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The Circulant Matrices

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Abstract: *In this research we introduce one of special matrices which is called circulate matrix. Also we study the invertible of some forms of it.*

1. Introduction

Many problems in applied mathematics and science lead to the solution of linear systems having circulant coefficients related to the periodicity of the problems, as the ones that appear when using the finite difference method to approximate elliptic equations with periodic boundary conditions, see [1]. Circulant matrices have a wide range of application in signal processing, image processing, digital image disposal, linear forecast, error correcting code theory, see [2, 3]. In the last years, there have been several papers on circulant matrices that attend to give an effective expression for the determinant, the eigenvalues and the inverse of the matrix, see for instance [4, 5, 6].

Fix a positive integer $n \geq 2$, and let

v = $(v_0.v_1.\dots.v_{n-1})$

be a row vector in Cn. Define the shift operator $T : C^n \to C^n$ by

$$T(v_0.v_1.\dots.v_{n-1}) = (v_{n-1}.v_0.\dots.v_{n-2})$$

The circulant matrix associated to v is the $n \times n$ matrix whose rows are given by iterations of the shift operator acting on v, that is to say, the matrix whose k-th row is given by $T^{k-1}v\,.k = 1.\dots.n$. Such a matrix will be denoted by

V = circ{v} = circ$(v_0.v_1.\dots.v_{n-1})$.

Many facts about these matrices can be proven using only basic linear algebra. This makes the area quite accessible to undergraduates looking for "research problems." Our note presents a general view of these type of matrices, and hopes to illustrates the latter point by including in it a number of problems that may be of interest to students.

In this research we have two chapters in chapter one discuss the definition of circulene matrix with some properties .

In chapter two we study the invertibal grculant matrix .

2. Basic Definitions

Definition (2.1) [2]

A matrix is an $m \times n$ array of from a given as below . The individual values in the matrix are called entries.

Examples.(2. 2)[2]

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The size of the array is–written as m × n, where

m × n

number of rows number of columns

Notation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n^2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \nwarrow \\ \leftarrow rows \\ \swarrow \end{array}$$

columns

A := uppercase denotes a matrix

a := lower case denotes an entry of a matrix

Definition (2. 3) [2]

Let A be an n x n matrix A is said to be invertible

if there is a matrix B such that

AB = BA = I.

In this case B is called the inverse of A, and the notation for the inverse is A–1 .

Examples.(2.4) [2]

(i) Let $\begin{matrix} A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \\ A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \end{matrix}$ $Then$

(ii) For n = 3 we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & -1 \\ -2 & 3 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 3 & -2 \\ -3 & 7 & -5 \end{bmatrix}$$

A square matrix need not have an inverse, as will be discussed in the next section. As examples, the two matrices below do not have inverses

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Eigenvalues(2.5) [2]

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The number is an eigenvalue of A if and only if AI is singular:

$$\det(A-\lambda I)=0$$

This "characteristic equation" det.A When A is n by n, the equation has degree n. Then A has n eigenvalues and each leads to x:

For each λ solve $(A-\lambda I)x=0 \ or \ Ax=\lambda x$ to find an eigenvector x:

Example (2.6) [2]

$A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ is already singular (zero determinant). Find its λ 's and x's.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation Ax= 0x has solutions. They are the eigenvectors for $\lambda = 0$. But $\det(A-\lambda I)=0$. is the way to find all λ 's and x's. Always subtract λ I from A:

Subtract from the diagonal to find $A-\lambda I=\begin{bmatrix}\mathbf{1-\lambda} & \mathbf{2}\\ \mathbf{2} & \mathbf{4-\lambda}\end{bmatrix}$

Take the determinant "ad - bc" *of this* 2 *by* 2 *matrix.* From $\mathbf{1-\lambda}$ times $-\boldsymbol{\lambda}$, the "ad " part is $\lambda^2-5\lambda+4$. The "bc" part, not containing , is 2 times 2.

$$\boldsymbol{det}\begin{bmatrix}\mathbf{1-\lambda} & \mathbf{2}\\ \mathbf{2} & \mathbf{4-\lambda}\end{bmatrix}=(\mathbf{1}-\boldsymbol{\lambda})(\mathbf{4}-\boldsymbol{\lambda})-(\mathbf{2})(\mathbf{2})=\boldsymbol{\lambda}^{\mathbf{2}}-\mathbf{5}\boldsymbol{\lambda}$$

Set this determinant $\boldsymbol{\lambda}^{\mathbf{2}}-\mathbf{5}\boldsymbol{\lambda}$ *to zero.* One solution is $\lambda_1=0$ (as expected, since A is singular). Factoring into $\boldsymbol{\lambda}$ times $\boldsymbol{\lambda}-\mathbf{5}$, the other root is $\lambda=5$

$\boldsymbol{det}(\boldsymbol{A}-\boldsymbol{\lambda I})=\boldsymbol{\lambda}^{\mathbf{2}}-\mathbf{5}\boldsymbol{\lambda}=\mathbf{0}$ yields the eigenvalues $\lambda_1=0$ and $\boldsymbol{\lambda_2}=\mathbf{5}$

Now find the eigenvectors. Solve $(\mathbf{A}-\boldsymbol{\lambda}\mathbf{I})\boldsymbol{x}=\mathbf{0}$ separately for $\lambda_1=0$ and $\boldsymbol{\lambda_2}=\mathbf{5}$

$(\mathbf{A}-\mathbf{0I})$ x = $\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ yields an eigenvector $\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$ for $\lambda_1=0$

$(\mathbf{A}-\mathbf{5I})$ x = $\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ yields an eigenvector $\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$ for $\lambda_2=5$

Summary (2. 7) [2]

To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. *Compute the determinant of* $\boldsymbol{A}-\boldsymbol{\lambda I}$ With λ subtracted along the diagonal, this determinant starts with $\lambda^n or -\lambda^n$. It is a polynomial in of degree n

2. *Find the roots of this polynomial,* by solving det $(\boldsymbol{A}-\boldsymbol{\lambda I})=\mathbf{0}$ The n roots are the n eigenvalues of A. They make $\boldsymbol{A}-\boldsymbol{\lambda I}$ singular

3. For each eigenvalue , $solve(\boldsymbol{A}-\boldsymbol{\lambda I})\boldsymbol{x}=\mathbf{0}$ to find an eigenvector x.

3. Circulant Matrices

A circulant matrix C is a matrix having the form

$$c = \begin{bmatrix} co & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & co & c_1 \; c_2 & & \vdots \\ \vdots & c_{n-1} & co \; c_1 & \ddots & c_2 \\ & \ddots & \ddots \; \ddots & & c_1 \\ c_1 & \cdots & c_{n-1} & & co \end{bmatrix}$$

where each row is a cyclic shift of the row above it. The structure can also be characterized by noting that the (k,j) entry of C, Ck,j, is given by

Ck,j = C(j-k) mod n.

Eigenvalues and Eigenvectors (3.1) [5]

The eigenvalues ψ_k and the eigenvectors $y^{(k)}$ of C are the solutions of

$$Cy=\psi_y \tag{1.2}$$

or, equivalently, of the n difference equations

$$\sum_{k=0}^{m-1} c_n - m + kyk + \sum_{k=m}^{n-1} c_k - myk = \psi\ ym; m = 0.1. \cdots .n-1 \tag{1.3}$$

Changing the summation dummy variable results in

$$\sum_{k=0}^{n-1-m} c_k yk + m + \sum_{k=n-m}^{n-1} c_k yk - (\, n-m) = \ \psi\ ym; m = 0.1. \cdots .n-1 \tag{1.4}$$

One can solve difference equations as one solves differential equations - by guessing an intuitive solution and then proving that it works. Since the equation is linear with constant coefficients a reasonable guess is

Yk = p^k (analogous to y(t) = e^{st} in linear time invariant differential equations). Substitution into (1.4) and cancellation of p^m yields

$$\sum_{k=0}^{n-1-m} c_k p^k + p^{-n} + \sum_{k=n-m}^{n-1} c_k p^k = \ \psi$$

Thus if we choose p^{-n} = 1, i.e., p is one of the n distinct complex n^{th} roots of unity, then we have an eigenvalue

$$\psi = \sum_{k=0}^{n-1} c_k p^k \tag{1.5}$$

with corresponding eigenvector

$$y = n^{-\frac{1}{2}}(1.p.p^2. \cdots .p^{n-1} \tag{1.6}$$

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where the prime denotes transpose and the normalization is chosen to give the eigenvector unit energy. Choosing Pm as the complex n^{th} root of unity,

Pm = $e^{-2\pi n i m/n}$, we have eigenvalue

$$\psi m = \sum_{k=0}^{n-1} c_k e^{-2\pi i m k/n} \tag{1.7}$$

3.1. Eigenvalues and Eigenvectors and eigenvector

$$y^{(m)} = \frac{1}{\sqrt{n}}\left(1, e^{-2\pi i m/n}, \cdots, e^{-2\pi i(n-1)/n}\right)$$

Thus from the definition of eigenvalues and eigenvectors,

$$c_y{}^{(m)} = \psi_m y^{(m)}.m = 0.1.\cdots.n-1 \tag{1.8}$$

Equation (1.7) should be familiar to those with standard engineering backgrounds as simply the discrete Fourier transform (DFT) of the sequence {ck}. Thus we can recover the sequence $\{c_k\}$ from theψ_k by the Fourier inversion formula. In particular,

$$\frac{1}{n} = \sum_{k=0}^{n-1} \psi_m e^{2\pi i l m} = \frac{1}{n}\sum_{m=0}^{n-1}\sum_{k=0}^{n-1}(c_k e^{-2\pi i m k/n})\, e^{2\pi i l m}$$

$$= \sum_{k=0}^{n-1} c_k \frac{1}{n} \sum_{m=0}^{n-1} e^{2\pi i(e-k)m/n} = c_l \tag{1.9}$$

where we have used the orthogonality of the complex exponentials:

$$\sum_{k=0}^{n-1} e^{2\pi i m k(e-k)/n} = n\delta k_{mod\, n} = \begin{cases} n & k \bmod n = 0 \\ 0 & otherwise \end{cases} \tag{1.10}$$

Where δ is the Kronecker delta,

$$\delta_m = \begin{cases} n & k \bmod n = 0 \\ 0 & otherwise \end{cases}$$

Thus the eigenvalues of a circulant matrix comprise the DFT of the first row of the circulant matrix, and conversely first row of a circulant matrix is the inverse DFT of the eigenvalues.

Eq. (1.8) can be written as a single matrix equation

$$\mathrm{CU} = \mathrm{U}\psi \tag{1.11}$$

where

$$\mathrm{U} = \langle y^{(0)} | y^{(1)} | \cdots | y^{(n-1)} \rangle$$

$$= n^{-1/2}[e^{\frac{2\pi imk(e-k)}{n}}; \ m.k = 0.1.\cdots.n-1]$$

is the matrix composed of the eigenvectors as columns, and ψ= diag(ψ_k) is the diagonal matrix with diagonal elements$\psi_0 . \psi_1 . \cdots . \psi_{n-1}$ .Furthermore, (1.10) implies that U is unitary. By way of details, denote that the (k,j)th element of UU* by $a_{k,j}$ and observe that ak,j will be the product of the kth row of U, which is {$e^{-2\pi imk}/n/\sqrt{n}$; m = 0,1,...,n – 1}, times the jth column of U*, which is {$e^{2\pi imj/n}/\sqrt{n}$; m=0,1,...,n – 1} so that

$$a_{k,j} = \frac{1}{n}\sum_{m=0}^{n-1} e^{2\pi im(j-k)/n} = \delta_{(k-j) \bmod n}$$

and hence UU* = I. Similarly, U*U = I. Thus (1.11) implies that

$$C = U\Psi U^* \quad \textbf{(1.12)}$$

$$\Psi = U^*CU. \quad \textbf{(1.13)}$$

Since C is unitarily similar to a diagonal matrix it is normal.

Theorem (3.2) [4]

Every circulant matrix C has eigenvectors $y^{(m)}$ =

$$\frac{1}{\sqrt{n}}\left(1, e^{-2\pi im/n}, \cdots, e^{-2\pi i(n-1)/n}\right)', \ m = 0, 1, \ldots, n-1,$$

and corresponding eigenvalues

$$\psi_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}$$

and can be expressed in the form C = U ψ U*, where U has the eigen – vectors as columns in order and ψ is diag(Uk). In particular all circulant matrices share the same eigenvectors, the same matrix U works for all circulant matrices, and any matrix of the form C=U ψ U* is circulant.

Let C = {c_{k}–j} and B = {b_{k}–j} be circulant n x n matrices with eigenvalues

$$\psi_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}, \quad \beta_m = \sum_{k=0}^{n-1} b_k e^{-2\pi imk/n},$$

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respectively. Then

(1) C and B commute and CB= BC = U γU* ,

where γ = diag($\psi_m \beta_m$), and CB is also a circulant matrix.

(2) C + B is a circulant matrix and C + B = U Ω U*,

where Ω = {($\psi_m \beta_m$)δ_{k-m}}

(3) If $\psi_m \neq 0$; m=0,1,...,n - 1, then C is nonsingular and

$C^{-1} = U\psi^{-1}U^*$.

Proof. We have C = UψU* and B = UΦU* where ψ = diag(ψ_m) and Φ = diag(β_m).

(1) CB=U ψU*UΦU* = UψΦU* = UΦψ U* = BC. Since ψΦ

is diagonal, the first part of the theorem implies that CB is circulant.

(2) C + B=U(ψ + Φ)U*.

(3) If ψ is nonsingular, then

$CU \psi^{-1}U^* = U \psi U^*U \psi^{-1}U^* = U \psi\psi^{-1}U^*$

$= UU^* = 1.$

4. Invertibility of some circulant matrices

We define a circular matrix by using linear operators on row vector space. Let T be the shift operator on C^n (written as row vectors) defined by

$T:x= (x_1, x_2, x_3 ..., x_n) \mapsto (x_n, x_1, x_2, ..., x_{n-1})$

Definition (4.1) [1]

Let A be an n x n complex matrix and a_i be the ith row of A. The matrix A is called circulant if $T(a_i) = a_{i+1}$ where the indices are taken in modulo n. Then we denote the circulant matrix A by circ(a_1).

Definition (4.2) [1]

Let A = circ(a_i, ..., a_n). The polynomial

$_{pA}(X) = \sum_i^n = 1 \; a_i \; X^{i-1}$ is called the representer polynomial of the circulant matrix A.

In what follows we list some basic properties of circulant matrix that we will use throughout . this paper. We denote by s(A) the sum of all entries in the first row of A.

Properties of cireculant matnces (4.3) [1]

Let A = circ(a) = circ(a_1,a_2, ..., a_n).

1-

(i) $(1,1,\ldots,1)A = s(A)(1,1,\ldots,1)$.

(ii) *If* $s(A) \neq 0$ *then* $(1,1,\ldots,1) = \frac{1}{s(A)} \sum_{i=1}^{n} T^i(a)$ *is in the row space of A.*

(iii) The circulant matrix A is invertible if and only if $(1,0,\ldots,0)$ *is in the row space of A. Moreover if* $e_1 = (1,0,\ldots,0) = \sum_{i=1}^{n} \alpha_i T^i(a)$ *for some real numbers* α_i *then*

$$A^{-1} = circ(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

(iv) Let $p_A(X)$ *be the representer matrix of A and* $\epsilon_j = e^{\frac{2\pi i(j-1)}{n}}$. *Then A is diagonalizable by the matrix*

$$F = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \epsilon_2 & \ldots & \epsilon_n \end{pmatrix}$$

3- *If* $s(A) \neq 0$ *and* $(1,\ldots,1) = \sum_{i=1}^{n} \alpha_i T^i(a)$ *where* $\alpha_{i_0} \neq \frac{1}{s(A)}$ *for some* i_0, *then A is singular.*

nd

4-

$$A = F\, Diag(p_A(\epsilon_1), p_A(\epsilon_2), \ldots, p_A(\epsilon_n))\ F^*.$$

F^*

(v) A is invertible if and only if $\gcd(p_A(X), X^n - 1) = 1$.

Proof(2) .By proposition 1.(1), $xA = 0$ has a non zero solution

Proof(3) By proposition1.(2) we have two different ways to write $(1.1.\cdots.1)$ as a linear combination of rows of A.

Theorem (4.4) [1]

The matrix $A = circ(\underbrace{1,1,\ldots,1}_{k},0,0,\ldots,0)$ f only if $(k,n) = 1$.

Proof. Clearly $p_A(1) = k \neq 0$. Suppose $(n, k) \neq 1$. Let $v \neq 1$ be an (n, k)th roof of unity. Then v is and nth root of unity and also a kth root of unity. Since $(1 - X)\, p_A(X) = x^k - 1$, then $pA(v) = 0$. Therefore A is not invertible.

If $(k, n) = 1$ then there exist $a, b \in Z$ such that $ak - bn = 1$. Let $a = (1,1,\ldots,1,0,\ldots,0)$ be the first row of A. Notice that

$a + T^k(a) + \ldots + T^{(a-1)k}(a) = (b+1, b, b, \ldots, b)$.

But then $(1,0,\ldots,0) = (b+1, b, b, \ldots, b) - b(1,1,\ldots)$ is in the row space of A, hence A is invertible.

In the case of $(n, k) = 1$ above we can write

$$(1.0.\cdots.0) = \sum_{j=1}^{a-1} T^{jk}\ (a) - \frac{b}{k}\sum_{i=1}^{n} T^{i-1}(a).$$

hence by **properties of circulant matrices (2.1.3)** we can find the explicit invers of A.

Theorem (4.5) [1]

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Let a = (1,0,...,0, -q,0,...,0) be the first row of a circulant matrix A. Let -q be on the k + 1th position. Then A = circ(a) is invertible if only if q is not $\frac{n}{(k.n)}$root of unity. In the case that A is invertible we have

$$A^{-1} = \frac{1}{1-q^{\frac{n}{d}}} circ\ \left(1.q.q^2 \cdots .q^{\frac{n}{d}}.0.\cdots.0\right) circ_k\ (e_1))$$

with circk $(e_1) = (e_1.T^k(e_1).T^{2k}(e_1).\cdots.T^{(n-1)k}(e_1))^t$

Proof. Let (k,n) = d. Define

$$b = a + \ qT^k\ (a) + q^2T^{2k}(a) + \cdots + q^{\frac{n}{d}-1}\,T^{\left(\frac{n}{d}-1\right)k}(a)$$

nonzero entry of b are 1 in the first coordinate dan $-q^{\frac{n}{d}}\, in\ the\ (k+1) + \left(\frac{n}{d} - 1\right) k = \frac{n}{d}\ k + 1 = 1th$ coordinate. Hence b = (1,0,...,0,– $q^{\frac{n}{d}}$,0,0,...,0). Therefore A invertible if and only if q is not an $\frac{n}{(k.n)}$th root of unity. If A is invertible then by using b we can write $e_1$as linear combination of rows of A. Hence we get the formula for A^{-1} as written above.

Corollary (4.5) [1]

Let $a.b \in R.k.i \in Z$ with $k < i$. Let $A = circ\ (\underbrace{a.\cdots.a}_{k}.\underbrace{0.\cdots.0}_{i-k}.\underbrace{b.\cdots.b}_{k}.0.\cdots.0)$

If (k, n) = 1 and $\left|\frac{b}{a}\right| \neq 1$ then A invertible.

Proof. Note that

$$pA^{(x)} = a\sum_{i=1}^{k} X^{i-1} + bX^{i+1}\sum_{j=1}^{k} X^{j-1}$$
$$= a\ (1 + \frac{b}{a}\ X^{i+1})(1 + X + \cdots + X^{k-1})$$

The polynomial 1+$\frac{b}{a}$ X^{i+1} is the representer polynomial of circ(1,0,...,0,$\frac{b}{a}$,0,0,...,0) and the polynomial 1+X+...+X^{k-1} is representer polynomial of the circ(1,...,1,0,...,0). Now the result follows from Theorem 1 and Theorem 2.

Theorem (4.6) [1]

Let $a = (\underbrace{1.0.\cdots.0}_{k}.\underbrace{-2_q.0.\cdots.0}_{k}.1.0.\cdots.0)$

Then A = circ(a) is invertible if only

$$if\ q\ \neq \cos\left(\frac{2\pi jk}{n}\right)\, for\ j = 0.1.\cdots.\frac{n}{(k.n)} - 1$$

Proof. We have

$$p_A(X) = 1 - 2qX^k + X^{2k}$$
$$= \left(1 - (q + i\sqrt{1-q^2})X^k\right)\left(1 - (q - i\sqrt{1-q^2})X^k\right).$$

The result follows.

5. The inverse of some circulant metrics

In what follows J will be the circulant matrix J = circ(1,1,...,1). We will consider several types of circulant matrices that can be written as A + cJ where A is an invertible circulant matrix and c is scalar.

Proposition (5.1) [3]

The matrix A = circ(0, 1, 2, ..., n − 1) is invertible and

$$A^{-1} = \frac{2}{n^2(n-1)}J - \frac{1}{n}circ(1,-1,0,\dots,0).$$

Proof Let $\mathbf{a} = (0,1,2,\dots,n-1)$. Then

$$-(\mathbf{a} - T(\mathbf{a})) = (n-1,-1,-1,\dots,-1)$$

$$\frac{2}{n(n-1)}\sum_{i=0}^{n-1} T^i(\mathbf{a}) - (\mathbf{a} - T(\mathbf{a})) = (n,0,0,\dots,0).$$

Therefore A invertible and

$$A^{-1} = \frac{2}{n^2(n-1)}J - \frac{1}{n}circ(1,-1,0,\dots,0).$$

Theorem (5.2) [3]

Let A be a circulant matrix of size n and c is scalar. If A is invertible and $s(A) + cn \neq 0$ then A +c J invertible and

$(A+cJ)^{-1} = A^{-1} - \frac{c}{s(A)(s(A)+cn)} J$

Proof. Since A is invertible then all eigen values of A are nonzero (including s(A). Then we can write A= F diag(s(A), λ_2,..., λ_n)F* where $\lambda_i \neq 0$. Now

$$\begin{aligned} A + cJ &= F\, diag(s(A),\lambda_2,\dots,\lambda_n)F^* + F\, diag(cn,0,\dots,0)F^* \\ &= F\, diag((s(A)+cn),\lambda_2,\dots,\lambda_n)F^*. \end{aligned}$$

Therefore $A + cJ$ invertible.

Write $(A+cJ)^{-1} = A^{-1} + X$. Then

$$\begin{aligned} (A+cJ)(A^{-1}+X) &= I \\ I + cJA^{-1} + (A+cJ)X &= I \\ (A+cJ)X &= -cJA^{-1} \\ X &= (A+cJ)^{-1}(-\frac{c}{s(A)}J) \\ X &= -\frac{c}{s(A)}\frac{1}{s(A)+cn}J. \end{aligned}$$

It follows that $(A+cJ)^{-1} = A^{-1} - \frac{c}{s(A)(s(A)+cn)}J$.

Theorem (2.2.3) [3]

Let a= (a, c,..., c, b, c,...,C) where a is in the first position and b is in the k+1th position while the rest are all c where a, b, c ∈R. If a − c≠0, s(A) ≠0 and q is not a$\frac{n}{(k.n)}$ th root of unity then A = circ(a) invertible and

$$A^{-1} = \frac{circ\left(\left(1, q, q^2, \ldots, q^{\frac{n}{(k,n)}-1}, 0, \ldots, 0\right) circ_k(e_1)\right)}{(a-c)(1-q^{\frac{n}{(k,n)}})} - \frac{c}{(a+b-2c)s(A)} J$$

for $q = -\frac{b-c}{a-c}$.

Theorem 5.4 [3]

Let $a, b, c \in \mathbb{R}$ *and* $\mathbf{a} = (a, \underbrace{c, \ldots, c}_{k-1}, b, \underbrace{c \ldots, c}_{k-1}, a, c, \ldots, c)$. *If* $a - c \neq 0, s(A) \neq 0$ *and* *q is not a* $\frac{n}{(k,n)}$*th root of unity then* $A = circ(\mathbf{a})$ *is invertible and*

$$A^{-1} = \frac{circ\,(1, 0 \ldots, 0, -2q, 0 \ldots, 0, 1, 0 \ldots, 0)^{-1}}{(a-c)} - \frac{c}{(2a+b-3c)(2a+b-(n-3)c)} J$$

for $q = -\frac{b-c}{2(a-c)}$.

Corollary 5.5 [3]

Let $A = circ(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{k}, c, \ldots, c)$ *where* $a, b, c \in \mathbb{R}$. *If* $(k, n) = 1$ *dan* $s(A) \neq 0$ *dan* $-\frac{a-c}{b-c}$ *is not an nth root of unity, then A invertible.*

Proof Write A as $A = (a-c)BC + cJ$ where

$$B = circ(1, \ldots, 1, 0, \ldots, 0) \text{ and}$$
$$C = circ\left(1, 0, \ldots, 0, -\left(-\frac{b-c}{a-c}\right), 0, \ldots, 0\right).$$

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