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Inequalities for L_p spaces, for $0 < P < 1$

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Abstract: *Few papers introduced about finding the lower bound to the L_1 -norm. If this lower bound in terms of $log\ N$ for number N is called Little wood's conjecture. The must result in this subject up to now is a terms of bound of order $\sqrt{log\ N}$. Here we generalize the Little wood's conjecture to functions in $L_p$spaces for $0 < P < \infty$ and get lower bound in terms of $(log\ N)^{2p}$.*

Keywords: L_1- norm, lower bound, Log N, Trigonometric polynomials, Dyadic System.

1. Introduction

The problem of estimating the bounds for L_1- norm of the trigonometric polynomials has the form

$$"\boldsymbol{F(x) = a_1 e^{ik_1 x} + a_2 e^{ik_2 x} + \cdots + a_n e^{ik_n x}}"$$

where $0 < k_1 < k_2 < \cdots < k_n$ are integers with $|a_j| \geq 1$, and "$j = 1, 2, \ldots N$,It has rich history see for example [1]. The existance of this bound in terms of log N with coefficients equal to 1 is called Little wood's conjecture".

In [1] and [2] the authors have the best results that are intermes of $(logN)^2$.

In our work we improve the results in [1] and [2] and prove the following theorem.

Theorem 1-1

"We can find $C(P)$ such that

$$\|F\| > C(P) \log N/(\log\log N)^2.$$

In our work we use some notations such as c(p) is refere to absolute positive constant depending on p and variy from one step to another.

We use the L_p $[0,2\pi]$ spaces for $0 < p < \infty$, and defined as

$$L_p[0,2\pi] = \{ f: [0,2\pi] \to R: C\left(\int_0^{2\pi} |f|^p\right)^{\frac{1}{p}} < \infty\}.$$
$$= \|f\|_p$$

$|E|$ denoted for the measure of the set E. If Γ is a finite sequence, $|\Gamma|$ will denote the number of the terms Γ ".

Definitions 1.1.

L_P space is space of all functions g satisfies $\|g\|_p < \infty$, define by $\|g\|_p = \|g\|_{L_p(I)} = (\int_I |g(x)|^p\, dx)^{\frac{1}{p}}$ where $0 < P < \infty$

Definitions 1.2.

The function $\omega(\delta)$ is called the modulus of continuity of f where

$$\omega(\delta) = \omega(\delta, f)_p = \sup_{0<|h|<\delta 1} \|f(x+h) - f(x)\|_p.$$

and $f(x)$ be defined in a closed interval.

Definitions 1.3 [3] **.**

$S[f]$ at $x = x_0$ is the same as the Fourier series at $t = 0$ of the even function.

$\frac{1}{2}[f(x_0+t)+f(x_0-t)]$, and $S[\acute{f}]$ at $x = x_0$ is the series conjugate to the Fourier series at $t = 0$ of the odd function $\frac{1}{2}[f(x_0+t)-f(x_0-t)]$.

Definitions 1.4. [6]

If E is a sub set of $[0,2\pi]$ we mean by $|E|$ the measure of the set E, $\grave{E}$ is the complement of E in $[0,2\pi]$.

Definitions 1.5 **(Big –oh: O(.))** [4]

Suppose that $f(n)$ and $g(n)$ be functions such that

$$\exists\, k > 0, \exists\, n_0, \forall\, n > n_0, |f(n)| \le |\, k \cdot g(n)|$$

then $f(n) \in O(g(n))$ or with some abuse of notation $f(n) = Og(n)$.

Definitions 1.6 **(Small –oh: o(.))** [4]**.**

Suppose that $f(n)$ *and* $g(n)$ be functions such that

$$\forall\, k > 0, \exists\, n_0, \forall\, n > n_0, |f(n)| \le k \cdot |\, g(n)|$$

then $f(n) \in o(g(n))$ or with some abuse of notation $f(n) = og(n)$.

Definitions 1.7 (Characteristic function) [5]**.**

For non-empty set y , and A sub set of y. a characteristic function of A is a mapping $\chi_A : y \to \{0,1\}$ which is defined as:

$$\chi_A(y) = \begin{cases} 1 & if\ y \in A \\ 0 & if\ y \in A^c \end{cases}$$

Definitions 1.8 **(Young function)** [7]

Let $\phi(u), u \ge 0$, and $\psi(v), v \ge 0$, be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each other. Then for $a, b \ge 0$

we have the following inequality.

$$ab \le \phi(a) + \psi(b), \text{ where } \phi(x) = \int_0^x \phi du,\ \psi(y) = \int_0^y \psi dv.$$

Definitions 1.9 **(Dyadic system)** [8]

The standard dyadic system denoted by $\mathcal{T}^0$ and defined as the form . $\mathcal{T}^0 = \cup_{j\in Z} \mathcal{T}_j^{\,0}, \mathcal{T}_j^{\,0} = \{2^j([0,1)+k): k \in Z\}$. A general dyadic system may be defined as a collection $\mathcal{T} = \cup_{j\in Z} \mathcal{T}_{j,}$ where $\mathcal{T}_j = \mathcal{T}_j + x_j$ for some $x_j \in R$ and the partition $\mathcal{T}_j$ refines $\mathcal{T}_{j+1}$ for each $j \in Z$.

2. Auxiliary Lemmas

In this section we give some auxiliary lemma that we need in our research .

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Lemma 2.1 [3]

If $\sum A_n(x)$ is $S'[f]$ then

$$f(x+h)-f(x-h) \sim \sum B_n \sin nh .$$

Where $A_n(x) = a_n \cos nx + b_n \sin nx$

and

$$B_n(x) = a_n \sin nx - b_n \cos nx \qquad n > 0.$$

Lemma 2.2 [3] (Jensen's inequality).

The Jensen's inequality given of the form

$$\frac{1}{b-a}(\emptyset \int_a^b f(x)dx) \le \frac{1}{b-a}(\int_a^b \varphi(f(x)dx).$$

with a, b are real's and be a function $f: [a, b] \to R$ defined on closed interval.

Lemma 2.3 (Young's inequality) [7]

If $c \ge 0$ and $d \ge 0$ are non-negative real numbers and if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$

Then

$$cd \le \frac{c^p}{p} + \frac{d^q}{q}.$$

"The Equality above is true $\Leftrightarrow c^p = d^q$".

Lemma 2.4 [3].

Let $n_{k+1}/n_k \ge q > 1$ for all k and$\gamma^2 = \sum(a_k^p + b_k^p)$. is finite, So that

$$\sum(a_k \cos n_k x + b_k \sin n_k x)(n_{k+1}/n_k \ge q > 1 \text{ is an } S[f]$$

Then

$$A_{r,q}\{\sum(a_k^p + b_k^p)\}^{\frac{1}{p}} \le \left\{c(p)\int_0^{2\pi}|f|^p dx\right\}^{\frac{1}{p}} \le B_{r,q}\{\sum(a_k^p + b_k^p)\}^{\frac{1}{p}}$$

for every $p > 0$, where $A_{r,q}$ and $B_{r,q}$ depend on r and q only.

If $\gamma \le 1$, Then also

$\int_0^{2\pi} \exp \mu f^p dx \le c$, provided $\mu \le \mu_0(q)$, with c an absolute constant.

Lemma 2.5 [3].

(i) $(\int|f+g|^p)^{\frac{1}{p}} \le 2^{\frac{1}{p}-1}(\int|f|^p)^{\frac{1}{p}} + (\int|g|^p)^{\frac{1}{p}}$ where $0 < p < 1$.

(ii) $(\int|f+g|^p) \le \int|f|^p + \int|g|^p$. where $0 < p < 1$.

(iii) $(\int|f+g|^p)^{\frac{1}{p}} > (\int|f|^p)^{\frac{1}{p}} + (\int|g|^p)^{\frac{1}{p}}$ where $0 < p < 1$.

Lemma 2.6 [6]

$\sum_{s=n}^{2n}\left(\frac{1}{2^s}\right)\binom{s}{n} = 1.$

3.Main Results

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In this section we give our main results.

Theorem 3.1.

If $f \in L_p$, $0 < p < \infty$ then S$[f]$ converges absolutely for $p = \frac{1}{2}$ this is not necessary true.

Proof

By using the Lemma 2.1 implies that

$$|\sum B_n \sin nh - f(x)| \leq C|f(x+h) - f(x-h)|$$

For $f \in L_p$

$$\|\sum B_n \sin nh - f(x)\|_p \leq \omega(f,h)_p . \quad ■$$

Theorem 3.2.

If E is a measurable set in $[0,2\pi]$ satisfy $0 < |E| < 1$ and assume that

$$G(x) = 1 + a_1 e^{ix} + \cdots + a_k e^{ikx}$$

Then

$$\left[|E|^{-1}\left(\int_E |G|^p\right)\right]^{|E|} . \left[|E'|^{-1}\left(\int_{E'} |G|^p\right)\right]^{|E'|} \geq 1 \qquad (2.1)$$

We mean by E' the complement of E in the interval $[0,2\pi]$ and $0 < p < \infty$.

proof.

Let χ_E , $\chi_{E'}$ are characteristic function of E and E' respectively. By Lemma 2.2 we have

$$0 \leq \left(\int_0^{2\pi} log|G|^p\right) = |E|\left(\int_0^{2\pi}(\chi_E/|E|)\log|G|^p\right) + |E'|\left(\int_0^{2\pi}(\chi_{E'}/|E'|)\log|G|^p\right)$$

adding $|E|$ to above inequality we have

$$1 \leq \left[|E|\left(\int_0^{2\pi}(\chi_E/|E|)\log|G|^p\right) + |E'|\left(\int_0^{2\pi}(\chi_{E'}/|E'|)\log|G|^p\right)\right] + |E|$$

Then taking e^0 to both sides of the inequality above we have prove inequality (2.1) ■

Theorem 3.3.

Let $G(x) = a_0 + a_1 e^{ix} + \cdots + a_k e^{ikx} = g(x) + i\tilde{g}(x)$ and write $exp(imx)G(x) = g_m(x) + i\tilde{g}_m(x)$.

Then

$$lim\|g_m\|_p = (2|\pi)\|G\|_p, \text{as } m \to \infty \qquad (3.1)$$

Proof.

Suppose that $I_j = (a_j, b_j) = (2\pi j/m, 2\pi(j+1)/m)$, $j = 0,1,\ldots,m-1$. We take m so big that the $(g - \tilde{g})$ in $I_j < \varepsilon$ $for\ all\ \varepsilon > 0$.

This imply

$$g_m(x) = g(x)\cos mx - \tilde{g}(x)\sin mx.$$

differs from $g(a_j)\cos mx - \tilde{g}\left(a_j\right)\sin mx$ such that

$g_m(x) - g\left(a_j\right) = [g(x)cosmx - \tilde{g}\,(x)sinmx)] - \left[g\left(a_j\right)cosmx - \tilde{g}\left(a_j\right)sinmx)\right] < 2\varepsilon$

By above equals $\left|G(\,a_j)\right|\cos\left(mx + t_j\right)$ where t_j satisfy

$$\tan t_j = \tilde{g}\,(\,a_j)/g(\,a_j)$$

It follows

$$\|g_m\|_p = \left(\int_{I_j} |g_m|^p\right)^{\frac{1}{p}} = \left|G\left(a_j\right)\right|\left(\int_{I_j} \left|\cos\left(m_x + t_j\right)\right|^p dx\right)^{\frac{1}{p}} + \left|I_j\right|O\varepsilon$$

$$= (2|\pi)\left|G(\,a_j)\right|\,\left|I_j\right| + \left|I_j\right|\,O\varepsilon\ > 1$$

make a summation for j =0, 1, …,m-1 and suppose $m \to \infty$ and $\varepsilon \to 0$ we obtain (3.1) ■

Theorem 3.4.

Let $n_{j+1}/n_j > \lambda > 1$, for all j. If a_n, b_n are the coefficients of an $f \in L_p$, $0 < p < 1$, the series $\sum\left(a_{n_j}^p + b_{n_j}^p\right)$ converges. The result holds if merely $|f|(log^+|f|)^{\frac{1}{p}}$ is integrable.

Proof

Let $N > 0$, for a suitable sequence $\alpha_{1,}\beta_1 \dots \alpha_n, \beta_n$ with $\sum \alpha_j^p + \beta_j^p = 1$

we have

$$\left\{\sum_{j=1}^{N}(a_{n_j}^p + b_{n_j}^p)\right\}^{\frac{1}{p}} \leq c(p)\|f\,(g)\|_p \tag{4.1}$$

Where

$$g = \sum(\alpha_j\, cosn_j t + \beta_j \sin n_j t)$$

Using the same lines of the proof Theorem (8.20) in [3], we get Lemma

$$c(p)\sum\left(a_k^p + b_k^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{2\pi}\int_0^{2\pi}|f|^p\right)^{\frac{1}{p}} \leq c(p)B_{r,q}\left(\sum a_k^p + b_k^p\right)^{\frac{1}{p}}$$

for $0 < p < 1$

There are positive constant γ, δ depending on λ only such that

$$\int_0^{2\pi} e^{\gamma q^2}\, \mathrm{dx} \leq \delta$$

We have $\|e^{\gamma}\|_p \leq \delta.$

Let $\Phi(u) = e^{\gamma u^2} - \gamma u^2 - 1$. The functions Φ $and \acute{\Phi}$ vanish for $u = 0$, and $\acute{\Phi}$ is strictly increasing for $u \geq 0$. Its complementary function $\Psi\,(v)$, as is easily seen, is $O\left(v\, log^{\frac{1}{2}} v\right)$ $for\ v \to \infty$. In other words

$$\Psi(v) \leq A_\lambda v(log^+ v)^{\frac{1}{p}} + B_\lambda \text{ for } v \geq 0.$$

By Young's inequality now show that the last term in (4.1) is not greater than

$$c(p)\{\|\Phi\,(|g|) + \Psi(|f|)\|_p\} \le \left\|e^{\gamma q^p}\right\|_p + A_\lambda \,\||f|(log^+|f|)\|_p + 2B_\lambda\,,$$

which gives $\left\{\sum_{j=1}^{N} a_{n_j}^p + b_{n_j}^p\right\}^{\frac{1}{p}} \le A_\lambda \||f|(log^+|f|\|_p + \acute{A}_\lambda.$

With $\acute{A}_\lambda = 2\,B_\lambda + \delta$. The inequality holds if $N \to \infty$. ■

Corollary 3.1.1.

Let $g \in L_p$ and let $m_{1,}m_{2\ldots}$ be a positive integer sequence satisfies any termes can be written uncial as $b_1m_1 + \cdots b_nm_n$, for an integer n, where $b_j \in \{-1,0,1\}, j = 1,2, \ldots$., Then if g has the form

$$g(x) = a_0 + a_1e^{ix} + \cdots + a_ne^{inx} + \cdots \text{ we have}$$

$$\left\{\sum_{j=1}^{\infty} |a_{mj}|^p\right\}^{\frac{1}{p}} \le c\,\||g|(log^+|g|)\|_p + c$$

$$\|a_{mj}\|_p \le c\,\||g|(log^+|g|)\|_p + c$$

where $log^+\,a = \log a\; if\; a \ge 1$ and $log^+\,a = 0\; if\;\; 0 < a \le 1$. ■

Theorem 3.5.

Let $E \subset [0,2\pi]$ is a set and positive constants $b_n, b_{n+1}, \ldots b_{2n}$ such that

$|E| = 1 - \frac{1}{2^n}, b_n + b_{n+1} + \cdots + b_{2n} = 1$, and

$\|f_q\|_p \ge \sum_{s=n}^{2n} b_s\|f_s\|_p + (\frac{1}{2^{n-1}})\|f_{n-1}\|_p$ where $0 < p < \infty$ (5.1)

Proof

Let $g_r(x) = sgnf_r(x)$. We see g_r and the spectrum $g_rg_s \ldots g_t\,, r < s < \cdots < t$, contains only odd multiples of 2^{k_r} (5.2)

The spectrum of the function f consisting of odd multiples of 2^k if and only if f is $(2\pi/2^k)$ - periodic and $f\left(\frac{\pi}{2^k} + x\right) = -f(x)$.

Let $\varepsilon = \{\varepsilon_0, \ldots, \varepsilon_{2n}\}$ is sequence of $+1s$ and $-1s$, and let $\emptyset$ be a family of sequences which contain more +1 than -1.

The function $\chi\,(x)$ can be defined as

$$\chi(x) = \sum_{\varepsilon\in\Phi}\prod_{k=0}^{2n}(\frac{1+\varepsilon_kg_k}{2}).$$

Since $\|f\| = 0$, we have

$$\|f_q\|_p = 2\|f\chi\|_p = 2\|(f_0 + \cdots + f_{2n})\chi\|_p \qquad (5.3)$$

Then

$$\|f_q\|_p > \sum_{s=0}^{2n}\|f_s\chi\|_p \qquad \text{where } 0 < p < 1$$

We shall show that all norms

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$A_s = \|f_s\chi\|_p$ are positive and that

$2(A_n + A_{n+1} + \cdots + A_{2n})$ and $2A_{n-1} >$ the 1st and 2nd terms of the right hand side of (5.1), So we examine A_s for affixed s.

A_s is the sume of the norms $\left\| f_s \prod_{k=0}^{2n} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p$ for all ε in Φ.

Now, we abserve that

$$\left\| f_s \prod_{k=0}^{2n} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p = \left\| \prod_{k<s}^{n} \left(\frac{1+\varepsilon_k g_k}{2}\right) f_s \left(\frac{1+\varepsilon_k g_k}{2}\right) . \prod_{k>s}^{n} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p \quad (5.4)$$

By (5.2) the first factor in the last norm will be only a factor $1/_{2^s}$.

Hence

$$\left\| f_s \prod_{k=0}^{2n} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p = \left(\frac{1}{2^s}\right) \left\| f_s \left(\frac{1+\varepsilon_s g_s}{2}\right) \prod_{k>s} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p$$

$$= \left(\frac{1}{2^{s+1}}\right) \left\| \varepsilon_s f_s \prod_{k>s} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p \quad (5.5)$$

In the last equality.

Now ,let as contribution of the sequence ε to A_s which have a definite pattern for $k > s$ and this pattern contains m , $0 \le m \le \min(n, 2n-s)$ -1elements .Let as divide Φ into $n+1$ classes $\Phi_0, \Phi_1, \ldots \Phi_n$ as the number -1 elements. The ε with the given pattern for $k > s$ distributed among the Φ_j as follows:

(i) There are no termes if $j < m$.

(ii) There is one term in Φ_m such that $\varepsilon_s = 1$.

(iii) There are $\binom{s}{j-m-1}$ with $\varepsilon_s = -1$ and $\binom{s}{j-m}$ with $\varepsilon = 1$ in Φ_j, $j > m$, then the contribution of ε_s to A_s with the given pattern for $k > s$ is

$$\left\{\left(1/_{2^{s+1}}\right) \left\| f_s \prod_{k>s} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p \right\} \left\{1 + \left\{\binom{s}{1} - \binom{s}{0}\right\} + \cdots + \left\{\binom{s}{n-m} - \binom{s}{n-m-1}\right\}\right\} =$$

$$\left(1/_{2^{s+1}}\right) \binom{s}{n-m} \left\| f_s \prod_{k>s} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p \quad (5.6)$$

Where $\binom{s}{a} = 0 \; if \; a > s$. A_s will be obtained if we make a summation for the termes under ε with apttern having m element, equal to -1 for $k > s$ and we make a summation under m with $\mathbf{max(0, n-s) \le m \le min(n, 2n-s)}$. Thus it is evident that all $A_s > 0$.

We examine now the cases $s = n-1 \; and \; s \ge n$

$s = n-1$. since $\binom{s}{n-m} = 1, A_{n-1}$ will exceed

$$\left(1/_{2^n}\right) \left\| f_{n-1} \sum \prod_{k>n-1} \left(\frac{1+\varepsilon_k g_k}{2}\right) \right\|_p$$

where the summation exceed over all choices of +1 and -1 for the $\varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_{2n}$ except the choices $\varepsilon_n = \varepsilon_{n+1} = \cdots = \varepsilon_{2n} = -1$ and

$\varepsilon_n = \varepsilon_{n+1} = \cdots = \varepsilon_{2n} = 1$. The summation above equal to 1 for all x except these x where $f_n, f_{n+1}, \ldots, f_{2n} \; are \; all \ge 0$ are all ≤ 0.

The last set has measure

$$\int \prod_{k>n-1}\left(\frac{1-g_k}{2}\right) + \int \prod_{k>n-1}\left(\frac{1+g_k}{2}\right) = {}^{1}\!/_{2^n}$$

and so it's the complement of E has also measure equal (1- (${}^{1}\!/_{2^n}$))
This shows that $2A_{n-1}$ greater than the last term in the right hand side of (5.1). $s \geq n$, Then all choices of $\varepsilon_k, k \geq n$, are permissible and all the factors $\binom{s}{n-m}$ are not < $\binom{s}{n}$. As the intervals previous case we abtain

$$A_s \geq 2\left({}^{1}\!/_{2^{s+1}}\right)\binom{s}{n}\|f_s\|_p = \left({}^{1}\!/_{2^s}\right)\binom{s}{n}\|f_s\|_p.$$

Writing $B_s = \left({}^{1}\!/_{2^s}\right)\binom{s}{n}, s = n, n+1, \ldots, 2n$ by using Lemma (2.6) we see that the condition $b_n + b_{n+1} + \cdots + b_{2n} = 1$ is satisfied. The Theorem proof completed.■

4. Theorem 1.1.

There exists a positive constant satisfies

$$\|F\|_P \geq c(p)(\log N)(\log\log N)^2$$

In this section we give the prove Theorem 1.1

$$\Gamma = \{n_1, n_2, \ldots n_N\},\ \ F(x) = f(x) + i f(x) = \sum_{j=1}^{N} e^{(\iota n_j x)}$$

Proof.

We will prove this Theorem using induction on N, Let

$$\|H\|_P \geq C_0 (\log M)/(\log\log M)^2$$

For any H exponential sum satisfy $M < N$ is nonzero conditions, *c(p)* is a constant depends on p. We also mention that it ere to avoid trivial proof, assume that $N \to \infty$.
We will first replace the Γ by one of their translations. This clearly does not affect F and allows for profitability use Theorem 3.5. We are writing numbers $n_1, n_2, \ldots, n_N$dyadic system and substract their joint tail (if found). This amounts to a translation make each $n_j$positive. Suppose 2^{k_0} is the highest power of the value of 2 which divides all elements of the translated sequence. There are definitely odd and even multiples of 2^{k_0}. If the odd number exceeds the number of even ones , then we add $2^{k_0}$for all Γ elements. We can write Γ_0 for the odd multiplier of 2^{k_0}. Now consider the set of even multiples of 2^{k_0} and again subtract their joint tail from all elements of Γ . Suppose $2^{k_1}(k_1 > k_0)$ is the highest power of 2 which divides all even multiples of 2^{k_0} If there exists more odd than even multiples of 2^{k_1} (among even multiples of 2^{k_0} then we add2^{k_1} to all Γ elements. Obviously, the operations in the second step do not affect $|\Gamma_0|$and 2^{k_0} We write Γ_1 for the odd multiples of 2^{k_1} and continue in the same manner until the sequence Γ is exhausted. Thus we obtain the sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ of discrete subsets of Γ such that Γ_j contains only odd multiples of $2^{k_j}, j = 0,1,\ldots,r$,$|\Gamma_j| \leq |\Gamma_{j+1}| + \cdots + |\Gamma_r|$, $j < r$ and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_r$. No let of translate the sequence Γ by of 2^{k_r} then L_p norm of any term of Γ is different from $\pi/2$ times the $p-$norm of its real part $< 1/N$. This is true Theorem 3.2. Now by using Theorem 3.5 we have

$$\|F\|_P = (\pi/2)\|f\|_p + O(N)^{-1} \geq (\pi/2)\,(1/2^{n-1})\left\|f_q\right\|_p + O(N)^{-1} \quad for\ 0 < p < \infty \quad (1.1)$$

$$\geq \left(\frac{\pi}{2}\right)\left(\frac{1}{2^{n-1}}\right)\sum_{s=n}^{2n} b_s\|f_s\|_p + \left(\frac{\pi}{2}\right)\left(\frac{1}{2^{n-1}}\right)\|f_{n-1}\|_p + O(N)^{-1}$$

$$= \sum_{s=n}^{2n} b_s\|F_s\|_p + (1/2^{n-1})\,\|f_{n-1}\|_p + O(N)^{-1}$$

Where $F_s = \sum_{m\in\Gamma_s} e^{(imx)}$ and E depends on the choice of f_s, $s = 0,1, \ldots 2n$.

Consider the cases:

Case (1)

Exist $2n+1$, with n satisfying the inequalities

$\log log \log N < n \log 2 < 1 + \log \log \log N$, defined Γ_j with $|\Gamma_j| \geq N(\log N)^4$. So our induction hypothesis and $b_n + b_{n+1} + \cdots + b_{2n} = 1$ impels that

$$\sum_{s=n}^{2n} b_s \|F_s\|_p \geq C\left(\frac{\log N - 4\log\log N}{(\log\log N)^2}\right)\ (1 + O(1/\log N)) \qquad (1.2)$$

where C is a positive constant.

$$= \left[C\frac{\log N}{(\log\log N)^2} - \left(C\frac{(4\log\log N)}{(\log\log N)^2}\right)\ (1 + O(1/logN))\right]$$

$$\geq \left\{C\left[\left(\frac{logN}{(\log\log N)^2} - \left(\frac{4C_0}{\log\log N}\right)\right]\ (1 + O(1/logN))\right\} \qquad (1.3)$$

By subtract $\left[C\frac{1}{\log\log N}\right]$ in equality above we have

$$\sum_{s=n}^{2n} b_s \|F_s\|_p \geq C\left[\left(\frac{logN}{(\log\log N)^2}\right) - \left(\frac{5C}{(\log\log N)}\right)\right]$$

Where abroches N→ ∞.

Now, let as prove we can choose C such that$(1/2^{n-1})\|f_{n-1}\|_p > 5C/\log\log N$.

Consider the remark:

(1) The norm of the sum of the terms of F is upper bound for $\|f_{n-1}\|$ which multiplies by $2^{k_{n-1}}$. And the norm of this sum bounded by $\|F\|$.
(2) We can assume $\|F\|_p$ bounded above by $logN$. Which is trivial and there is nothing to prove.
(3) $(1/|E|)^{|E|}$ and $(1/|\acute{E}|)^{|\acute{E}|}$are bounded.

By using these remarks and applying Theorem(3.2) we abtain

$1 \leq C(\|f_{n-1}\|_p)^{|E|}(\|f_{n-1}\|_p)^{|\acute{E}|} \leq C(\|f_{n-1}\|_p)^{|E|}$, where in the last inequality we used(2) and the estimate

$|\acute{E}| = 1/2^n \leq 1/\log\log N$.

Then

$$\|f_{n-1}\|_p \geq C \qquad (1.4)$$

of (1.1),(1.2),(1.4) and the inequality$1/2^{n-1} > \log\log N$ yield

$$\|F\|_p \geq C\left(\frac{\log N}{(\log\log N)^2}\right) - \frac{5C}{\log\log N} + O(N)^{-1} + \frac{C}{2^{n-1}}$$

$$\geq C\left(\frac{logN}{(\log\log N)^2}\right)$$

For $N \to \infty$, The case (1) proof completed.

Case (2)

At most $4\ log\ log\ N\Gamma_j$ are such that $|\Gamma_j| \geq N/(log\ N)^4$.

Suppose that $\Gamma_{j_1}, \Gamma_{j_2}, \ldots, \Gamma_{j_k}$, $j_1 < j_2 < \cdots < j_k$, $k < 4\ log\ log\ log\ N$ be the Γ_j with more than $N/(\log N)^4$ elements. We prove in this case there are more than $(\log N)^3$ classesΓ_j.

Suppose $j_1, j_2 - j_1, \ldots, j_k - j_{k-1}$ are less than $(logN)^3$.

We write $\Gamma'_j = \Gamma_{j+1} \cup \ldots \cup \Gamma_r$, i.e the set of even multiples of 2^{k_j}.

Since $|\Gamma'_j| \geq |\Gamma_j|$ we get

$$|\Gamma_{j_1}| \geq \left(\frac{1}{2}\right)|\Gamma_{j_1} \cup \Gamma_{j_1}| \geq \frac{1}{2}\left\{N - \frac{N}{(logN)^4}(\log N)^3\right\}$$
$$= \left(\frac{1}{2}\right)\left\{N - \frac{N}{logN}\right\}.$$

Similarity we get $|\Gamma_{j_h}| \geq \left(\frac{1}{2^h}\right)\left\{N - \frac{N}{logN}\ (1 + 2 + \cdots + 2^{h-1}\right\}, 1 \leq h \leq\ k.$

assume $h = k$ in above we see that $|\Gamma_{j_k}|$ grater than

$$CN/(\log\log N)^4.$$

Since $|\Gamma_j| \leq N/(\log N)^4$ if $j \geq\ j_k$, the number of Γ_j with j $> j_k$ grater than

$$\frac{C(logN)^4}{(\log logN)^4} \geq\ (logN)^3.$$

On choosing now one frequency from each such class we abtain a subsequence of Γ satisfies the hypotheses of Theorem 3.3. Using Theorem 3.3 $log|F|\ \leq log\ N$ we have

$$(logN)^{\frac{1}{2}}\,\|F\|_P\ \geq C(\log N)^{\frac{3}{2}}.$$

provided than $N \to \infty$, We chosen $C_0 < C$ in above, then 1.1 satisfy.

From case 1 and 2 we complete the proof.

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