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(α,β,γ)-(Γ-derivation) and Generalized Γ-derivation of Γ-Lie Algebra

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Abstract : *(α,β,γ)-(Gamma -derivation) and Generalized Gamma-derivation of Γ-Lie Algebra. of finite dimensions on the domain of complex numbers. Using certain complex parameters . We also provide all the generalized Γ-derivations of Γ-Lie algebras.*

Keywords: Gamma Lie algebras, (α,β,γ) −Gamma derivations , Generalized Gamma derivations

1. Introduction

The limited-dimensions complex theory In Lie theory, it is an essential component of Lie theory. With the growing body of theory and applications, making sure of the applicable tools to deal with it became important. They also find uses finite-dimensional complex number theory [1]. Casimir factors are discussed, lower and higher central sequences, Lie algebra of derivations, ideals When affinities between Lie algebras are possible ,and sub algebras [2,3] are all discussed .In [4] these characteristics are extremely important .

The linear maps $f : \mathcal{L} \to \mathcal{L}$ of a Γ-Lie algebra $\mathcal{L}$ such that $f([x,y]_{\lambda}) = [f(x),y]_{\lambda} + [x,f(y)]_{\lambda}$ applies to any $x,y \in \mathcal{L}$, denoted by $Der_{\lambda}(\mathcal{L})$ [5].

Leger mentioned in his article that $GenDer(S). = QDer(S) + C(S)$. when S is a pseudo-simple Lie algebra in characteristic 0 [6] . We introduce the following notion of $(\alpha,\beta,\gamma) - (\Gamma - \text{derivations})$, which connects with the automorphism group when specializing in the case where α and β and γ are automorphisms [7].

In this article , in section two let $\mathcal{L}$ stand for the finite-dimensional Γ-Lie algebra over the Γ-field of complex numbers $\mathbb{C}$ and $End(\mathcal{L})$ for the associative Γ-algebra of all linear operators on the Γ-vector space $\mathcal{L}$. $End(\mathcal{L})$ is a space with a standard Γ-Lie commutator $[N,M]_{\lambda} = N \cdot_{\lambda} M - M \cdot_{\lambda} N$, is denoted as usual by $gl(\mathcal{L})$. Through this method, any Γ-Lie sub algebra of $End(\mathcal{L})$ and $gl(\mathcal{L})$ has a Γ-Lie sub algebra we also rely on the notation of the center $Z(\mathcal{L})$ and for the $\Gamma -$ derived Γ-Lie algebra $\mathcal{L}^2 = [\mathcal{L},\mathcal{L}]_{\lambda}$.

In section three , when we take the values of each of (α,β,γ) Equal to natural numbers, you write (f, f', f'') to express the difference in functions after multiplying them by the natural numbers .then it is equal to Γ-derivation, and if (α,β,γ) is not equal to one, we deduce another application of Γ-Lie algebra, which is Generalized Γ-derivation, quasi Γ-derivation, quasi centroid and centroid, and these are special cases. of (α,β,γ)-(Γ-derivations) of Γ-Lie algebra.

2. (α,β,γ)-(Γ-derivations) of Γ-Lie

In this section, we introduce (α,β,γ)-Γ-derivations and show their pertinent properties. All possible intersections of spaces containing these Γ-derivations are investigated**.**

2.1 Definition (α,β,γ)-(Gamma-derivations) of Γ-Lie algebra

Let $\mathcal{L}$ is a Γ-Lie algebra and a linear operator $f \in End(\mathcal{L})$ .is called (α,β,γ)-(Γ-derivations) of $\mathcal{L}$ if there is $\alpha,\beta,\gamma \in \mathbb{C}$ such that.

$$\alpha f([x,y]_{\lambda}) = \beta[f(x),y]_{\lambda} + \gamma[x,f(y)]_{\lambda} \qquad (1)$$

For all $x,y \in \mathcal{L}$.

2.2 Properties and structure of (α,β,γ)-(Γ-derivations)

For given $\alpha,\beta,\gamma \in \mathbb{C}$ we denote the set of all. (α,β,γ)-(Γ-derivations) as $D(\alpha,\beta,\gamma)$, i.e.

$$D(\alpha,\beta,\gamma) = \{f \in End(\mathcal{L}) | \alpha f([x,y]_{\lambda}) = \beta[f(x),y]_{\lambda} + \gamma[x,f(y)]_{\lambda} \forall x,y \in \mathcal{L}\} \qquad (2)$$

Obvious that $D(\alpha,\beta,\gamma)$ is Γ-Lie subspace of $End(\mathcal{L})$ and by (1) We conclude from that $\mathcal{E} \in \mathbb{C} \setminus \{0\}$ it holds that.

$$D(\alpha,\beta,\gamma) = D(\alpha\mathcal{E}, \beta\mathcal{E}, \gamma\mathcal{E}) = D(\alpha,\gamma,\beta) \qquad (3)$$

It is clear that

$$D(\alpha,\beta,\gamma) = D(\alpha\mathcal{E}, \beta\mathcal{E}, \gamma\mathcal{E})$$

And through the structure of all three equations, we can deduce

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$D(\alpha, \beta, \gamma)$ we get.

$$\alpha f([x,y]_{\lambda}) = \beta[f(x),y]_{\lambda} + \gamma[x,f(y)]_{\lambda} \quad \text{(a)}$$

$D(\alpha\mathcal{E}, \beta\mathcal{E}, \gamma\mathcal{E})$ we get.

$$\alpha\mathcal{E}f([x,y]_{\lambda}) = \beta\mathcal{E}[f(x),y]_{\lambda} + \gamma\mathcal{E}[x,f(y)]_{\lambda} \quad \text{(b)}$$

$D(\alpha, \gamma, \beta)$ we get.

$$\alpha f([x,y]_{\lambda}) = \gamma[f(x),y]_{\lambda} + \beta[x,f(y)]_{\lambda} \quad \text{(c)}$$

Since $\alpha f([x,y]_{\lambda})$ equals $\alpha f([x,y]_{\lambda})$ from that we deduce the equality of the two equations Eq(a) and Eq(c) implies that.

$$\alpha f([x,y]_{\lambda}) = \beta[f(x),y]_{\lambda} + \gamma[x,f(y)]_{\lambda}$$

And

$$\alpha f([x,y]_{\lambda}) = \gamma[f(x),y]_{\lambda} + \beta[x,f(y)]_{\lambda}$$

Thus

$$D(\alpha, \beta, \gamma) = D(\alpha, \gamma, \beta)$$

2.3 Lemma

For every $\alpha, \beta, \gamma \in \mathbb{C}$ (4)

$$D(\alpha, \beta, \gamma) = D(0, \beta-\gamma, \gamma-\beta) \cap D(2\alpha, \beta+\gamma, \beta+\gamma)$$

Proof:- Suppose any $\alpha, \beta, \gamma \in \mathbb{C}$ are given. Then for $f \in D(\alpha, \gamma, \beta)$ and arbitrary $x, y \in \mathcal{L}$ we have

$$\alpha f([x,y]_{\lambda}) = \beta[f(x),y]_{\lambda} + \gamma[x,f(y)]_{\lambda} \quad (5)$$

$$\alpha f([y,x]_{\lambda}) = \beta[f(y),x]_{\lambda} + \gamma[y,f(x)]_{\lambda} \quad (6)$$

We get (7) and (8) from adding and subtracting equations (5) and (6)

$$0 = (\beta-\gamma)([f(x),y]_{\lambda}) + (\gamma-\beta)([x,f(y)]_{\lambda}) \quad (7)$$

$$2\alpha f([y,x]_{\lambda}) = (\beta+\gamma)([f(x),y]_{\lambda} + [x,f(y)]_{\lambda}) \quad (8)$$

And thus

$$D(\alpha, \beta, \gamma) \subseteq D(0, \beta-\gamma, \gamma-\beta) \cap D(2\alpha, \beta+\gamma, \beta+\gamma)$$

Similarly, beginning with Eq (7) and (8) we obtain Eq(5) and Eq (6) and the remaining inclusion is proven.

2.4 Theorem:

There exists $\delta \in \mathbb{C}$ and a $\alpha, \beta, \gamma \in \mathbb{C}$ such that the subspace $D(\alpha, \beta, \gamma) \subseteq End(\mathcal{L})$ is equal to some of the four following subspaces:

1- $D(\delta, 0,0)$.

2- $D(\delta, 1, -1)$.

3- $DD(\delta, 1,0)$.

4- $D(\delta, 1,1)$.

Proof:

(1) $D(\delta, 0,0)$:

(a) If $\delta = 0$ We get it trivially $D(0,0,0)$ of $End(\mathcal{L})$.

(b) If $\delta \neq 0$ the space $D(1,0,0)$ is a Γ-Lie sub algebra of $End(\mathcal{L})$, mapping derived Γ-Lie algebra.

(2) $D(\delta, 1, -1)$:

(a) If $\delta = 0$ we get a Γ-Lie sub algebra of $\mathcal{L}$

$$D(0,1,-1) = \{f \in End(\mathcal{L}) | [f(x),y]_{\lambda} = [x,f(y)]_{\lambda} \forall x,y \in \mathcal{L}\}$$

(b) If $\delta \neq 0$ we get Γ-Lie algebra $(1,1,-1) \subseteq \mathcal{L}$, Intersection of two a Γ- Lie algebras.

$$D(\delta,1,-1) = D(0,1,-1) \cap D(\delta,0,0) = D(0,1,-1) \cap D(1,0,0) = D(1,1,-1)$$

(3) $D(\delta, 1,0)$:

(a) For $\delta = 0$ we get a Γ-Lie algebra of every linear operator of the vector space $\mathcal{L}$

$$D(0,1,0) = \{f \in End(\mathcal{L}) | f(\mathcal{L}) \subseteq Z(\mathcal{L})\},$$

(b) If $\delta = 1$ So space $D(1,1,0)$ is an adjoint representation centralizer $ad(\mathcal{L})\ in\ gl_{\lambda}(\mathcal{L})$

(c) For the rest of the values δ of space $D(\delta, 1,0)$ The $End\,(\mathcal{L})$ vector subspace of the Γ-Lie algebra $\mathcal{L}$, only the vector subspace in the general case, There is a set of vector spaces for the residual values of one-parameter space shapes.

$$D(\delta, 1,0) = D(0,1,-1) \cap D(2\delta, 1,1).$$

(4) $D(\delta, 1,1)$:

(a) If $\delta = 0$ we have a Γ- Lie algebra

$$D(0,1,1) = \{f \in End(\mathcal{L}) | [f(x),y]_{\lambda} = -[x,f(y)]_{\lambda} \forall x,y \in \mathcal{L}\}$$

(b) If $\delta = 1$ We now have the Γ-algebra of Γ-derivations of $\mathcal{L}$,

$D(1,1,1) = Der_{\lambda}(\mathcal{L})$

For the rest of the values δ of space $D(\delta,1,1)$ The $End\ (\mathcal{L})$ vector subspace of the Γ-Lie algebra $\mathcal{L}$, only the vector subspace in the general case.

3. Generalized Γ-Derivations of Γ-Lie algebra

In this section contains some elementary observations about generalized Γ-derivations, quasi Γ-derivations, quasi Γ-centroids, some of which are technical results. The section also includes a characterization of Γ-Lie algebras $\mathcal{L}$ for which $GenDer_{\lambda}(\mathcal{L}) \subseteq gl_{\lambda}(\mathcal{L})$. For such Γ-algebras we have $QDer_{\lambda}(\mathcal{L}) \subseteq gl_{\lambda}(\mathcal{L})$.

Suppose $\mathcal{L}$ is a finite-dimensional Γ-Lie algebra with multiplication $\mathcal{L} \propto \mathcal{L} \to \mathcal{L}$. Let $\nabla(A)$ denote the set of triples $(f, f, f")$, with $f, f', f" \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ As explained in the following definitions

3.1 Definition:- (Quasi Γ-derivation)

A linear map $f \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ is called a Quasi Γ-derivation of $\mathcal{L}$ if there exists map $f' \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$, such that.

$$[f(x),\mathrm{y}]_{\lambda} + [\mathrm{x}, f(y)]_{\lambda} = f'([\mathrm{x},\mathrm{y}]_{\lambda})$$

For all $x, y \in \mathcal{L}$. We denote by $QDer_{\lambda}(\mathcal{L})$ the set of all quasi Γ-derivation of $\mathcal{L}$.

3.2 Definition:- (Generalized Γ-derivation)

A linear map $f \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ is called a Generalized Γ-derivation of $\mathcal{L}$. if there exists maps $f', f" \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$, such that

$$[f(x),\mathrm{y}]_{\lambda} + [\mathrm{x}, f'(y)]_{\lambda} = f''([\mathrm{x},\mathrm{y}]_{\lambda})$$

For all $, y \in \mathcal{L}$.

For all $x, y \in \mathcal{L}$. We denote by $GenDer_{\lambda}(\mathcal{L})$ the set of all generalized Γ-derivation of $\mathcal{L}$.

3.3 Definition:- (Quasi Γ-centroid)

A linear map $f \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ is called Quasi Γ-centroid of $\mathcal{L}$ if .

$$[f(x), y]_{\lambda} = [x, f(y)]_{\lambda}$$

for all $x, y \in \mathcal{L}$.

For all $x, y \in \mathcal{L}$. We denote by $QC(\mathcal{L})$ the set of all quasi Γ-centroid of $\mathcal{L}$.

3.4 Lemma:-

Let $\mathcal{L}$ be a Γ-Lie algebra. Then $GenDer_{\lambda}(\mathcal{L})$ is Γ-Lie algebra.

Proof:-

Let $f_1, f_2 \in GenDer_{\lambda}(\mathcal{L})$, then there exists $f'_1, f"_1, f'_2 f"_2 \in Hom(\mathcal{L},\mathcal{L})$.

Since $f_1, f_2 \in GenDer_{\lambda}(\mathcal{L})$ we can write the following two equations by [3.2].

$$f"_1([x,y]_{\lambda}) = [f_1(x), y]_{\lambda} + [x, f'_1(y)]_{\lambda} \qquad (a)$$

$$f"_2([x,y]_{\lambda}) = [f_2(x), y]_{\lambda} + [x, f'_2(y)]_{\lambda} \qquad (b)$$

Now we will prove that $GenDer_{\lambda}(\mathcal{L})$ is Γ-Lie algebra

$$[[f_1, f_2]_{\lambda}(x), y]_{\lambda} = [(f_1 f_2 - f_2 f_1)(x), y]_{\lambda}$$
$$= [f_1 f_2(x) - f_2 f_1(x), y]_{\lambda}$$
$$= [f_1 f_2(x), y]_{\lambda} - [f_2 f_1(x), y]_{\lambda}$$

Then

$$[[f_1, f_2]_{\lambda}(x), y]_{\lambda} = [f_1 f_2(x), y]_{\lambda} - [f_2 f_1(x), y]_{\lambda} \qquad (c)$$

i- $[f_1 f_2(x), y]_{\lambda} =$

$$f"_1([f_2(x), y]_{\lambda}) - [f_2(x), f'_1(y)] \qquad \text{by Eq (a)}$$

$$= \left. \begin{matrix} f"_1 f"_2([x,y]_{\lambda}) - f"_1([x, f'_2(y)]_{\lambda} \\ -f"_2([x, f'_1(y)]) + [x, f'_2 f'_1(y)]_{\lambda} \end{matrix} \right) \quad \text{by Eq(b)}$$

Then

$[f_1 f_2(x), y]_{\lambda} =$

$$f"_1 f"_2([x,y]_{\lambda}) - f"_1([x, f'_2(y)]_{\lambda}$$
$$-f"_2([x, f'_1(y)]) + [x, f'_2 f'_1(y)]_{\lambda}$$

ii- $[f_2 f_1(x), y]_{\lambda} =$

$$f"_2([f_1(x), y]_{\lambda}) - [f_1(x), f'_2(y)] \qquad \text{by Eq(b)}$$

$$= \left. \begin{matrix} f"_2 f"_1([x,y]_{\lambda}) - f"_2([x, f'_1(y)]_{\lambda} \\ -f"_1([x, f'_2(y)]_{\lambda}) + [x, f'_1 f'_2(y)]_{\lambda} \end{matrix} \right) \quad \text{by Eq(a)}$$

Then

$[f_2 f_1(x), y]_{\lambda} =$

$$= f"_2 f"_1([x,y]_{\lambda}) - f"_2([x, f'_1(y)]_{\lambda})$$
$$-f"_1([x, f'_2(y)]_{\lambda}) + [x, f'_1 f'_2(y)]_{\lambda}$$

Substituting the values of (i), (ii) into equation (c) we get

$[[f_1, f_2]_{\lambda}(x), y]_{\lambda} =$

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$$f"_1 f"_2([x,y]_ג) - f"_1([x, f'_2(y)]_ג - f"_2([x, f'_1(y)]_ג)) + [x, f'_2 f'_1(y)]_ג$$

$$-f"_2 f"_1([x,y]_ג) + f"_2([x, f'_1(y)]_ג) + f"_1([x, f'_2(y)]_ג) - [x, f'_1 f'_2(y)]_ג$$

$$= f"_1 f"_2([x,y]_ג) + [x, f'_2 f'_1(y)]_ג - f"_2 f"_1([x,y]_ג) - [x, f'_1 f'_2(y)]_ג$$

$$= (f"_1 f"_2([x,y]_ג) - f"_2 f"_1([x,y]_ג))$$

$$-([x, f'_1 f'_2(y)]_ג - [x, f'_2 f'_1(y)]_ג$$

$$= [f"_1, f"_2]_ג([x,y]_ג - [x, [f'_1, f'_2]_ג(y)]_ג$$

Thus

$$[[f_1, f_2]_ג(x), y]_ג = [f"_1, f"_2]_ג([x,y]_ג - [x, [f'_1, f'_2]_ג(y)]_ג$$

Then $[f_1, f_2]_ג \in GenDer_ג(\mathcal{L})$, we have $GenDer_ג(\mathcal{L})$ is Γ-Lie algebra. Since $GenDer_ג(\mathcal{L}) \subseteq gl_ג(\mathcal{L})$,we have $GenDer_ג(\mathcal{L})$is Γ-Lie sub algebra of $gl_ג(\mathcal{L})$.

3.5 Lemma:-

Let $\mathcal{L}$ be a Γ-Lie algebra. Then

1- $QDer_ג(\mathcal{L})$ is a Γ-Lie sub algebra of $GenDer_ג(\mathcal{L})$.

2- $[Der_ג(\mathcal{L}), C(\mathcal{L})]_ג \subseteq C(\mathcal{L})$.

3- $[QDer_ג(\mathcal{L}), QC(\mathcal{L})]_ג \subseteq QC(\mathcal{L})$.

4- $C(\mathcal{L}) \subseteq QDer_ג(\mathcal{L})$.

5- $[QC(\mathcal{L}), QC(\mathcal{L})]_ג \subseteq QDer_ג(\mathcal{L})$

Proof:-

1- Γ- algebra $QDer_ג(\mathcal{L}) \subseteq GenDer_ג(\mathcal{L})$ by [3.1 & 3.2].

Let $f_1, f_2 \in QDer_ג(\mathcal{L})$' then there exists $f'_1 f'_2 \in Hom(\mathcal{L}, \mathcal{L})$

Since $f_{1,} f_2 \in QDer_ג(\mathcal{L})$ we can write the following two equations by [Definition 2.4.1]

$$f'_1([x,y]_ג) = [f_1(x), y]_ג + [x, f_1(y)]_ג \qquad (a)$$

$$f'_2([x,y]_ג) = [f_2(x), y]_ג + [x, f_2(y)]_ג \qquad (b)$$

$$\left[[f_1, f_2](x), y\right]_ג = [(f_1 f_2 - f_2 f_1)(x), y]_ג$$

$$= [f_1 f_2(x) - f_2 f_1(x), y]_ג$$

$$= [f_1 f_2(x), y]_ג - [f_2 f_1(x), y]_ג \qquad (c)$$

i- $[f_1 f_2(x), y]_ג = f'_1([f_2(x), y]_ג) - [f_2(x), f_1(y)]_ג$ by Eq (a)

$$\left. \begin{array}{l} = f'_1 f'_2([x,y]_ג) - f'_1([x, f_2(y)]_ג) \\ -f'_2([x, f_1(y)]_ג) + [x, f_2 f_1(y)]_ג \end{array} \right) \text{ by Eq (b)}$$

Then

$$[f_1 f_2(x), y]_ג = f'_1 f'_2([x,y]_ג) - f'_1([x, f_2(y)]_ג)$$

$$-f'_2([x, f_1(y)]_ג) + [x, f_2 f_1(y)]_ג$$

ii- $[f_2 f_1(x), y]_ג =$

$f'_2([f_2(x), y]_ג) - [f_1(x), f_2(y)]_ג$ by Eq (b)

$$\left. \begin{array}{l} = f'_2 f'_1([x,y]_ג) - f'_2([x, f_1(y)]_ג) \\ -f'_1([x, f_2(y)]_ג) + [x, f_1 f_2(y)]_ג \end{array} \right) \text{ by Eq (a)}$$

Then

$$[f_1 f_2(x), y]_ג = f'_2 f'_1([x,y]_ג) - f'_2([x, f_1(y)]_ג)$$

$$-f'_1([x, f_2(y)]_ג) + [x, f_1 f_2(y)]_ג$$

Substituting the values of (i), (ii) into equation (c) we get

$[[f_1, f_2]_ג(x), y]_ג =$

$$f'_1 f'_2([x,y]_ג) - f'_1([x, f_2(y)]_ג) - f'_2([x, f_1(y)]_ג) + [x, f_2 f_1(y)]_ג$$

$$-f'_2 f'_1([x,y]_ג) + f'_2([x, f_1(y)]_ג) + f'_1([x, f_2(y)]_ג) - [x, f_1 f_2(y)]_ג$$

$$= f'_1 f'_2([x,y]_ג) + [x, f_2 f_1(y)]_ג - f'_2 f'_1([x,y]_ג) - [x, f_1 f_2(y)]_ג$$

$$= f'_1 f'_2([x,y]_ג) - f'_2 f'_1([x,y]_ג) - ([x, f_1 f_2(y)]_ג - [x, f_2 f_1(y)]_ג)$$

$$= [f'_1, f'_2]_ג([x,y]_ג) - [x, [f_1, f_2]_ג(y)]_ג$$

Then

$$[[f_1, f_2]_ג(x), y]_ג = [f'_1, f'_2]_ג([x,y]_ג) - [x, [f_1, f_2]_ג(y)]_ג$$

Then $[f_1, f_2]_ג \in QDer_ג(\mathcal{L})$, we have $QDer_ג(\mathcal{L})$ is Γ-Lie sub algebra of $GenDer_ג(\mathcal{L})$.

2- To prove $[Der_ג(\mathcal{L}), C(\mathcal{L})]_ג \subseteq C(\mathcal{L})$ Let $f_1 \in Der_ג(\mathcal{L})$ and , $f_2 \in C(\mathcal{L})$ for all $x, y \in \mathcal{L}$

Since $f_{1,} \in Der_ג(\mathcal{L})$, $f_2 \in C(\mathcal{L})$ we can write the following two equations By Defined By. Rezaei In [8]

$$f_1([x,y]_ג) = [f_1(x), y]_ג + [x, f_1(y)]_ג \qquad (a)$$

$f_2([x,y]_\lambda) = [f_2(x),y]_\lambda = [x,f_2(y)]_\lambda$ (b)

$[[f_1,f_2]_\lambda(x),y]_\lambda = [(f_1f_2 - f_2f_1)(x),y]_\lambda$

$= [f_1f_2(x) - f_2f_1(x),y]_\lambda$

$= [f_1f_2(x),y]_\lambda - [f_2f_1(x),y]_\lambda$ (c)

i- $[f_1f_2(x),y]_\lambda =$

$f_1([f_2(x),y]_\lambda) - [f_2(x),f_1(y)]_\lambda$ by Eq (a)

$= f_1f_2([x,y]_\lambda) - f_2([x,f_1(y)]_\lambda)$ by Eq (b)

Then

$$[f_1f_2(x),y]_\lambda = f_1f_2([x,y]_\lambda) - f_2([x,f_1(y)]_\lambda)$$

ii- $[f_2f_1(x),y]_\lambda = f_2([f_1(x),y]_\lambda)$ by Eq (b)

$= f_2f_1([f_1(x),y]_\lambda) - f_2([x,f_1(y)]_\lambda)$ by Eq (a)

Then

$$[f_2f_1(x),y]_\lambda = f_2f_1([f_1(x),y]_\lambda) - f_2([x,f_1(y)]_\lambda)$$

Substituting the values of (i), (ii) into equation (c) we get

$$[[f_1,f_2]_\lambda(x),y]_\lambda = f_1f_2([x,y]_\lambda) - f_2([x,f_1(y)]_\lambda)$$

$$-f_2f_1([f_1(x),y]_\lambda) + f_2([x,f_1(y)]_\lambda)$$

$$= f_1f_2([x,y]_\lambda) - f_2f_1([f_1(x),y]_\lambda)$$

$$= [f_1,f_2]_\lambda([x,y]_\lambda)$$

Then

$$[[f_1,f_2]_\lambda(x),y]_\lambda = [f_1,f_2]_\lambda([x,y]_\lambda)$$

Then $[f_1,f_2] \in C(\mathcal{L})$, we have $[Der_\lambda(\mathcal{L}) , C(\mathcal{L})] \subseteq C(\mathcal{L})$.

3- To prove $[QDer_\lambda(\mathcal{L}) , QC(\mathcal{L})]_\lambda \subseteq QC(\mathcal{L})$ Let $f_1 \in QDer_\lambda(\mathcal{L})$ then there exists $f'_1 \in Hom(\mathcal{L},\mathcal{L})$ and $f_2 \in QC(\mathcal{L})$ for all $x,y \in \mathcal{L}$.

Since $f_1 \in QDer_\lambda(\mathcal{L})$, $f_2 \in QC(\mathcal{L})$ we can write the following two equations by [3.1 & 3.3]

$f'_1([x,y]_\lambda) = [f_1(x),y]_\lambda + [x,f_1(y)]_\lambda$ (a)

$[f_2(x),y]_\lambda = [x,f_2(y)]_\lambda$ (b)

$[[f_1,f_2]_\lambda(x),y]_\lambda = [(f_1f_2 - f_2f_1)(x),y]_\lambda$

$= [f_1f_2(x) - f_2f_1(x),y]_\lambda$

$= [f_1f_2(x),y]_\lambda - [f_2f_1(x),y]_\lambda$ (c)

i- $[f_1f_2(x),y]_\lambda =$

$f'_1([f_2(x),y]_\lambda) - [f_2(x),f_1(y)]_\lambda$ by Eq (a)

$= f'_1([x,f_2(y)]_\lambda) - [x,f_2f_1(y)]_\lambda$ by Eq (b)

Then

$$[f_1f_2(x),y]_\lambda = f'_1([x,f_2(y)]_\lambda) - [x,f_2f_1(y)]_\lambda$$

ii- $[f_2f_1(x),y]_\lambda = [f_1(x),f_2(y)]_\lambda$ by Eq (b)

$= f'_1([x,f_2(y)]_\lambda) - [x,f_1f_2(y)]_\lambda$ by Eq (a)

Then

$$[f_2f_1(x),y]_\lambda = f'_1([x,f_2(y)]_\lambda) - [x,f_1f_2(y)]_\lambda$$

Substituting the values of (i), (ii) into equation (c) we get

$[[f_1,f_2]_\lambda(x),y]_\lambda =$

$$f'_1([x,f_2(y)]_\lambda) - [x,f_2f_1(y)]_\lambda - f'_1([x,f_2(y)]_\lambda) + [x,f_1f_2(y)]_\lambda$$

$$= [x,f_1f_2(y)]_\lambda - [x,f_2f_1(y)]_\lambda$$

$$= [x,[f_1,f_2]_\lambda(y)]_\lambda$$

Thus

$$[[f_1,f_2]_\lambda(x),y]_\lambda = [x,[f_1,f_2]_\lambda(y)]_\lambda$$

Thus $[f_1,f_2]_\lambda \in QC(\mathcal{L})$

Then $[QDer_\lambda(\mathcal{L}) , QC(\mathcal{L})]_\lambda \subseteq QC(\mathcal{L})$.

4- To prove $C(\mathcal{L}) \subseteq QDer_\lambda(\mathcal{L})$ Let $f \in C(\mathcal{L})$ for all $x,y \in \mathcal{L}$

Since $f \in C(\mathcal{L})$ we can write the following equation, By Defined By. Rezaei In [8]

$$f([x,y]_\lambda) = [f(x),y]_\lambda = [x,f(y)]_\lambda$$

Hence,

$$[f(x),y]_\lambda + [x,f(y)]_\lambda = 2f([x,y]_\lambda)$$

Therefore , $f \in QDer_\lambda(\mathcal{L})$ since $f' = 2f \in Hom(\mathcal{L},\mathcal{L})$.

$$C(\mathcal{L}) \subseteq QDer_\lambda(\mathcal{L}).$$

5- To prove $[QC(\mathcal{L}) , QC(\mathcal{L})]_\lambda \subseteq QDer_\lambda(\mathcal{L})$ Let $f_1,f_2 \in QC(\mathcal{L})$ for all $x,y \in \mathcal{L}$

Since $f_1,f_2 \in QC(\mathcal{L})$ we can write the following two equations. By [3.3]

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$[f_1(x), y]_\lambda = [x, f_1(y)]_\lambda$ (a)

$[f_2(x), y]_\lambda = [x, f_2(y)]_\lambda$ (b)

$$[[f_1, f_2]_\lambda(x), y]_\lambda = [(f_1 f_2 - f_2 f_1)(x), y]_\lambda$$

$$= [f_1 f_2(x) - f_2 f_1(x), y]_\lambda$$

$$= [f_1 f_2(x), y]_\lambda - [f_2 f_1(x), y]_\lambda \quad (c)$$

i- $[f_1 f_2(x), y]_\lambda = [f_2(x), f_1(y)]_\lambda$ by Eq (a)

$= [x, f_2 f_1(y)]_\lambda$ by Eq (b)

Then

$$[f_1 f_2(x), y]_\lambda = [x, f_2 f_1(y)]_\lambda$$

ii- $[f_2 f_1(x), y]_\lambda = [f_1(x), f_2(y)]_\lambda$ by Eq (b)

$= [x, f_1 f_2(y)]_\lambda$ by Eq(a)

Then

$$[f_2 f_1(x), y]_\lambda = [x, f_1 f_2(y)]_\lambda$$

Substituting the values of (i), (ii) into equation (c) we get

$$[[f_1, f_2]_\lambda(x), y]_\lambda = [x, f_2 f_1(y)]_\lambda - [x, f_1 f_2(y)]_\lambda$$

$$= [x, f_2 f_1(y) - f_1 f_2(y)]_\lambda$$

$$= [x, (f_2 f_1 - f_1 f_2)(y)]_\lambda$$

$$= [x, [f_2, f_1]_\lambda(y)]_\lambda$$

$$= -[x, [f_1, f_2]_\lambda(y)]_\lambda$$

Then

$$[[f_1, f_2]_\lambda(x), y]_\lambda = -[x, [f_1, f_2]_\lambda(y)]_\lambda$$

Hence

$$[[f_1, f_2]_\lambda(x), y]_\lambda + [x, [f_1, f_2]_\lambda(y)]_\lambda = 0$$

Therefore, $[f_1, f_2]_\lambda \in QDer_\lambda(\mathcal{L})$ since $f' = 0 \in Hom(\mathcal{L}, \mathcal{L})$

$[QC(\mathcal{L}), QC(\mathcal{L})]_\lambda \subseteq QDer_\lambda(\mathcal{L})$.

3.6 Proposition :-

Let $\mathcal{L}$ be a Γ-Lie algebra, then

1- $GenDer_\lambda(\mathcal{L}) = QDer_\lambda(\mathcal{L}) + QC(\mathcal{L})$.

2- $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_\lambda$ is an ideal in the Γ-Lie algebra $GenDer_\lambda(\mathcal{L})$

Proof:-

1- We will prove that $GenDer_\lambda(\mathcal{L}) = QDer_\lambda(\mathcal{L}) + QC(\mathcal{L})$ Let $f_1 \in QDer_\lambda(\mathcal{L}), f_2 \in QC(\mathcal{L})$ for all $x, y \in \mathcal{L}$**.**

Since $f_1 \in QDer_\lambda(\mathcal{L}), f_2 \in QC(\mathcal{L})$ we can write the following two equations, by [Definition 2.4.1 , 2.4.4]

$[f_1(x), y]_\lambda + [x, f_1(y)]_\lambda = f_1'([x, y]_\lambda)$ (a)

$[f_2(x), y]_\lambda = [x, f_2(y)]_\lambda$ (b)

$$[(f_1 + f_2)(x), y]_\lambda = [(f_1(x) + f_2(x), y]_\lambda$$

$$= [f_1(x), y]_\lambda + [f_2(x), y]_\lambda \quad (c)$$

i- $[f_1(x), y]_\lambda = f_1'([x, y]_\lambda) - [x, f_1(y)]_\lambda$ by Eq (a)

ii- $[f_2(x), y]_\lambda = [x, f_2(y)]_\lambda$ by Eq (b)

Substituting the values of (i), (ii) into equation (c) we get

$$[(f_1 + f_2)(x), y]_\lambda = f_1'([x, y]_\lambda) - [x, f_1(y)]_\lambda + [x, f_2(y)]_\lambda$$

$$= f_1'([x, y]_\lambda) - [x, (f_1 - f_2)(y)]_\lambda$$

Thus

$$f_1'([x, y]_\lambda) = [(f_1 + f_2)(x), y]_\lambda + [x, (f_1 - f_2)(y)]_\lambda$$

Since $(f_1 - f_2), f_1' \in Hom(\mathcal{L}, \mathcal{L})$, then $f_1 + f_2 \in GenDer_\lambda(\mathcal{L})$ implies that

$$GenDer_\lambda(\mathcal{L}) \supseteq QDer_\lambda(\mathcal{L}) + QC(\mathcal{L})$$

If we assume $f_1 + f_2 \in GenDer_\lambda(\mathcal{L}), f_1 - f_2, f_1' \in Hom(\mathcal{L}, \mathcal{L})$ and using the [Definition 2.4.3] we will get the second direction implies that

$$GenDer_\lambda(\mathcal{L}) \subseteq QDer_\lambda(\mathcal{L}) + QC(\mathcal{L})$$

This gives

$$GenDer_\lambda(\mathcal{L}) = QDer_\lambda(\mathcal{L}) + QC(\mathcal{L}).$$

2- To prove $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_\lambda$ is an ideal in the Γ-Lie algebra $GenDer_\lambda(\mathcal{L})$ We will install first $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_\lambda$ is a Γ-Lie sub algebra of $GenDer_\lambda(\mathcal{L})$

By [Lemma 3.5 (5)] and [Proposition 2.4.7 (1)] we get

$$QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_\lambda \subseteq QC(\mathcal{L}) + QDer_\lambda(\mathcal{L})$$

$$= GenDer_\lambda(\mathcal{L})$$

Then

$$QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_\lambda \subseteq GenDer_\lambda(\mathcal{L})$$

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Now to prove $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$ is Ґ-Lie sub algebra of $GenDer_{\lambda}(\mathcal{L})$

$[QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}, QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

By [Lemma 3.5 (5)] we get

$\subseteq [QC(\mathcal{L}) + QDer_{\lambda}(\mathcal{L}), QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

$= [QC(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda} +$

$[QDer_{\lambda}(\mathcal{L}), QC(\mathcal{L})]_{\lambda} + [QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

By [Lemma 2.4.6 (3)] and Jacobi identity of Ґ-Lie algebra we get

$$\subseteq [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda} + QC(\mathcal{L}) + [QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$$

It is easy to verify $[QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} = 0$ by the Jacobi

identity of Lie algebra. Thus

$[QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} =$

$-[QC(\mathcal{L}), [QDer_{\lambda}(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

$-[QC(\mathcal{L}), [QC(\mathcal{L}), QDer_{\lambda}(\mathcal{L})]_{\lambda}]_{\lambda}$

$= -[QC(\mathcal{L}), [QDer_{\lambda}(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} +$

$[QC(\mathcal{L}), [QDer_{\lambda}(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} = 0$

Then

$$[QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda} + QC(\mathcal{L}) + [QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$$

$$= [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda} + QC(\mathcal{L})$$

Thus

$[QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}, QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

$\subseteq QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$

Then $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$ Ґ-Lie sub algebra.

Now to prove $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$ is an ideal in the Ґ-Lie algebra $GenDer_{\lambda}(\mathcal{L})$

$$[GenDer_{\lambda}(\mathcal{L}), QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$$

By [Lemma 3.5 (1)] we get

$= [QDer_{\lambda}(\mathcal{L}) + QC(\mathcal{L}), QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

$= [QDer_{\lambda}(\mathcal{L}), QC(\mathcal{L})]_{\lambda} + [QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

$+[QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda} + [QC(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda}$

By [Lemma 2.4.6 (5)] and Jacobi identity of Ґ-Lie algebra and $[QDer_{\lambda}(\mathcal{L}), [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} = 0$ as we explained above we get

$$\subseteq QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$$

Thus

$$[GenDer_{\lambda}(\mathcal{L}), QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}]_{\lambda} \subseteq QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$$

Then $QC(\mathcal{L}) + [QC(\mathcal{L}), QC(\mathcal{L})]_{\lambda}$ is an ideal in the Lie algebra $GenDer_{\lambda}(\mathcal{L})$.

3.7 Lemma

If $\mathcal{L} = H \oplus K$ is the direct sum of ideals, and • $Ann(\mathcal{L}) = 0$ then.

(1) $GenDer_{\lambda}(\mathcal{L}) = GenDer_{\lambda}(H) \oplus GenDer_{\lambda}(K)$.

(2) $QDer_{\lambda}(\mathcal{L}) = QDer_{\lambda}(H) \oplus QDer_{\lambda}(K)$..

(3) $C(\mathcal{L}) = C(H) \oplus C(K)$.

(4) $QC(\mathcal{L}) = QC(H) \oplus QC(K)$.

Proof:-

(1) To prove .

$$GenDer_{\lambda}(\mathcal{L}) = GenDer_{\lambda}(H) \oplus GenDer_{\lambda}(K).$$

For $f \in GenDer_{\lambda}(H)$ extend it to a linear transformation on $\mathcal{L}$ by setting $f(x) = f(h+k) = f(h) \quad \forall\, x \in \mathcal{L}, h \in H, k \in K$.

Then $f \in GenDer_{\lambda}(\mathcal{L})$ and $enDer_{\lambda}(H) \subseteq GenDer_{\lambda}(\mathcal{L})$. Similarly, $GenDer_{\lambda}(H) \subseteq GenDer_{\lambda}(\mathcal{L})$.

Let $f_1, f_2 \in Hom(\mathcal{L}, \mathcal{L})$ difine by $f_1(x) = f_1(h+k) = f(h)$ and $f_2(x) = f_2(h+k) = f(k)$, then $f_1 \in GenDer_{\lambda}(H)$ and $f_2 \in GenDer_{\lambda}(K)$

Hence

$f(x) = f(h+k) = f(h) + f(k) = f_1(h+k) + f_2(h+k) = f_1(x) + f_2(x)$

Then $f = f_1 + f_2 \in GenDer_{\lambda}(H) + GenDer_{\lambda}(K)$

Thus $GenDer_{\lambda}(H) + GenDer_{\lambda}(K) = GenDer_{\lambda}(\mathcal{L})$.

Let $h \in H, k \in K$ and $f \in GenDer_{\lambda}(\mathcal{L})$ then

$$f''([h,k]_\lambda) = [f(h),k]_\lambda + [h,f'(k)]_\lambda$$

$$[f(h),k]_\lambda = f''([h,k]_\lambda) - [h,f'(k)]_\lambda \in H \cap K = 0$$

Suppose $f(h) = a + b$ where $a \in H, b \in K$

$$0 = [f(h),k]_\lambda = [a,k]_\lambda + [b,k]_\lambda$$

So $[b,k]_\lambda = 0$ and $b \in Ann(\mathcal{L})$, $b = 0$. Hence $f'(h) = a \in H$, Therefore $f(H) \subseteq H$. Similarly, $f(K) \subseteq K$

(2), (3), (4) Similar to the proof of (1).

3.8 Remark:-

If $\mathcal{L}$ is commutative, then • $f, f', f'' \in Hom(\mathcal{L},\mathcal{L})$‘ implies that Eq(a) = Eq(b) listed below, i.e. (f,f',f'') , $(f',f,f'') \in \nabla(A)$

$$f''([x,y]_\lambda) = [f(x),y]_\lambda + [x,f'(y)]_\lambda \quad \text{(a)}$$

$$f''([x,y]_\lambda) = [f'(x),y]_\lambda + [x,f(y)]_\lambda \quad \text{(b)}$$

Because $f, f', f'' \in Hom(\mathcal{L},\mathcal{L})$ for all $x, y \in \mathcal{L}$ then

$$f''([x,y]_\lambda) = [f(x),y]_\lambda + [x,f'(y)]_\lambda$$

Sense $\mathcal{L}$ is commutative then $[x,y]_\lambda = [y,x]_\lambda$ implies that

$$f''([x,y]_\lambda) = f''([y,x]_\lambda$$

Since

$$f''([y,x]_\lambda = [f(y),x]_\lambda + [y,f'(x)]_\lambda$$

Then

$$f''([y,x]_\lambda) = [f(y),x]_\lambda + [y,f'(x)]_\lambda = f''([x,y]_\lambda)$$

implies that

$$f''([x,y]_\lambda) = [f(y),x]_\lambda + [y,f'(x)]_\lambda$$

$$f''([x,y]_\lambda) = [f'(x),y]_\lambda + [x,f(y)]_\lambda$$

Thus Eq(a) = Eq(b) and (f,f',f'') , $(f',f,f'') \in \nabla(A)$.

3.9 Theorem

If $\mathcal{L}$ is an Γ-Lie algebra, then $[C(\mathcal{L}), QC(\mathcal{L})] \subseteq Hom(\mathcal{L}, Z(\mathcal{L}))$. Moreover, if $Z(\mathcal{L}) = \{0\}$, then $[C(\mathcal{L}), QC(\mathcal{L})] = \{0\}$.

Proof:-

Let $f_1 \in C(\mathcal{L}), f_2 \in QC(\mathcal{L})$, then for all $x, y \in \mathcal{L}$,

Since $f_1 \in C(\mathcal{L})$, $f_2 \in QC(\mathcal{L})$ we can write the following two equations By Defined By. Rezaei In [8] &, [3.3]

$$f_1([x,y]_\lambda) = [f_1(x),y]_\lambda = [x,f_2(y)]_\lambda \quad \text{(a)}$$

$$[f_2(x),y]_\lambda = [x,f_2(y)]_\lambda \quad \text{(b)}$$

We can write an equation (b) in the following form

$$[f_2(x),y]_\lambda - [x,f_2(y)]_\lambda = 0 \quad \text{(c)}$$

$$[[f_1,f_2]_\lambda(x),y]_\lambda = [(f_1f_2 - f_2f_1)(x),y]_\lambda$$

$= [f_1f_2(x) - f_2f_1(x),y]_\lambda$

$= [f_1f_2(x),y] - [f_2f_1(x),y]_\lambda$

$= f_1([f_2(x),y]) - [f_2f_1(x),y]_\lambda$ by Eq (a)

$= f_1([f_2(x),y]) - [f_1(x),f_2(y)]_\lambda$ by Eq (b)

$= f_1([f_2(x),y]) - f_1([x,f_2(y)]_\lambda)$ by Eq (a)

$= f_1([f_2(x),y]) - f_1([x,f_2(y)]_\lambda)$ by Eq (a)

$= f_1([f_2(x),y] - [x,f_2(y)]_\lambda)$

$= f_1(0) = 0$ by Eq (c).

Hence $[f_1,f_2]_\lambda(x) \in Z(\mathcal{L})$ and $[f_1,f_2]_\lambda \in Hom(\mathcal{L}, Z(\mathcal{L})$ as desired. Furthermore, if $Z(\mathcal{L}) = \{0\}$, it is clear that $[C(\mathcal{L}), QC(\mathcal{L})]_\lambda = \{0\}$.

REFERENCES

[1] Abellanas, L., & Martinez Alonso, L. (1975). A general setting for Casimir invariants. Journal of Mathematical Physics, 16(8), 1580-1584.

[2] Jacobson, N. (1979). Lie algebras (No. 10). Courier Corporation..

[3] Rand, D., Winternitz, P., & Zassenhaus, H. (1988). On the identification of a Lie algebra given by its structure constants. I. Direct decompositions, Levi decompositions, and nilradicals. Linear algebra and its applications, 109, 197-246.

[4] Popovych, R. O., Boyko, V. M., Nesterenko, M. O., & Lutfullin, M. W. (2003). Realizations of real low-dimensional Lie algebras. Journal of Physics A: Mathematical and General, 36(26), 7337.

[5] Alzaiadi,Ali,A,H., & Shaheen. Rajaa .C. Derivations on Γ-Lie Algebras and Some Related Concepts , Msc.thesis ,Department of mathematic, college of education, AI-Qadisiyah University.(2021)

[6] Leger, G. F., & Luks, E. M. (2000). Generalized derivations of Lie algebras. Journal of Algebra, 228(1), 165-203.

[7] De Filippis, V., & Wei, F. (2018). b-generalized (α, β)-derivations and b-generalized (α, β)-biderivations of Prime Rings. Taiwanese Journal of Mathematics, 22(2), 313-323

[8] Rezaei, A. H., & Davvaz, B. (2018). Construction of $\Gamma $-algebra and $\Gamma $-Lie admissible algebras. Korean Journal of Mathematics, 26(2), 175-189.