

Numerical Technique for Solving Systems of Linear Equations

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Abstract: The main objective of this research is to use the iterative method (Gauss-Siedl method) to solve systems of linear equations using an algorithm in the MATLAB language, where we obtained good results of accuracy.

Keywords— linear system of equations , iterative methods , numerical technique , Gauss-Siedl method.

1. INTRODUCTION

Iterative methods are techniques used in various fields of mathematics, science, and engineering to approximate solutions to problems by iteratively improving the initial estimate. These methods involve a step-by-step process where each iteration improves the approximation, usually converging toward the true solution. It is particularly useful when it is difficult or impossible to obtain direct analytical solutions.[5]

Iterative methods are commonly used to solve systems of linear equations and optimization problems and to find the roots of equations from these methods the Gauss-Seidel method is an iterative numerical technique used to solve systems of linear equations. It is named after German mathematicians Carl Friedrich Gauss and Philipp Ludwig von Seidel. This method is particularly useful when dealing with large systems of equations, where direct methods such as Gaussian elimination can be computationally expensive or memory intensive.

2.linear System of Equations [7]

We can write the general system that content m equations and n variables as follow:-

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \right\} \dots(1)$$

By using matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Now, we can rewrite (1) as follows :- $Ax = b$

3.Gauss-Seidel Method[7]

The Gauss-Seidel method is an iterative approach to solving systems of linear equations. It updates each component of the solution vector iteratively using current estimates of the other components. This iterative process continues until the solution converges to an acceptable accuracy.

Rewrite the linear system $Ax=b$ as

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)}}{a_{22}}$$

$$\vdots$$

$$x_n^{(k+1)} = \frac{b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \dots - a_{n,n-1}x_{n-1}^{(k+1)}}{a_{nn}}, \quad \text{Where } k=0, 1, \dots$$

Stop condition is $|x_i^{(k+1)} - x_i^{(k)}| < \epsilon$ for any k and all $i=1, 2, \dots, n$.

To solve above system, we need an approximate solution $x^{(0)} = [x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}]^T$

In general

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(k)} \right) \quad \dots(2)$$

where , $i = 1, 2, \dots, n$, $k = 1, 2, \dots$, $a_{ii} \neq 0$, $\forall i = 1, 2, \dots, n$

4. Convergence condition for the Gauss-Seidel method

Condition of convergence is

$$\max_{j \neq i} \sum_{j=1}^n \left| \frac{a_{ij}}{a_{ii}} \right| < 1, \quad i = 1, 2, \dots, n$$

5. Algorithm the Gauss-Seidel method

- Initialization: Start with an initial guess for the solution vector $x^{(0)} = [x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}]^T$.
- Iteration: For each equation i in the system, update the i th component of the solution vector using the current estimates for the other components. This updating is done iteratively until the solution converges.

For equation i the update formula is:
$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(k)} \right)$$

Where:

a_{ii} is the coefficient of the i th variable in the i th equation.

$x_j^{(k+1)}$ is the estimate of the j th variable at the $k+1$ th iteration.

$x_j^{(k)}$ is the estimate of the j th variable at the k th iteration.

b_i is the constant term in the i th equation.

n is the number of variables in the system.

- Convergence and Termination: The method iterates until the solution vector x converges within a specified tolerance or until a maximum number of iterations is reached. Convergence can be monitored by tracking the changes in the solution vector between iterations.

6. Advantages [5]

- 1- Can converge faster than other iterative methods like the Jacobi method, especially for systems with strong diagonal dominance.
- 2- Can be applied to both symmetric and non-symmetric systems of linear equations.
- 3- Memory usage is relatively low since it updates variables in-place.

7. Examples and results

Example (1):-

Solve the following system of linear equations using the Gauss-Seidel method

$$6x_1 - 3x_2 + x_3 = 14$$

$$2x_1 + 10x_2 - x_3 = 15$$

$$x_1 - 2x_2 - 9x_3 = -11$$

Solution To begin, write the system in the form

$$x_1^{k+1} = (1/6)(14 + 3x_2^k - x_3^k)$$

$$x_2^{k+1} = (1/10)(15 - 2x_1^{k+1} + x_3^k)$$

$$x_3^{k+1} = (-1/9)(-11 - x_1^{k+1} + 2x_2^{k+1})$$

Where $k=0, 1, \dots$, Initial approximation $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

When applying the Gauss-Seidel method to the above-mentioned system, we obtained the results shown in Table (1)

Table (1)			
k	x_1	x_2	x_3
0	0	0	0
1	2.333333	1.033333	1.251852
2	2.641358	1.096914	1.271948
3	2.669799	1.093235	1.275925
4	2.667297	1.094133	1.275448
5	2.667825	1.09398	1.275541
6	2.667733	1.094007	1.275524
7	2.66775	1.094002	1.275526
8	2.667747	1.094003	1.275527
9	2.667747	1.094003	1.275527

We note that the iterations in the three columns are identical, so $x_1 = 2.667747$, $x_2 = 1.094003$,

$x_3 = 1.275527$ when $k=8$ and $k=9$.

Example (2):

Solve the following system of linear equations using the Gauss-Seidel method

$$12x_1 + 4x_2 - x_3 + 3x_4 = 8$$

$$2x_1 + 10x_2 + x_3 - 3x_4 - 2x_5 = 25$$

$$-4x_1 + x_2 + 13x_3 - x_4 - 5x_5 = -14$$

$$x_2 + 2x_3 + 10x_4 - 5x_5 = 16$$

$$x_1 - 2x_2 - 3x_3 + x_4 + 9x_5 = -13$$

Solution To begin, write the system in the form

$$x_1^{k+1} = (1/12)(8 - 4x_2^k + x_3^k - 3x_4^k)$$

$$x_2^{k+1} = (1/10)(25 - 2x_1^{k+1} - x_3^k + 3x_4^k + 2x_5^k)$$

$$x_3^{k+1} = (1/13)(-14 + 4x_1^{k+1} - x_2^{k+1} + x_4^k + 5x_5^k)$$

$$x_4^{k+1} = (1/10)(16 - x_2^{k+1} + x_3^{k+1} + 5x_5^k)$$

$$x_5^{k+1} = (1/9)(-13 - x_1^{k+1} + 2x_2^{k+1} + 3x_3^{k+1} - x_4^{k+1})$$

Where $k=0, 1, \dots$, Initial approximation $x_1 = 0, x_2 = 0, x_3 = 0$, when applying the Gauss-Seidel method to the above-mentioned system, we obtained the results shown in Table (2).

k	x1	x2	x3	x4	x5
0	0	0	0	0	0
1	0.666667	2.366667	-1.05385	1.574103	-1.51877
2	-0.60357	2.894574	-1.94836	0.940826	-1.48813
3	-0.69576	2.818609	-2.00781	0.975635	-1.51845
4	-0.6841	2.8266	-2.01382	0.960877	-1.51834
5	-0.68357	2.822691	-2.01444	0.961451	-1.51954
6	-0.68246	2.822465	-2.0145	0.960886	-1.51967
7	-0.68225	2.822233	-2.01451	0.960846	-1.51974
8	-0.68217	2.82219	-2.01452	0.960814	-1.51976
9	-0.68214	2.822173	-2.01452	0.960807	-1.51976
10	-0.68214	2.822168	-2.01452	0.960805	-1.51976
11	-0.68213	2.822167	-2.01452	0.960804	-1.51977
12	-0.68213	2.822167	-2.01452	0.960804	-1.51977

We note that the iterations of values x_3 are identical when $k=8$ and $k=9$ while the iterations of values x_2, x_3, x_4 are identical when $k=11$ and $k=12$, so $x_1 = -0.68213, x_2 = 2.822167, x_3 = -2.01452, x_4 = 0.960804, x_5 = -1.51977$.

Example (3):

Solve the following system of linear equations using the Gauss-Seidel method

$$4x_1 + x_2 - x_3 = 3$$

$$x_1 + 6x_2 - 2x_3 + x_4 - x_5 = -6$$

$$x_2 + 5x_3 - x_5 + x_6 = -5$$

$$2x_2 + 5x_4 - x_5 = 0$$

$$-x_3 - x_4 + 6x_5 - x_6 = 12$$

$$-x_3 - x_5 + 5x_6 = -12$$

Solution To begin, write the system in the form

$$x_1^{k+1} = (0.25)(3 - x_2^k + x_3^k)$$

$$x_2^{k+1} = (1/6)(-6 - x_1^{k+1} + 2x_3^k - x_4^k + x_5^k)$$

$$x_3^{k+1} = (1/5)(-5 - x_2^{k+1} + x_5^k - x_6^k)$$

$$x_4^{k+1} = (1/5)(-2x_2^{k+1} + x_5^k)$$

$$x_5^{k+1} = (1/6)(12 + x_3^{k+1} + x_4^{k+1} + x_6^k)$$

$$x_6^{k+1} = (1/5)(-12 + x_3^{k+1} + x_5^{k+1})$$

Where $k=0, 1, \dots$, Initial approximation $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0$ when applying the Gauss-Seidel method to the above-mentioned system, we obtained the results shown in Table (3).

Table (3)						
k	x1	x2	x3	x4	x5	x6
0	0	0	0	0	0	0
1	0.75	-1.125	-0.775	0.225	1.908333	-2.17333
2	0.8375	-1.11736	0.039806	0.605139	1.745269	-2.04299
3	1.039292	-0.96993	-0.04836	0.543039	1.741948	-2.06128
4	0.98039	-0.9797	-0.04341	0.54433	1.739939	-2.06069
5	0.984072	-0.97921	-0.04403	0.543831	1.739851	-2.06078
6	0.983796	-0.97931	-0.044	0.543831	1.739832	-2.06082
7	0.983826	-0.9793	-0.04401	0.543827	1.739831	-2.06083
8	0.983825	-0.97931	-0.04401	0.543827	1.739831	-2.06083
9	0.983825	-0.97931	-0.04401	0.543827	1.739831	-2.06083

We note that the iterations of values x_1, x_2 are identical when $k=8$ and $k=9$ while the iterations of values x_3, x_4, x_5, x_6 are identical when $k=7$ and $k=8$, so $x_1 = 0.983825, x_2 = -0.97931, x_3 = -0.04401, x_4 = 0.543827, x_5 = 1.739831, x_6 = -2.06083$.

8. Conclusion

The Gauss-Seidel method is an iterative technique used to solve systems of linear equations. They are particularly useful when solving large systems of equations where direct methods such as Gaussian elimination may be computationally expensive or memory intensive. The method updates the solution vector variables recursively, including the current and updated values of the other variables. This iterative process continues until the solution converges within a specified tolerance. This method has diverse applications as its ability to iteratively solve linear systems makes it a valuable tool for various industries and fields in which equations are prevalent.

9. Applications of the Gauss-Seidel method

The Gauss-Seidel method, an iterative numerical technique, finds applications in various fields of science, engineering, and technology. Here are some examples of how the Gauss-Seidel method is used in real-life applications:

1. Engineering Analysis and Simulations:

- Structural Engineering: Gauss-Seidel can be used to analyze complex structures and solve equations related to stress distribution, load balancing, and material behavior.
- Electrical Engineering: It's applied to solve circuit analysis problems, especially in networks with resistors, capacitors, and inductors.
- Fluid Dynamics: Gauss-Seidel can help solve equations describing fluid flows, heat transfer, and turbulence in engineering simulations.

2. Physics and Scientific Research:

- Quantum Mechanics: The method is used in solving linear equations that arise in computational quantum mechanics simulations.
- Numerical Modeling: Gauss-Seidel helps solve partial differential equations that describe physical phenomena like diffusion, wave propagation, and electromagnetic fields.

3. Economics and Finance:

- Economic Modeling: Gauss-Seidel can be used in solving systems of linear equations that model economic scenarios, such as supply and demand analysis.
- Portfolio Optimization: In finance, it's applied to solve optimization problems related to portfolio diversification and risk management.

4. Environmental Studies:

Groundwater Flow Modeling: The method can be used to simulate the flow of groundwater in geological formations, helping with environmental impact assessments and resource management.

5. Image Processing and Computer Graphics:

- Image Deblurring: Gauss-Seidel can help restore images degraded by blurring, as it's used to solve inverse problems in image processing.
- Computer Graphics: In rendering algorithms, it can help solve equations related to lighting and shading.

6. Medical Imaging

Image Reconstruction: In medical imaging modalities like MRI and CT scans, Gauss-Seidel can be used in image reconstruction to improve image quality and reduce artifacts.

7. Machine Learning and Data Science:

Optimization Problems: Gauss-Seidel can be applied to solve certain optimization problems that arise in machine learning algorithms.

8. Numerical Methods and Software Development:

Numerical Libraries: Gauss-Seidel is implemented in numerical libraries and software packages used for scientific computing and simulations.

These are just a few examples of the diverse applications of the Gauss-Seidel method. Its ability to solve linear systems iteratively makes it a valuable tool for various industries and fields where equations are prevalent. However, it's important to consider the convergence properties, accuracy requirements, and specific characteristics of the problem when applying the method to real-world scenarios.

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