International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 1 January - 2024, Pages: 48-52

About Existence solution of functional quadratic integral equation in Banach space

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Abstract— *In this article, we will study existence solution for functional quadratic integral equation, after apply the Leray-Schauder alternative fixed point theorem, we considered the necessary sufficient conditions for the existence of solutions for equation, the solution was considered only in the space. In this way we will also show that the examples has solution.*

Keywords— Functional quadratic integral equation, Leray-Schauder alternative theorem, Cauchy sequence, Banach space, complete space.

1. Introduction

Integral equations are a crucial and essential element of mathematical analysis and its practical applications (see [6], [7],[13], [19], [20] among other works). The study of integral equations has greatly progressed with the aid of diverse tools such as functional analysis, topology, and fixed-point theory where the last estimate have held great significance in various fields, particularly in the realms of differential, integral, and functional equations. These theorems serve as a topological tool for investigating the solutions of both linear and nonlinear equations.

Nonlinear functions integral equations have long been discussed in detail in the literature and Various forms of functional integral equations are considered a special and prestigious branch of nonlinear analysis, and a variety of methods are sought to represent many practical and real-world problems (see [1], [4],[5], [14-18],[22]).

Our work is presented in the following manner: In part 2, we provide some notations, definitions, and important details. Moving on to Part 3 is dedicated to proving the existence of a solution using the powerful generalized Leray-Schauder alternative theorem on [0,1]. As a demonstration, we also provide two examples to further clarify our theorem. Lastly, in part 4, it is contain our conclusion.

2. Preliminaries and Notation

In this part, we will provide some theorems, definitions and auxiliary facts that will be necessary for the following portions.
The solution was considered only in the Banach space, so we denoted it as $\wp$ with the norm $(\|.\|)$ and zero element ϑ. the Leray-Schauder alternative theorem is as follows

Theorem (the Leray-Schauder alternative) 2.1. [10] Suppose the set $\mathcal{S}$ is open subset of a nonempty convex set $\mathcal{C}$ in a Banach space $\wp$, let the element zero belong to $\mathcal{S}$ and the mapping $\mathcal{T}: \bar{\mathcal{S}} \to \mathcal{C}$ is continuous and compact. Then either

1- $\mathcal{T}$ has at least one fixed point in $\mathcal{S}$, or
2- There exist a belong to the interval (0,1) and $\kappa \in \gamma\mathcal{S}$ where $a\mathcal{T}\kappa$, such that $\gamma\mathcal{S}$ is boundary of K.

Leray-Schauder alternative is theorem of the most important theorem of nonlinear functional analysis, proved by topological degrees in 1934. Nowadays, some types of Leray-Schauder type alternatives have been proven using various techniques (see [2],[9],[10],[11],[21]). Another important fixed point theorem often used in the theory of nonlinear differential and integral equations is the following generalization of Banach's contraction map principle, which was proved in Browder [8].

Definition 2.1. [12] A sequence of points $\{\psi\}_{i=1}^{\infty} \subset \mathbf{\Psi}$ in normed vector space $(\mathbf{\Psi}, \|.\|)$ is a Cauchy sequence if and only if any positive number ε such that there is $\aleph > \mathbb{N}$ and $m, n > \aleph$ indicate to

$$\|\psi_m - \psi_n\| < \varepsilon$$

Definition 2.2. [12] Let $(\mathbf{\Psi}, \|.\|)$ is anormed vector space, it is so called a Banach space if $(\Psi, \|.\|)$ is complete, this means that when a sequence is Cauchy and convergent with reference to the norm $(\|.\|)$.

3. Main results

We studied a type of problems referred to as quadratic integral equations for this paper. The subject of this paper focuses on functional quadratic integral equations, this is a type of equation of the following form

$$\mathcal{G}\psi(\tau) = \alpha(\tau, \beta\left(\tau, \varsigma, \psi\big(u(\varsigma)\big)\right) \int_0^1 \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma) \qquad \tau \in [0,1] \qquad (\circledast)$$

Such that $\tau \in I$ and $u, v: I \to I$. we will analyze the equation $(\circledast)$ based on the following condition, which are listed as follows:

1. The function $\alpha: I \times \mathcal{R} \to \mathcal{R}$ is continuous and fulfilled the Lipschitz condition i.e there are $\phi \geq 0$ such that:
$$|\alpha(\tau, \mathcal{g}) - \alpha(\tau, \mathcal{h})| \leq \phi|\mathcal{g} - \mathcal{h}|$$
hold for all $\mathcal{g}, \mathcal{h} \in \mathcal{R}$ and $\tau \in I$.
2. The function $\beta: I \times I \times \mathcal{R} \to \mathcal{R}$ is continuous and $\mathcal{H}_\beta: [0,1] \to \mathcal{R}$ is nondecreasing continuous function such that
$$|\beta(\tau, \varsigma, \psi)| \leq \mathcal{H}_\beta(\varsigma)(|\psi|)$$
for each $\psi \in \mathcal{R}$ and $\tau \in I$.
3. The function $\gamma: I \times I \times \mathcal{R}^2 \to \mathcal{R}$ is continuous and $\mathcal{F}: [0,1]^2 \to \mathcal{R}$ is nondecreasing continuous function, also $\mathcal{H}_\gamma: [0,1] \to \mathcal{R}$ is nondecreasing continuous function such that
$$|\gamma(\tau, \varsigma, \psi, \psi_*)| \leq \mathcal{H}_\gamma(\varsigma)\mathcal{F}(|\psi|, |\psi_*|)$$
For all $\psi, \psi_* \in \mathcal{R}$ and $\tau \in I$.

Theorem 3.1. If the assumption 1-3 are fulfilled then the equation $(\circledast)$ has at least one solution on interval $[0,1]$.

Proof: Suppose the set $\mathcal{C}$ is define by

$$\mathcal{C} = \{\psi \in [0,1]: \|\psi\| = max|\psi| \leq \ell\}$$

This mean that it is nonempty, bounded, closed and convex which it has been proved in [3].

Now, let the set $\mathcal{S}$ define as the following

$$\mathcal{S} = \{\psi \in [0,1]: \|\psi\| = max|\psi| < \ell\}$$

We conclude that it is open subset of $\mathcal{C}$, which it is so called open convex set. Next step, we must prove that the mapping $\mathcal{G}: \bar{\mathcal{S}} \to \mathcal{C}$ is continuous and compact. So, let τ_1, τ_2 are arbitrary constant belong to the set $\bar{\mathcal{S}}$ such that $\tau_1 < \tau_2$ and $|\tau_{2-}\tau_1| \leq \varepsilon$, we obtain:

$$|\mathcal{G}\psi(\tau_2) - \mathcal{G}\psi(\tau_1)| \leq \Big|\alpha(\tau_2, \beta(\tau_2, \varsigma, \psi(u(\varsigma))) \textstyle\int_0^1 \gamma(\tau_2, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma) - \alpha(\tau_1, \beta(\tau_1, \varsigma, \psi(u(\varsigma))) \int_0^1 \gamma(\tau_1, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma)\Big|$$

$$\begin{aligned}|\mathcal{G}\psi(\tau_2) - \mathcal{G}\psi(\tau_1)| \leq \Big|&\alpha(\tau_2, \beta(\tau_2, \varsigma, \psi(u(\varsigma))) \textstyle\int_0^1 \gamma(\tau_2, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma) \\ &-\alpha(\tau_2, \beta(\tau_2, \varsigma, \psi(u(\varsigma))) \textstyle\int_0^1 \gamma(\tau_1, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma) \\ &+\alpha(\tau_2, \beta(\tau_2, \varsigma, \psi(u(\varsigma))) \textstyle\int_0^1 \gamma(\tau_1, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma) \\ &-\alpha(\tau_1, \beta(\tau_1, \varsigma, \psi(u(\varsigma))) \textstyle\int_0^1 \gamma(\tau_1, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma)\Big|\end{aligned}$$

$$\begin{aligned}|\mathcal{G}\psi(\tau_2) - \mathcal{G}\psi(\tau_1)| \leq \phi_1[\,|\beta(\tau_2, \varsigma, \psi(u(\varsigma)))| \int_0^1 |\gamma(\tau_2, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma))) - \gamma(\tau_1, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))|\, d\varsigma] \\ + \phi_2[|\beta(\tau_2, \varsigma, \psi(u(\varsigma))) - \beta(\tau_1, \varsigma, \psi(u(\varsigma)))| \textstyle\int_0^1 |\gamma(\tau_1, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))|\, d\varsigma]\end{aligned}$$

That implying

$$|\mathcal{G}\psi(\tau_2) - \mathcal{G}\psi(\tau_1)| \leq \phi_1\mathcal{H}_\beta(|\psi|)\omega(\gamma, \delta) + \phi_2\omega(\gamma, \beta)\mathcal{H}_\gamma(|\psi|, |\psi_*|)$$

This means that the mapping $\mathcal{G}$ is continuous.

We consider that $\mathcal{G}(\psi_m)$ is a Cauchy sequence, so let > 0 , since $\mathcal{G}$ is continues mapping then $|\mathcal{G}_n - \mathcal{G}| \to 0$ and there is some $\sigma \in \mathcal{N}$ where $|\mathcal{G}_\sigma - \mathcal{G}| \leq \frac{\varepsilon}{3\ell}$ moreover, if $\mathcal{G}_\sigma(\psi_m)$ is Cauchy sequence then there is some $\mathcal{N}^* > 0$ such that $|\mathcal{G}_\sigma(\psi_s) - \mathcal{G}_\sigma(\psi_t)| < \frac{\varepsilon}{3}$ where $s, t > \mathcal{N}^*$ we get

$$\begin{aligned}|\mathcal{G}(\psi_s) - \mathcal{G}(\psi_t)| &= |\mathcal{G}(\psi_s) - \mathcal{G}_\sigma(\psi_s) + \mathcal{G}_\sigma(\psi_s) - \mathcal{G}_\sigma(\psi_t) + \mathcal{G}_\sigma(\psi_t) - \mathcal{G}(\psi_t)| \\ &\leq |\mathcal{G}(\psi_s) - \mathcal{G}_\sigma(\psi_s)| + |\mathcal{G}_\sigma(\psi_s) - \mathcal{G}_\sigma(\psi_t)| + |\mathcal{G}_\sigma(\psi_t) - \mathcal{G}(\psi_t)| \\ &< |\mathcal{G} - \mathcal{G}_\sigma||\psi_s| + \frac{\varepsilon}{3} + |\mathcal{G}_\sigma - \mathcal{G}||\psi_t| \\ &< |\mathcal{G} - \mathcal{G}_\sigma|\ell + \frac{\varepsilon}{3} + |\mathcal{G}_\sigma - \mathcal{G}|\ell \\ &< \frac{\varepsilon}{3\ell}\ell + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\ell}\ell = \varepsilon\end{aligned}$$

The last estimate is proved that the sequence $\mathcal{G}(\psi_m)$ is Cauchy. Since the convex set $\mathcal{C}$ subset of a Banach space $\wp$ which it is complete space then the set $\mathcal{C}$ is also complete, so $\mathcal{G}(\psi_m)$ converges of under image $\mathcal{G}$ that imply to $\mathcal{G}$ is compact.

4. Examples

In this part, we extend two examples to further clarify our theorem.

Example 4.1. Suppose the functional quadratic differential equation

$$\psi(\tau) = \tau^2 + \frac{(\tau+\varsigma)}{2\psi(u)} \int_0^1 e^\tau \varsigma \,(\psi(\varsigma) + \psi_*(v))\, d\varsigma$$

Now, we set

a. $\tau^2 + \frac{(\tau+\varsigma)}{\psi} \int_0^1 e^\tau \varsigma \,(\psi + \psi_*)\, d\varsigma = \alpha(\tau, \beta\left(\tau, \varsigma, \psi\big(u(\varsigma)\big)\right) \int_0^1 \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma)$

For all $\psi, \psi_* \in \mathcal{R}$ and $\tau, \varsigma \in I$, so we have

$$\left|\tau^2 + \frac{(\tau+\varsigma)}{2\psi} \int_0^1 e^\tau \varsigma \,(\psi + \psi_*) d\varsigma - \tau^2 - \frac{(\tau+\varsigma)}{2\bar{\psi}} \int_0^1 e^\tau \varsigma \,(\bar{\psi} + \overline{\psi_*})\, d\varsigma\right| =$$

$$\leq \frac{1}{2}\left|\frac{(\tau+\varsigma)}{\psi} \int_0^1 e^\tau \varsigma \,(\psi + \psi_*) d\varsigma - \frac{(\tau+\varsigma)}{\bar{\psi}} \int_0^1 e^\tau \varsigma \,(\bar{\psi} + \overline{\psi_*})\, d\varsigma\right|$$

Then $\phi = \frac{1}{2}$ for all $\tau \in I$, means that that the assumption (1) is hold.

b. $\frac{(\tau+\varsigma)}{\psi} = \beta\left(\tau, \varsigma, \psi\big(u(\varsigma)\big)\right)$, so we have

$$\left|\frac{(\tau+\varsigma)}{\psi}\right| = |(\tau+\varsigma)| \left|\frac{1}{\psi}\right|$$

For each $\psi \in \mathcal{R}$ and $\tau \in I$, we note that $|(\tau+\varsigma)| = \mathcal{H}_\beta$, since $\mathcal{H}_\beta$ is nondecreasing function means that the assumption (2) is hold.

c. $\int_0^1 e^\tau \varsigma \,(\psi + \psi_*) d\varsigma = \int_0^1 \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma$, so we have

$$e^\tau \varsigma(\psi + \psi_*) = \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma))),$$

For all $\psi, \psi_* \in \mathcal{R}$ and $\tau \in I$, such that

$$|e^\tau \varsigma(\psi + \psi_*)| = |e^\tau \varsigma||\psi + \psi_*|,$$

We note that $|e^\tau \varsigma| = \mathcal{H}_\gamma$, since $\mathcal{H}_\gamma$ is nondecreasing function means that the assumption (3) is hold.

Example 4.2. Suppose the functional quadratic differential equation

$$\psi(\tau) = \ln(1+\tau^2) + \frac{2}{3}\left(\varsigma^{1-\tau}\psi^2(u)\right) \int_0^1 \frac{\psi(\varsigma)\psi_*(v)}{\tau+\varsigma}\, d\varsigma,$$

Now, we set

a. $\ln(1+\tau^2) + \frac{2}{3}\left(\varsigma^{1-\tau}\psi^2(u)\right) \int_0^1 \frac{\psi(\varsigma)\psi_*(v)}{\tau+\varsigma}\, d\varsigma = \alpha(\tau, \beta\left(\tau, \varsigma, \psi\big(u(\varsigma)\big)\right) \int_0^1 \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma)$

For all $\psi, \psi_* \in \mathcal{R}$ and $\tau, \varsigma \in I$, so we have

$$\left|\ln(1+\tau^2) + \frac{2}{3}\left(\varsigma^{1-\tau}\psi^2(u)\right) \int_0^1 \frac{\psi(\varsigma)\psi_*(v)}{\tau+\varsigma}\, d\varsigma - \ln(1+\tau^2) - \frac{2}{3}\left(\varsigma^{1-\tau}\bar{\psi}^2(u)\right) \int_0^1 \frac{\bar{\psi}(\varsigma)\overline{\psi_*}(v)}{\tau+\varsigma}\, d\varsigma\right| =$$

$$\leq \frac{2}{3}\left|\left(\varsigma^{1-\tau}\psi^2(u)\right) \int_0^1 \frac{\psi(\varsigma)\psi_*(v)}{\tau+\varsigma}\, d\varsigma - \left(\varsigma^{1-\tau}\bar{\psi}^2(u)\right) \int_0^1 \frac{\bar{\psi}(\varsigma)\overline{\psi_*}(v)}{\tau+\varsigma}\, d\varsigma\right|,$$

Then $\phi = \frac{2}{3}$ for all $\tau \in I$, means that that the assumption (1) is hold.

b. $(\varsigma^{1-\varsigma}\psi^2(u)) = \beta\left(\tau, \varsigma, \psi\big(u(\varsigma)\big)\right)$, so we have

$$|(\varsigma^{1-\tau}\psi^2(u))| = |\varsigma^{1-\tau}||\psi^2(u)|$$

For each $\psi \in \mathcal{R}, \in I$, we note that $|\varsigma^{1-\tau}| = \mathcal{H}_\beta$, since $\mathcal{H}_\beta$ is nondecreasing function means that the assumption (2) is hold.

c. $\int_0^1 \frac{\psi(\varsigma)\psi_*(v)}{\tau+\varsigma}\, d\varsigma = \int_0^1 \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma)))d\varsigma$, so we have

$$\frac{\psi(\varsigma)\psi_*(v)}{\tau+\varsigma} = \gamma(\tau, \varsigma, \psi(\varsigma), \psi_*(v(\varsigma))),$$

For all $\psi, \psi_* \in \mathcal{R}$ and $\tau \in I$, such that

$$\left|\frac{\psi\psi_*}{\tau+\varsigma}\right| = \left|\frac{1}{\tau+\varsigma}\right| |\psi\psi_*|,$$

We note that $|e^{\tau}\varsigma| = \mathcal{H}_{\gamma}$, since $\mathcal{H}_{\gamma}$ is nondecreasing function means that the assumption (3) is hold.

5. Conclusion

Through application of the generalized Leray-Schauder alternative theorem in the Banach space, we have successfully demonstrated the existence of the considered nonlinear functional integral equation on the interval $[0, 1]$. Additionally, our result is showcased with two examples to emphasize the level of accuracy solution achieved with this method.

6. Reference

[1] A. Deep, Deepmala, J. R. Roshan, Solvability for generalized nonlinear functional integral equations in Banach spaces with applications, J. Integral Equ. Appl., 33 (2021), 19–30.
[2] A. J. B. Potter, An elementary version of the Leray–Schauder theorem, J. London Math. Soc. 5 (1972), 414–416.
[3] A. M. Moter, The Existence of Maximal and Minimal Solution of Quadratic Integral Equation, (2017) 1973.
[4] B. Matani, J. R. Roshan, Multivariate generalized Meir-Keeler condensing operators and their applications to systems of integral equations, J. Fixed Point Theory Appl., 22 (2020), 1–28.
[5] B. Matani, J. R. Roshan, N. Hussain, An extension of Darbo's theorem via measure of noncompactness with its application in the solvability of a system of integral equations, Filomat, 33(2019), 6315–6334.
[6]. C. Corduneanu, Integral Equations and Applications, Cambridge Univ. Press, Cambridge, (1991).
[7]. D. O'Regan and M. Meehan, Existence Theory for Nonlinear Integral and Integrodifferential Equations, Kluwer Academic, Dordrecht, (1998).
[8] F. E. Browder, Nonlinear operators and nonlinear equations of evolutions in Banach spaces, Proc. Symp. pure Math. Amer. Math. Soc. Providence, Rhode Island 1976.
[9] H. Ben-El-Mechaiekh, S. Chebbi and M. Florenzano, A Leray–Schauder type theorem for approximable maps. A simple proof, Proc. Amer. Math. Soc. 126 (1998), 2345–2349.
[10] H. Ben-El-Mechaiekh and A. Idzik, A Leray–Schauder type theorem for approximable maps, Proc. Amer. Math. Soc. 122 (1994), 105–109.
[11] J. Dugundji and A. Granas, Fixed Point Theory, Warsaw, 1982.
[12] J. B. Cooper, Functional Analysis—Banach spaces, Johannes Kepler Universit.
[13] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, (1985).
[14]. L. N. Mishra, R. P. Agarwal, On existence theorems for some nonlinear functional-integral equations, Dyn. Syst. Appl., 25 (2016), 303–320.
[15]. L. N. Mishra, R. P. Agarwal, M. Sen, Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erd´elyi-Kober fractional integrals on the unbounded interval, Progr. Fract. Differ. Appl., 2 (2016), 153–168.
[16] L. N. Mishra, M. Sen, On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order, Appl. Math. Comput., 285 (2016), 174–183.
[17] L. N. Mishra, M. Sen, R. N. Mohapatra, On existence theorems for some generalized nonlinear functional-integral equations with applications, Filomat, 31 (2017), 2081–2091.
[18]. L. N. Mishra, H. M. Srivastava, M. Sen, Existence results for some nonlinear functional-integral equations in Banach algebra with applications, Int. J. Anal. Appl., 11 (2016), 1–10.
[19]. M.A. Krasnosel'skii, P.P. Zabrejko, J.I. Pustyl'nik and P.J. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leyden, (1976).
[20] P.P. Zabrejko, A.I. Koshelev, M.A. Krasnosel'skii, S.G. Mikhlin, L.S. Rakovschik and V.J. Stetsenko, *Integral Equations*, Noordhoff, Leyden, (1975).
[21] S. Park, Generalized Leray–Schauder principles for compact admissible multifuctions, Topol. Methods Nonlinear Anal. 5 (1995), 271–277
[22] V. K. Pathak, L. N. Mishra, Application of fixed point theorem to solvability for nonlinear fractional Hadamard functional integral equations, Mathematics, 10 (2022), 1–16.

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