta_{-\tau}\omega)x = y \in \gamma^+_M(\omega)$.

From Case I and Case II, we have $y \in \gamma^+_M(\omega) \cup \Gamma^+_M(\omega)$. Hence

$$\overline{\gamma^+_M(\omega)} \subset \gamma^+_M(\omega) \cup \Gamma^+_M(\omega).$$

Therefore, $\overline{\gamma^+_M(\omega)} = \gamma^+_M(\omega) \cup \Gamma^+_M(\omega)$ ■

From above theorem and Proposition (1.6.2)[7], we get the following Corollary**.**

Corollary 3.8: for all $x \in X$ and $y \in \Gamma^+_x(\theta_{-t}\omega)$ then $\Gamma^+_y(\omega) \cup \Gamma^+_y(\omega) \subseteq \overline{\gamma^t_y(\omega)} \subseteq \Gamma^+_x(\omega),\ \omega \in \Omega$.

Proof: $\gamma^+_y(\omega) \cup \Gamma^+_y(\omega) = \overline{\gamma^+_y(\omega)}\ \subseteq \overline{\gamma^{\mathbb{R}}_y(\omega)}$

Since $y \in \Gamma^+_x(\theta_{-t}\omega)$, then $\forall t \in \mathbb{R}$: $\varphi(t,\theta_{-t}\omega)\mathrm{y} \in \varphi(t,\theta_{-t}\omega)\Gamma^+_x(\theta_{-t}\omega) = \Gamma^+_x(\omega)$

Then $\overline{\gamma^{\mathbb{R}}_y(\omega)} \subseteq \Gamma^+_x(\omega)$.

So, $\gamma_y^+(\omega) \cup \Gamma_y^+(\omega) \subseteq \overline{\gamma_y^{\mathbb{R}}(\omega)} \subseteq \Gamma_x^+(\omega)$

Then $\Gamma_y^+(\omega) \subseteq \overline{\gamma_y^{\mathbb{R}}(\omega)} \subseteq \Gamma_x^+(\omega)$

Similarly

$$\Gamma_y^-(\omega) \subseteq \overline{\gamma_y^{\mathbb{R}}(\omega)} \subseteq \Gamma_x^+(\omega)$$

Then $\Gamma_y^+(\omega) \cup \Gamma_y^-(\omega) \subseteq \overline{\gamma_y^{\mathbb{R}}(\omega)} \subseteq \Gamma_x^+(\omega)$. ■

Theorem 3.9: If M is a nonempty negatively invariant compact random set, then $M \subseteq J_{\Gamma_X^+}^+(\omega)$.

Proof: Let $x \in M(\omega)$. Then by Definition 2.5 (i), there is a trajectory $\psi(\cdot,\cdot,x): \mathbb{R} \times \Omega \longrightarrow X$ with $\psi(0,\omega)x = x$ and for every $t \geq 0, \psi t, \omega)x \in M(\vartheta_{-\ t}\omega)$. Put $z := \psi(t, \vartheta_{-\ t}\omega)x$ .Since $\Gamma_z^-(\omega) \neq \emptyset$, it is closed and $\Gamma_z^-(\omega) \subseteq M(\vartheta_{-\ t}\omega)$. It is thus compact. Let $y \in \Gamma_z^-(\theta_{-\ t}\omega)$, then $\Gamma_y^+(\omega) \subseteq \Gamma_z^-(\omega)$.

If $p \in \Gamma_y^+(\omega) \subseteq \Gamma_X^+(\omega)$, then there exist $t_n \longrightarrow +\infty$ such that

$$x_k = \varphi\big(-t_k, \theta_{-\ t_k}\omega\big)y \longrightarrow p, \qquad x = \varphi\big(\ t_k, \theta_{-\ t_k}\omega\big)\phi(-t_k),$$

and consequently, $x \in J_p^+(\omega) \subseteq J_{\Gamma_X^+}^+(\omega)$. ■

Corollary 3.10: If (ϑ, ψ) is a point dissipative RDS, then n the set $\Gamma_X^+(\omega)$ is compact, and $\Gamma_X^+(\omega) \subseteq J_{\Gamma_X^+}^+(\omega)$, for all $\omega \in \Omega$.

Proof: since $\Gamma_x^+(\omega)$ is invariant $\forall\ x \in X$ and $\Gamma_X^+(\omega) := \overline{\cup\{\Gamma_x^+(\omega) \mid x \in X\}}$, it follows $\Gamma_X^+(\omega)$ also invariant, and according to Theorem 3.9, $\Gamma_X^+(\omega) \subseteq J_{\Gamma_X^+}^+(\omega)$. ■

Lemma 3.11: If $D_{\Gamma_X^+}^+(\omega)$ (respectively, $J_{\Gamma_X^+}^+(\omega)$) is compact in point dissipative RDS (θ, φ), then $D^+(\Gamma_X^+(\omega)) = D^+(D^+(\Gamma_X^+(\omega)))$ (respectively, $J^+(\Gamma_X^+(\omega)) = J^+(J^+(\Gamma_X^+(\omega)))$.

Proof: Since $\Gamma_x^+(\omega) \subseteq D_{\Gamma_X^+}^+(\omega)$, we have $D_{\Gamma_X^+}^+(\omega) \subseteq D_{\Gamma_X^+}^+(D_{\Gamma_X^+}^+(\omega))$, and consequently to show that $D_{\Gamma_X}^+(D_{\Gamma_X}^+(\omega)) \subseteq D_{\Gamma_X}^+(\omega)$. Let $x \in D_{\Gamma_X}^+(\omega)$. Then $\Gamma_x(\omega) \subseteq D_{\Gamma_X}^+(\omega)$, and if $y \in \Gamma_x^+(\omega) \subseteq \Gamma_X^+(\omega)$, then according to Lemma 3.3 and Corollary 3.4, $J_x^+(\omega) \subseteq J_y^+(\omega) = D_y^+(\omega) \subseteq D_{\Gamma_X}^+(\omega)$. Since

$$D_x^+(\omega) = \gamma_x^+(\omega) \cup J_x^+(\omega) \subset D_{\Gamma_X}^+(\omega) \cup D_{\Gamma_X}^+(\omega) = D_{\Gamma_X}^+(\omega) \text{ for all } x \in D_{\Gamma_X}^+(\omega)$$

and $D_{\Gamma_X}^+(\omega)$ is compact, it follows from Theorem 3.9 that $D_{\Gamma_X}^+\left(D_{\Gamma_X}^+(\omega)\right) = \cup\{D_x^+(\omega) | x \in D_{\Gamma_X}^+(\omega)\} \subseteq D_{\Gamma_X}^+(\omega)$. The second equation is proved similarly, taking into consideration $\Gamma_X^+(\omega) \subseteq J_{\Gamma_X}^+(\omega)$. ■

Definition 3.12: Consider an RDS (ϑ, ψ). A random set M is called **orbitally stable** whenever for any (t.r.v) ε and any non-negative number t, there exists (t.r.v) δ such that

$$d\big(x, M(\omega)\big) < \delta(\omega) \text{ implies } d\big(\psi(t, \vartheta_{-t}\omega)x, M(\omega)\big) < \varepsilon(\omega).$$

Lemma 3.13: If the RDS (ϑ, ψ) is compact dissipative, then

$$D_{\Gamma_X}^+\left(D_{\Gamma_X}^+(\omega)\right) = D_{\Gamma_X}^+(\omega) \text{ (respectively, } J_{\Gamma_X}^+(J_{\Gamma_X}^+(\omega)) = J_{\Gamma_X}^+(\omega)).$$

Proof: Let $L_X(\omega)$ be an RLS for the compact dissipative RDS (ϑ, ψ) . Hence $\Gamma_x^+(\omega) \subseteq L_X(\omega)$. By the orbitally stable of $L_X(\omega)$, we have $D_{L_X}^+(\omega) = L_X(\omega)$ (respectively, $J_{L_X}^+(J_X(\omega)) \subseteq L_X(\omega)$), and consequently, $D_{L_X}^+(\omega) = L_X(\omega)$ (respectively, $J_{L_X}^+(\omega) = L_X(\omega)$. By Lemma 3.11 and closedness of $D_{L_X}^+(\omega)$ (respectively, $J_{\Gamma_X}^+(\omega)$ and compactness of $L_X(\omega)$, we get the compactness of $D_{L_X}^+(\omega)$ (respectively, $J_{\Gamma_X}^+(\omega)$. ■

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Lemma 3.14: Consider a random compact dissipative RDS (ϑ, ψ). The forward invariant compact random set M is orbitally stable if and only if $D_M^+(\omega) = M$.

Proof: Suppose that $D_M^+(\omega) = M$. Assume contrary that M is not orbitally stable. Then there exist tempered random variable $\varepsilon_0(\omega) > 0$, $x_n \to x \in M$, and $t_n \geq 0$ such that

$$d\big(\varphi(\ t_n, \theta_{-\ t_n}\omega)x_n, M\big) \geq \ \varepsilon_0(\omega) \qquad (3.4)$$

Since an RDS (θ, φ) is compact dissipative, the set $\gamma_K^+(\omega)$ is precompact, where $K := \{x_n\}$, and consequently, the sequence $\{\varphi\big(t_n, \theta_{-\ t_n}\omega\big)x_n\}$ can be considered convergent. Assume

$$y := \lim_{n \to +\infty} \varphi(\ t_n, \theta_{-\ t_n}\omega)x_n$$

Then on the one hand, $y \in D_M^+(\omega) = M$. Now (3.4) implies that

$$d(y, M(\theta_{-\ t_n}\omega)) \geq \varepsilon_0(\omega) > 0.$$

This contradiction. The converse is follow immediately ∎

Theorem 3.15: If (θ, φ) is compact dissipative RDS, then $L_X(\omega) = J_{\Gamma_X}^+(\omega)$.
Proof: Since $\Gamma_x(\omega) \subseteq L_X(\omega)$ and $J_X(\omega)$ is asymptotically stable, then $J_{\Gamma_X}^+(\omega) \subseteq L_X(\omega)$. Because $D_{\Gamma_X}^+(J_{\Gamma_X}^+(\omega)) = J_{\Gamma_X}^+(\omega)$, the orbitally stability of the set $J_{\Gamma_X}^+(\omega)$ follows from Lemmas 2.13 and 2.14 . Let $x \in L_X(\omega) \setminus J_{\Gamma_X}^+(\omega)$ and $d_X := d(x, J_{\Gamma_X}^+(\omega)) \geq 0$. If $d_X = 0$ for all $x \in L_X(\omega) \setminus J_{\Gamma_X}^+(\omega)$. ∎

Conversely, assume that for some $x_0 \in L_X(\omega) \setminus J_{\Gamma_X}^+(\omega)$ we have $d_{x_0} > 0$. By orbital stability of $J_{\Gamma_X}^+(\omega)$, for a (t.r.v) $0 < \varepsilon(\omega) < \frac{1}{2} d_{x_0}$, choose $\delta(\varepsilon, \omega) > 0$. By Theorem 2.7[15] we can find a continuous function $\phi : S \to L_X(\omega)$ with the property that

$$\psi(t, \vartheta_{-t}\omega)\phi(s) = \ \phi(t\ +\ s) \text{ for all } t \in T, \omega \in \Omega,\ s \in S, \text{ and } \phi(0) = x_0.$$

Since $L_X(\omega)$ is compact, then $\Gamma_{\phi_{x_0}}^-(\omega)$ is not a void and compact, and $\Gamma_X^+(\omega) \cap \Gamma_{\phi_{x_0}}^-(\omega) \neq \emptyset$, and consequently, there exists $t_n \to -\infty$ such that $d(\psi\big(t_n, \vartheta_{-t_n}\omega\big)x_0, \Gamma_x(\omega)) \to 0$. Choose n_0 such that

$$d(\psi\big(t + t_n, \vartheta_{-(t+t_n)}\omega\big)x_0, \Gamma_x(\omega)) < \delta\ (n\ \geq n_0).$$

Then we have $d(\psi\big(t_n, \vartheta_{-t_n}\omega\big)x_0,\ J_{\Gamma_X}^+(\omega)) < \delta(\omega)$, and consequently,

$$d(\psi\big(t + t_n, \vartheta_{-(t+t_n)}\omega\big)x_0,\ J_{\Gamma_X}^+(\omega) < \varepsilon(\omega) \text{ for all } t \geq 0 \text{ and } n \geq n_0.$$

In particular, when $t = -t_n$ we have

$d_{x_0} = d(x_0 J_{\Gamma_X}^+(\omega)) < \varepsilon(\omega) < \frac{1}{2} d_{x_0}$. This contradiction. ∎

Corollary 3.16: If an RDS (θ, φ) is compact dissipative, then $L_X(\omega) = D_{\Gamma_X}^+(\omega)$.
Proof. The proof follows from Theorem 3.15, taken into consideration

$$L_X(\omega) \subseteq J_{\Gamma_X}^+(\omega) \subseteq D_{\Gamma_X}^+(\omega) \subseteq L_X(\omega).$$

Corollary 3.17: If (ϑ, ψ) is compact dissipative RDS, then the orbitally stability of the set $\Gamma_x^+(\omega)$ is equivalent to the condition $L_X(\omega) = \Gamma_x^+(\omega)$.

Proof. The proof follows from Theorem 3.15 and Lemma 3.14 .

Definition 3.18: The set $W_M^s(\omega)$ defined by

$$W_M^s(\omega) := \left\{x \in X: \lim_{t \to +\infty} d(\psi(t, \vartheta_{-t}\omega)x, M(\omega)) = 0\right\}$$

is the **stable manifold** of the random set M.

Definition 3.19: Consider the RDS (ϑ, ψ) .A closed random set M in locally compact space X is called **globally asymptotically stable** if M asymptotically stable and $W_M^s(\omega) = X$.

Theorem 3.20: An RDS (θ, φ) is compact dissipative, if and only if there is a nonvoid compact random set K with the property that

$\forall$ (t.r.v) $\varepsilon > 0$ and $x \in X$, $\exists \delta(\varepsilon, x) > 0$, $l(\varepsilon, x) > 0$ so that

$$\psi(t, \vartheta_{-t}\omega)B\big(x, \delta(\varepsilon, x)\big) \subseteq B(K(\vartheta_{-t}\omega), \varepsilon) \qquad (3.5)$$

for all $t \geq l(\varepsilon, x)$, $\omega \in \Omega$.

Proof: Let $L_X(\omega)$ be an RLC for the compact dissipative (ϑ, ψ). Let $\varepsilon(\omega) > 0$, and $x \in X$. In view of Theorem 2.7 [17], $L_X(\omega)$ is orbitally stable. Thus we can define a number $(\varepsilon, \omega) > 0$. Since $L_X(\omega)$ is globally asymptotically stable, so for $\rho(\varepsilon) > 0$ and $x \in X$ there is $l(\varepsilon, x)$ for which $\varphi(t, \theta_{-t}\omega)x \in B(L_X(\omega), \rho)$ for all $t \geq l(\varepsilon, x)$. Since $B(L_X(\omega), \rho)$ is open, there is $\alpha = \alpha(\varepsilon, x) > 0$ such that $B((\psi(l(\varepsilon, x), \vartheta_{-l(\varepsilon,x)}\omega)x, \alpha) \subseteq B(L_X(\omega), \rho)$. By the assumption, the mapping

$\psi(l, \vartheta_{-l}\omega) : X \to X$ is continuous and so for every $x \in X, \alpha > 0$, there is $\delta = \delta(\varepsilon, x) > 0$ with

$$\psi\big((l, \vartheta_{-l}\omega), B(x, \delta)\big) \subseteq B(\psi(l, \vartheta_{-l}\omega)x, \alpha) \qquad (3.6)$$

From the inclusion (3.6) and by the choice of ρ, we have $\varphi(t,, \theta_{-t}\omega)B(x, \delta) \subseteq B(L_X(\omega), \varepsilon)$ for all $t \geq l(\varepsilon, x)$. Suppose that $K \subseteq X$ be a nonvoid compact random subset adequate (3.5). If M is a nonvoid compact random subset of X and $\varepsilon > 0$, then for every $x \in M$ there is $\delta(\varepsilon, x) > 0$ and $l(\varepsilon, x) > 0$ so that (3.5) satisfies . The collection $\{B(x, \delta(\varepsilon(\omega), x)) \mid x \in M\}$ form an open covering of the set M. Since M is compact and X is complete , this covering admits finite subcovering $\big\{B\big(x_i, \delta(\varepsilon(\omega), x_i)\big)\big| \, i = 1, \dots, n\big\}$. Assume $L(\varepsilon(\omega), M) := max\{l(\varepsilon(\omega), x_i) | \, i = 1, \dots, n\}$. From (3.5) it follows that

$$\psi(t, \vartheta_{-t}\omega)M \subseteq B(K, \varepsilon(\omega)) \text{ for all } t \geq L(\varepsilon(\omega), M). \qquad \blacksquare$$

Theorem 3.21: A point dissipative RDS (ϑ, ψ) is compact dissipative, if and only if there exist a nonviod compact random set M satisfying :

i. $\Gamma_X(\omega) \subseteq M$.
ii. M is an orbitally stable .

Proof: It is plentiful to prove the sufficiency of (i) and (ii). As in Theorem 3.20, we create that $\forall$ (t.r.v) $\varepsilon(\omega) > 0$ and $x \in X$ there is $\delta(\varepsilon(\omega), x) > 0$ and $l(\varepsilon(\omega), x) > 0$ filling (3.5). Here, we should take as K the set M from Theorem 3.21. From Theorem 3.20 the compact dissipativity of (ϑ, ψ) follows. To show that $L_X(\omega) \subseteq M$. From the orbitally stability of M and the fact that $\Gamma_X(\omega) \subseteq M$, we get

$$D^+(\Gamma_X(\omega)) \subseteq D_M^+(\omega) = M.$$

Now, refer to Corollary 3.17 to end the proof.

Theorem 3.22: A point dissipative RDS (ϑ, ψ) is compact dissipative, if and only if the set $D^+_{\Gamma_X}(\omega)$ (respectively, $J^+_{\Gamma_X}(\omega)$) be compact and orbitally stable. Here , $L_X(\omega) = D^+_{\Gamma_X}(\omega)$ (respectively, $L_X(\omega) = J^+_{\Gamma_X}(\omega)$).
Proof. The proof follows from Theorems 3.15, 3.20, 3.21 and Corollary 3.16.

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Theorem 3.23: A point dissipative RDS (ϑ, ψ) is compact dissipative, if and only if the trajectory $\gamma_K^+(\omega)$ of every compact random $K: \Omega \longrightarrow 2^X$ is precompact.

Proof: By Theorem 3.10 [17] we have the necessity. To prove the sufficiency. Under the conditions of the theorem, $J^+_{\Gamma_X}(\omega)$ is nonempty and compact. By Corollary 3.10, $\Gamma_X(\omega) \subseteq J^+_{\Gamma_X}(\omega)$, and consequently, $J^+_{\Gamma_X}(\omega) \neq \emptyset$. We get $J^+_{\Gamma_X}(\omega)$ is compact. Let $\{y_k\} \subseteq J^+_{\Gamma_X}(\omega)$ and $\varepsilon_k(\omega) \downarrow 0$. Then there exist $p_k \in \Gamma_X(\omega)$, $y_k \in J^+_{p_k}(\omega)$, $\bar{p}_k \in B\big(p_k, \varepsilon_k(\omega)\big)$, and $t_k > 0$ such that

$$d\big(y_k, \varphi\big(t_k, \theta_{-t_k}\omega\big)\bar{p}_k\big) < \varepsilon_k(\omega) \tag{3.7}$$

Since $\{p_k\}$ in a compact set $\Gamma_X(\omega)$ and $\varepsilon_k(\omega) \downarrow 0$, then $\{p_k\}$ is relatively compact, and so, by assumptions , the sequence $\{\varphi\big(t_k, \theta_{-t_k}\omega\big)\bar{p}_k\}$ is precompact. Hence by (3.7) we have $\{y_k\}$ is precompact, hence the compactness of $J^+_{\Gamma_X}(\omega)$ follows. To see that the set $J^+_{\Gamma_X}(\omega)$ attracts every random compact set in X. Let K be a nonvoid random compact set in X. By assumptions , $\gamma_K^+(\omega)$ is precompact. If follows from Theorem 3.10 [17], the set $\Gamma_X(\omega)$ is nonvoid with

$$\lim_{t \to +\infty} d(\varphi(t, \theta_{-t}\omega)K, \Gamma_X(\omega)) = 0 \tag{3.8}$$

According to Theorem 3.5 [17], $\Gamma_K(\omega)$ is an invariant, and by Lemma 3.7, $\Gamma_K(\omega) \subseteq J^+_{\Gamma_X}(\omega)$, and consequently, from (3.8) it follows that $J^+_{\Gamma_X}(\omega)$ attracts K. ■

In the end, we give an example of an RDS to show that the compact dissipativity does not implied by a point dissipativity .

Example 3.24 : Consider the RDS generated by (Markov Chain) [7]. Let $X = \mathbb{R}$ and $\Gamma_0 = \{0, 1\}$. Suppose that the continuous functions f_0 and f_1 satisfy the inequality

$$|f_i(x)| \leq a|x| + b \text{ for some } 0 \leq a < 1, b \geq 0 .$$

Here, Γ is the set of two-sided sequences $\omega = \{\omega_i \mid i \in \mathbb{Z}\}$ involving of 0's and 1's and
$\psi(n, \omega) = f_{\omega_{n-1}} \circ f_{\omega_{n-2}} \circ \ldots \circ f_{\omega_1} \circ f_{\omega_0}$, $\omega = \{\omega_i \mid \mathrm{i} \in \mathbb{Z}\}$, $\mathrm{n} \in \mathbb{N}$.
By the cocycle property we get

$$|\psi(n+1, \omega)x| \leq a \cdot |\psi(n, \omega)x| + b, \qquad n \in \mathbb{Z}_+ \tag{3.9}$$

By n iterations yield

$$|\psi(n, \omega)x| \leq a^n \cdot |x| + b \cdot (1-a)^{-1}, n \in \mathbb{Z}_+ \tag{3.10}$$

Let $\mathcal{D}$ be the collection of all tempered (with respect to ϑ) random closed sets in $\mathbb{R}$. Let $D \in \mathcal{D}$ and $D(\omega) \subset \{x: |x| \leq r(\omega)\}$, where $r(\omega)$ is a t.r.v. Then by (3.9) we have

$$|\psi(n, \theta_{-n}\omega)x(\vartheta_{-n}\omega)| \leq a^n r(\vartheta_{-n}\omega) + b \cdot (1-a)^{-1}, \forall\, x(\omega) \in D(\omega) .$$

Because $0 \leq a < 1$, it yield from (tempered random variable property) that $a^n r(\vartheta_{-n}\omega) \to 0$ as $n \to +\infty$.
Therefore, $\forall\, \omega \in \Omega$ there exists $n_0(\omega)$ such that $a^n r(\vartheta_{-n}\omega) \leq 1$ for $n \geq n_0(\omega)$. Consequently,

$$\psi(n, \theta_{-n}\omega)D(\vartheta_{-n}\omega) \subset B := [-1 - \frac{b}{(1-a)}, 1 + \frac{b}{(1-a)}]$$

for $n \geq n_0(\omega)$. So the RDS is dissipative in $\mathcal{D}$ the universe of all tempered random closed sets from $\mathbb{R}$. By (3.9) when $n = 0$ we see that B is a forward invariant random set in $\mathcal{D}$.

Let $x \in X^\Omega$ and $(\omega) \in \{y: |y| \leq r(\omega)\}$, with $r(\omega)$ satisfy

$$\sup\{e^{-\gamma|t|}|r(\vartheta_t\omega)|: t \in \mathbb{R}\} < \infty,\ \forall\, \omega \in \Omega \text{ and } \gamma > 0 \tag{3.11}$$

$r(\omega)$ is a (t.r.v.) with respect to θ, then for all x , (3.11) implies that

$$|\psi(n,\vartheta_{-n}\omega)x(\vartheta_{-n}\omega)| \le a^n\, r(\vartheta_{-n}\omega) + \frac{b}{(1-a)}.$$

Because $0 \le a < 1$ from (3.11) yield that $a^n\, r(\vartheta_{-n}\omega) \to 0$ when $n \to \infty$.

So, $\forall\ \omega \in \Omega$ there exists $n_0(\omega)$ such that $a^n\, r(\vartheta_{-n}\omega) \le 1$, for $n \ge n_0(\omega)$. Consequently, we have

$$\psi(n,\vartheta_{-n}\omega)x(\vartheta_{-n}\omega) \in B = \left[-1-\frac{b}{1-a}, 1+\frac{b}{1-a}\right], n \ge n_0(\omega)$$

Thus, the RDS is point dissipative. Let $D = [-r(\omega), r(\omega)]$ is compact random set and $k(\omega) = (2+\frac{b}{1-a}, 3+\frac{b}{1-a})$

$\psi(n,\vartheta_{-n}\omega)D(\vartheta_{-n}\omega) \subset B$

$$d(\psi(n,\vartheta_{-n}\omega)D(\vartheta_{-n}\omega), k(\omega)\} > 0$$

Hence, (ϑ,ψ) is not compact dissipative. ■

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