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Generalized Approximation Space and Topological Structures in Graph Theory

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Abstract : *This paper introduces the concept of a generalized approximation space in the context of graph theory and topology. Two relations are defined as topological structures derived from graphs: the first is the H_1-incidence composed (resp H_1-non incidence composed), which represents a supra topology, and the second is the H_2-incidence composed (resp H_2-non incidence composed), which represents a topology. The study establishes that the H_1 relation, as a supra topology, can be utilized with graphs to define the lower and upper sets in the generalized approximation space. Furthermore, it explores various relationships between these sets, contributing to a deeper understanding of their interactions within this framework.*

Keywords : graph , supra topology, lower and upper approximations.

1. INTRODUCTION :

Two factors make graph theory a significant and intriguing area of mathematics that is primarily utilized in discrete mathematics. In terms of mathematics, the graph is appealing. They can be used to depict a wide range of mathematical graphs, including topographic space and harmonic objects, even though they are simple relation graphs. The second reason is that when many concepts are empirically represented by graphs, they will be incredibly helpful in practice. The concepts of topological graph theory [1, 2, 3, 4, 5, 8, and 9] are a subfield of mathematics that have numerous applications in both theoretical and practical contexts. We predict that topological graph structure will play a key role in bridging the topology and applications divide. For all graph theory slang and notation, we refer to Harary [6] and all terminology and notation in topology , we refer to Moller [7].Some basic concepts of graph theory [10] are presented . A undirected graph or graph is pair $\mathfrak{G} = \left(\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G})\right)$ where $\mathbb{W}(\mathfrak{G})$ is a non-empty set whose elements are called points or vertices (called vertex set) and $\breve{E}(\mathfrak{G})$ is the set of unordered pairs of elements of $\mathbb{W}(\mathfrak{G})$ (called edge set). An edge of a graph that joins a vertex to itself is called a loop. A star graph Sn is an undirected graph consisting of one central vertex connected to peripheral vertices, with no edges between the peripheral vertices. It has vertices and edges, forming a tree. An antisymmetric graph, let $\mathfrak{G} = \left(\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G})\right)$ be a graph if $(\breve{\varphi}, \tilde{u}) \in \breve{E}(\mathfrak{G})$ and $(\tilde{u}, \breve{\varphi}) \in \breve{E}(\mathfrak{G})$ implies $\tilde{u} = \breve{\varphi}$ Then $\mathfrak{G}$ is called antisymmetric graph. The incidence vertex edges set of $\breve{\varphi}$ is denoted by $\mathrm{INVE}(\breve{\varphi})$ and defined by $:\mathrm{INVE}(\breve{\varphi}) = \{\ddot{\imath} \in \breve{E}(\mathfrak{G}) : \ddot{\imath} = (\breve{\varphi}, \partial)$ for some $\partial \in \mathbb{W}(\mathfrak{G})\}$. The non-incidence vertex edges set of $\breve{\varphi}$ is denoted by $\mathrm{NINVE}(\breve{\varphi})$ and defined by: $\mathrm{NINVE}(\breve{\varphi}) = \{\ddot{\imath} \in \breve{E}(\mathfrak{G}) : \ddot{\imath} = (\partial, \gamma)$ and $\partial, \gamma \neq \breve{\varphi}$ for som $\partial, \gamma \in \mathbb{W}(\mathfrak{G})\}$. The incidence vertex edges system (resp. non-incidence vertex edges system) of a vertex $\breve{\varphi} \in \mathbb{W}(\mathfrak{G})$ is denoted by $\mathrm{INVES}(\breve{\varphi})$ (resp. $\mathrm{NINVES}(\breve{\varphi})$) and defined by : $\mathrm{INVES}(\breve{\varphi}) = \{\mathrm{INVE}(\breve{\varphi})\}$ (resp.$\mathrm{NINVES}(\breve{\varphi}) = \{\mathrm{NINVE}(\breve{\varphi})\}$) .The admixture vertex edges system of a vertex $\breve{\varphi} \in \mathbb{W}(\mathfrak{G})$ is denoted by $\mathrm{AVES}(\breve{\varphi})$ and defined by: $\mathrm{AVES}(\breve{\varphi}) = \{\mathrm{INVES}(\breve{\varphi}), \mathrm{NINVES}(\breve{\varphi})\}$, such that such that $\mathrm{AVE}(\breve{\varphi}) \in \mathrm{AVES}(\breve{\varphi})$. The A-space is the pair $(\mathfrak{G}, \mathfrak{h}_a)$ such that $\mathfrak{G}$ is a graph and $\mathfrak{h}_a : \mathbb{W}(\mathfrak{G}) \to P\left(P\left(\breve{E}(\mathfrak{G})\right)\right)$ is a mapping which assigns for each $\breve{\varphi}$ in $\mathbb{W}(\mathfrak{G})$ its admixture vertex edges system in $P\left(P\left(\breve{E}(\mathfrak{G})\right)\right)$. The a-derived graph of an sub graph H is denoted by $\left[\breve{E}(H)\right]_a^{`}$ , and defined by: $\left[\breve{E}(H)\right]_a^{`} = \left\{\begin{matrix} \ddot{\imath} \in \breve{E}(\mathfrak{G}); \forall \mathrm{AVE}(\breve{\varphi}) \text{ where } \breve{\varphi} \text{ incidence on } \ddot{\imath} \\ , \mathrm{AVE}(\breve{\varphi}) \cap \left(\breve{E}(H) - \{\ddot{\imath}\}\right) \neq \phi \end{matrix}\right\}$.

The family of a-closed of an sub graph H in a-space is defined by : $\dot{F}_{\mathfrak{h}_a} = \left\{\breve{E}(H) \subseteq \breve{E}(\mathfrak{G});\ [\breve{E}(H)]_a^{`} \subseteq \breve{E}(H)\right\}$.

The family of a-open of an sub graph H in a-space is defined by :

$\mho_{\mathfrak{h}_a} = \{\breve{E}(S) \subseteq \breve{E}(\mathfrak{G}); \breve{E}(S) = \breve{E}(\mathfrak{G}) - \breve{E}(H)$ such that $\breve{E}(H) \in \dot{F}_{\mathfrak{h}_a}\}$. Then $\mho_{\mathfrak{h}_a}$ represents a supra topology on $\breve{E}(\mathfrak{G})$ because (1)$\emptyset$, $\breve{E}(\mathfrak{G}) \in \mho_{\mathfrak{h}_a}$.

(2) If $\{S_i : i \in I\} \in \mho_{\mathfrak{h}_a}$ then $\cup_i S_i \in \mho_{\mathfrak{h}_a}$. An approximation space is a structure $(\mathfrak{G}, T)$, where $\mathfrak{G}$ is a non-empty set called the universe, and T is a binary relation on $\mathfrak{G}$. It is used in rough set theory to define lower and upper approximations of subsets of $\mathfrak{G}$, based on the indistinguishability or similarity defined by T. A Generalized Approximation Space is a structure (G,T), where G is a universal set, and T is a generalized structure such as a relation supra topology used to define lower and upper approximations of subsets of G , enabling flexible and broad applications beyond traditional equivalence based approximation spaces.

2. A-Lower and A-Upper Approximations

In this section we introduce the supra topological spaces i-space and n-space. The unguarded. We defined i-interior, n-interior, i-closure, and n-closure. Lastly, using i-interior (resp. n-interior and a-interior), to define the a-lower and a-upper approximations in generalized rough set theory and look into some of its aspects.

Definition 2.1: Let $\mathfrak{E} = \left(\mathbb{W}(\mathfrak{E}), \breve{E}(\mathfrak{E})\right)$ be a graph and suppose that $\mathfrak{h}_i : \mathbb{W}(\mathfrak{E}) \longrightarrow P\left(P\left(\breve{E}(\mathfrak{E})\right)\right)$ (resp. $\mathfrak{h}_n : \mathbb{W}(\mathfrak{E}) \longrightarrow P\left(P\left(\breve{E}(\mathfrak{E})\right)\right)$ is a mapping which assigns for each $\ddot{\wp}$ in $\mathbb{W}(\mathfrak{E})$ it's incidence (resp. non incidence) vertex edges system in $P\left(P\left(\breve{E}(\mathfrak{E})\right)\right)$. The pair $(\mathfrak{E}, \mathfrak{h}_i)$ (resp. $(\mathfrak{E}, \mathfrak{h}_n)$) is called an i-space (resp. n-space).

Definition 2.2: Let $(\mathfrak{E}, \mathfrak{h}_i)$ be an i-space and $(\mathfrak{E}, \mathfrak{h}_n)$ be an n-space and let $H \subseteq \mathfrak{E}$. Then :

(1) The i-derived and n-derived of an sub graph H are defined respectively by :

$$[\breve{E}(H)]_i^{\grave{}} = \left\{ \begin{matrix} \ddot{\iota} \in \breve{E}(\mathfrak{E}); \text{INVE}(\ddot{\wp}) \cap \left(\breve{E}(H) - \{\ddot{\iota}\}\right) \neq \phi, \\ \text{where } \ddot{\wp} \text{ incidence on } \ddot{\iota} \end{matrix} \right\}.$$

$$[\breve{E}(H)]_n^{\grave{}} = \left\{ \begin{matrix} \ddot{\iota} \in \breve{E}(\mathfrak{E}); \text{NINVE}(\ddot{\wp}) \cap \left(\breve{E}(H) - \{\ddot{\iota}\}\right) \neq \phi, \\ \text{where } \ddot{\wp} \text{ incidence on } \ddot{\iota} \end{matrix} \right\}.$$

(2) The families of i-closed and n-closed of an sub graph H in i-space and n-space are defined respectively by :

$$\dot{F}_{\mathfrak{h}_i} = \left\{ \breve{E}(H) \subseteq \breve{E}(\mathfrak{E});\ \left[\breve{E}(H)\right]_i^{\grave{}} \subseteq \breve{E}(H) \right\}.$$

$$\dot{F}_{\mathfrak{h}_n} = \left\{ \breve{E}(H) \subseteq \breve{E}(\mathfrak{E});\ \left[\breve{E}(H)\right]_n^{\grave{}} \subseteq \breve{E}(H) \right\}.$$

(3) The families of i-open and n-open of an sub graph H in i-space and n-space are defined respectively by :

$$\mho_{\mathfrak{h}_i} = \left\{ \begin{matrix} \breve{E}(S) \subseteq \breve{E}(\mathfrak{E}); \breve{E}(S) = \breve{E}(\mathfrak{E}) - \breve{E}(H) \\ \text{such that } \breve{E}(H) \in \dot{F}_{\mathfrak{h}_i} \end{matrix} \right\}.$$

$$\mho_{\mathfrak{h}_n} = \left\{ \begin{matrix} \breve{E}(S) \subseteq \breve{E}(\mathfrak{E}); \breve{E}(S) = \breve{E}(\mathfrak{E}) - \breve{E}(H) \\ \text{such that } \breve{E}(H) \in \dot{F}_{\mathfrak{h}_n} \end{matrix} \right\}.$$

(4) The i-interior and n-interior of an sub graph H are defined respectively by :

$$\text{Int}_i\left(\breve{E}(H)\right) = \cup \left\{ \breve{E}(S) \in \mho_{\mathfrak{h}_i}; \breve{E}(S) \subseteq \breve{E}(H) \right\}.$$

$$\text{Int}_n\left(\breve{E}(H)\right) = \cup \left\{ \breve{E}(S) \in \mho_{\mathfrak{h}_n}; \breve{E}(S) \subseteq \breve{E}(H) \right\}.$$

(5) The i-closure and n-closure of an sub graph H are defined respectively by :

$$\text{Cl}_i\left(\breve{E}(H)\right) = \cap \left\{ \breve{E}(K) \in \dot{F}_{\mathfrak{h}_i}; \breve{E}(H) \subseteq \breve{E}(K) \right\}.$$

$$\text{Cl}_n\left(\breve{E}(H)\right) = \cap \left\{ \breve{E}(K) \in \dot{F}_{\mathfrak{h}_n}; \breve{E}(H) \subseteq \breve{E}(K) \right\}.$$

(6) The i-boundary and n-boundary of an sub graph H are defined respectively by :

$$[\breve{E}(H)]_i^b = \text{Cl}_i\left(\breve{E}(H)\right) - \text{Int}_i\left(\breve{E}(H)\right).$$

$$[\breve{E}(H)]_n^b = \text{Cl}_n\left(\breve{E}(H)\right) - \text{Int}_n\left(\breve{E}(H)\right).$$

Theorem 2.3: Let $(\mathfrak{E}, \mathfrak{h}_i)$ (resp. $(\mathfrak{E}, \mathfrak{h}_n)$) be is an i-space (resp. n-space) and $H \subseteq \mathfrak{E}$, then H is an i-open (resp. n-open) if and only if it contains at least one incidence vertex edges (resp. non incidence vertex edges) of $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ for each $\ddot{\iota} \in \breve{E}(H)$.

Proof : Let $(\mathfrak{E}, \mathfrak{h}_i)$ be an i-space and H be an i-open graph contained in $\mathfrak{E}$ and $\ddot{\iota} \in \breve{E}(H)$. Suppose that for each incidence vertex edges of $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ such that $\text{INVE}(\ddot{\wp}) \nsubseteq \breve{E}(H)$ where $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ for each $\ddot{\iota} \in \breve{E}(H) \implies \text{INVE}(\ddot{\wp}) \cap \left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right] \neq \emptyset$ where $\ddot{\wp}$ incidence on $\ddot{\iota} \in \breve{E}(H) \implies \ddot{\iota} \in \left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]_i^{\grave{}}$ since $\breve{E}(H)$ is i-open $\therefore \breve{E}(\mathfrak{E}) - \breve{E}(H)$ is i-closed, from definition (2.2(2)) we obtain $\left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]_i^{\grave{}} \subseteq \left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]$ thus $\ddot{\iota} \in \left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]$. Therefor $\ddot{\iota} \notin \breve{E}(H)$ which contradicts with $\ddot{\iota} \in \breve{E}(H)$ and consequently if $H \subseteq \mathfrak{E}$ is i-open and $\ddot{\iota} \in \breve{E}(H)$, then at least one incidence vertex edges of $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ for each $\ddot{\iota} \in \breve{E}(H)$ which is contained in $\breve{E}(H)$.

Conversely, let $\ddot{\iota} \in \breve{E}(H)$ and H contains at least one incidence vertex edges of $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ for each $\ddot{\iota} \in \breve{E}(H)$ i.e (for each $\ddot{\iota} \in \breve{E}(H)\ \exists \ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ s.t $\text{INVE}(\ddot{\wp}) \subseteq \breve{E}(H)$). Let $\varrho \in \left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]_i^{\grave{}}$ by definition ((3.2) (1)) $\implies \text{INVE}(u) \cap \left[\left(\breve{E}(\mathfrak{E}) - \breve{E}(H)\right) - \{\varrho\}\right] \neq \emptyset$ where u incidence on ϱ then $\varrho \notin \breve{E}(H)$. If $\varrho \in \breve{E}(H)$ there would be an incidence vertex edges of u, such that $\text{INVE}(u) \subseteq \breve{E}(H)$ and this would imply that : $\text{INVE}(u) \cap \left[\left(\breve{E}(\mathfrak{E}) - \breve{E}(H)\right) - \{\varrho\}\right] = \emptyset \therefore \varrho \notin \breve{E}(H) \implies \varrho \in \breve{E}(\mathfrak{E}) - \breve{E}(H)$, thus $\left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]_a^{\grave{}} \subseteq \left[\breve{E}(\mathfrak{E}) - \breve{E}(H)\right]$. Hence $\breve{E}(\mathfrak{E}) - \breve{E}(H)$ is i-closed and $\breve{E}(H)$ is i-open . Similarly, we can prove that H is n-open if and only if it contains the non incidence vertex edges of $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ for each $\ddot{\iota} \in \breve{E}(H)$.

Definition 2.4: Let $ß = \left(\mathbb{W}(\mathfrak{E}), \breve{E}(\mathfrak{E})\right)$ be a generalized approximation space and $H \subseteq \mathfrak{E}$. Then is called $H_1 -$ incidence composed (resp. $H_1 -$ non incidence composed) if H contains at least one incidence vertex edges (resp. non incidence vertex edges) of $\ddot{\wp} \in V(H)$ incidence on $\ddot{\iota}$ for each $\ddot{\iota} \in \breve{E}(H)$.

Definition 2.5: Let $ß = (\mathbb{W}(\mathfrak{E}), \breve{E}(\mathfrak{E}))$ be a generalized approximation space, then the families of all $H_1 -$ incidence composed (resp. $H_1 -$ non incidence composed) graph are denoted by $\mathcal{T}_i$ (resp. $\mathcal{T}_n$) and defined by:

$$\mathcal{T}_i = \left\{ \begin{matrix} H \subseteq \mathfrak{E};\ \forall \ddot{\iota} \in \breve{E}(H)\ \exists \ddot{\wp} \in V(H) \text{ incidence} \\ \text{on } \ddot{\iota} \text{ such that } \text{INVE}(\ddot{\wp}) \subseteq \breve{E}(H) \end{matrix} \right\}$$

(resp. $\mathcal{T}_n = \left\{ \begin{matrix} H \subseteq \mathfrak{G};\ \forall \mathfrak{i} \in \breve{E}(H)\ \exists\ \breve{\varphi} \in V(H) \text{ incidence} \\ \text{on } \mathfrak{i} \text{ such that NINVE}(\breve{\varphi}) \subseteq \breve{E}(H) \end{matrix} \right\}$).

Theorem 2.6: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space, then $\mathcal{T}_i$ (resp. $\mathcal{T}_n$) forms a supra topology on $\breve{E}(\mathfrak{G})$.

Proof : Now to prove that $\mathcal{T}_i$ is supra topology on $\breve{E}(\mathfrak{G})$: (1) $\breve{E}(\mathfrak{G}), \emptyset \in \mathcal{T}_i$.

(2) Let $\breve{E}(H_i) \in \mathcal{T}_i\ \forall\ i \in I$. let $\mathfrak{i} \in \cup_i \breve{E}(H_i)$ imply that $\exists\ i_o \in I$ such that $\mathfrak{i} \in \breve{E}(H_{i_o}) \subseteq \cup_i \breve{E}(H_i)$, since $\breve{E}(H_{i_o}) \in \mathcal{T}_i \exists\ \breve{\varphi} \in V(H)$ incidence on $\mathfrak{i}$ such that $\text{INVE}(\breve{\varphi}) \subseteq \breve{E}(H_{i_o})$ hence $\text{INVE}(\breve{\varphi}) \subseteq \cup_i \breve{E}(H_i) \implies \cup_i \breve{E}(H_i) \in \mathcal{T}_i\ \ \forall i \in I$. We prove with the same technique $\mathcal{T}_n$ is supra topology on $\breve{E}(\mathfrak{G})$.

Theorem 2.7: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space, if $\mathfrak{G}$ is a star graph then $\mathcal{T}_i = P\left(\breve{E}(\mathfrak{G})\right)$ (resp. $\mathcal{T}_n = P\left(\breve{E}(\mathfrak{G})\right)$) .**Proof :** Let $\mathfrak{G}$ be a star graph with center vertex $\breve{\varphi}_c$ and leaves $\{\breve{\varphi}_1, \breve{\varphi}_2, \dots \dots, \breve{\varphi}_n\}$ the edges set is given by :

$$\breve{E}(\mathfrak{G}) = \{\mathfrak{i}_1 = (\breve{\varphi}_c, \breve{\varphi}_1), \mathfrak{i}_2 = (\breve{\varphi}_c, \breve{\varphi}_2), \dots, \mathfrak{i}_n = (\breve{\varphi}_c, \breve{\varphi}_n)\}.$$

For any $H \subseteq \mathfrak{G}$ let $\mathfrak{i}_m = (\breve{\varphi}_c, \breve{\varphi}_m) \in H$ then $\text{INVE}(\breve{\varphi}_m) = \{\mathfrak{i}_m\} \subseteq H$ thus $H \in \mathcal{T}_i$ so $\mathcal{T}_i = P\left(\breve{E}(\mathfrak{G})\right)$. Now to proof $\mathcal{T}_n = P\left(\breve{E}(\mathfrak{G})\right)$. For any $H \subseteq \mathfrak{G}$ let $\mathfrak{i}_k = (\breve{\varphi}_c, \breve{\varphi}_k) \in H$ then $\text{NINVE}(\breve{\varphi}_c) = \emptyset \subseteq H$ thus $H \in \mathcal{T}_n$ so $\mathcal{T}_n = P\left(\breve{E}(\mathfrak{G})\right)$.

Theorem 2.8: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space, if $\mathfrak{G}$ is an antisymmetric graph then $\mathcal{T}_i = P\left(\breve{E}(\mathfrak{G})\right)$ and $\mathcal{T}_n = \left(\emptyset, \breve{E}(\mathfrak{G})\right)$.

Proof : Let $\mathfrak{G}$ be antisymmetric graph and H is every subgrah of $\mathfrak{G}$ then every $\mathfrak{i}_n \in \breve{E}(H)$ we can write it $\mathfrak{i}_n = (\breve{\varphi}_n, \breve{\varphi}_n)$ and $\text{INVE}(\breve{\varphi}_n) = \{\mathfrak{i}_n\} \subseteq H$ thus $H \in \mathcal{T}_i$ there fore $\mathcal{T}_i = P\left(\breve{E}(\mathfrak{G})\right)$. And we can prove $\mathcal{T}_n = \left(\emptyset, \breve{E}(\mathfrak{G})\right)$ because not there exist H proper sub graph of antisymmetric graph $\mathfrak{G}$ it achieves property definition (2.5) class $\mathcal{T}_n$ thus only$\emptyset, \breve{E}(\mathfrak{G}) \in \mathcal{T}_n$.

Theorem 2.9: Let $(\mathfrak{G}, \mathfrak{h}_i)$ be an i-space then $\text{INVE}(\breve{\varphi}_i)$ for all $i = 1, \dots, |V(\mathfrak{G})|$ is i-open.

Proof : By using theorem (2.3) for each $\mathfrak{i} \in \text{INVE}(\breve{\varphi}_i)$ such that $\mathfrak{i} = (\breve{\varphi}_i, \breve{\varphi}_j)$ thus there exist $\breve{\varphi}_i \in V(H)$ incidence on $\mathfrak{i}$ and $\text{INVE}(\breve{\varphi}_i) \subseteq \text{INVE}(\breve{\varphi}_i)$ hence $\text{INVE}(\breve{\varphi}_i)$ is i-open.

Proposition 2.10: Let $ß = \left(\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G})\right)$ be a generalized approximation space, then $\mathcal{T}_i = \mho_{\mathfrak{h}_i}$ and $\mathcal{T}_n = \mho_{\mathfrak{h}_n}$.

Proof : The proof of $\mathcal{T}_i = \mho_{\mathfrak{h}_i}$ and $\mathcal{T}_n = \mho_{\mathfrak{h}_n}$ is immediately follows from definition (2. 4), (2. 5) and theorem (2..3).

Remark 2.11: An immediate consequence of proposition (3.10) and theorem (2. 6) we have $\mho_{\mathfrak{h}_i}$ and $\mho_{\mathfrak{h}_n}$ form supra topologies on $\mathfrak{G}$.

Definition 2.12: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space, then the families of all $H_2 -$ incidence composed (resp. H_2 −non incidence composed) graph are denoted by $\dagger_i$ (resp. $\dagger_n$) and defined by:

$$\dagger_i = \left\{ \begin{matrix} H \subseteq \mathfrak{G};\ \forall \mathfrak{i} \in \breve{E}(H) \text{ where } \breve{\varphi} \in V(H) \text{ incidence} \\ \text{on } \mathfrak{i} \text{ such that INVE}(\breve{\varphi}) \subseteq \breve{E}(H) \end{matrix} \right\}$$

(resp. $\dagger_n = \left\{ \begin{matrix} H \subseteq \mathfrak{G};\ \forall \mathfrak{i} \in \breve{E}(H) \text{ where } \breve{\varphi} \in V(H) \text{ incidence} \\ \text{on } \mathfrak{i} \text{ such that NINVE}(\breve{\varphi}) \subseteq \breve{E}(H) \end{matrix} \right\}$).

Theorem 2.13: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space, then $\dagger_i$ (resp. $\dagger_n$) forms a topology on $\breve{E}(\mathfrak{G})$.

Proof : Now to prove that $\dagger_i$ is topology on $\breve{E}(\mathfrak{G})$:

(1) $\breve{E}(\mathfrak{G}), \emptyset \in \dagger_i$.

(2) Let $\breve{E}(H), \breve{E}(K) \in \dagger_I$ and for each $\mathfrak{i} \in \breve{E}(H) \cap \breve{E}(K)$so $\mathfrak{i} \in \breve{E}(H)$ where $\breve{\varphi} \in V(H)$ incidence on $\mathfrak{i}$ such that $\text{INVE}(\breve{\varphi}) \subseteq \breve{E}(H)$ and $\mathfrak{i} \in \breve{E}(K)$ where $\breve{\varphi} \in V(K)$ incidence on $\mathfrak{i}$ such that $\text{INVE}(\breve{\varphi}) \subseteq \breve{E}(K)$ thus $\text{INVE}(\breve{\varphi}) \subseteq \breve{E}(H) \cap \breve{E}(K)\ \ \therefore \breve{E}(H) \cap \breve{E}(K) \in \dagger_i$.

(3) Let $\breve{E}(H_i) \in \dagger_i\ \forall\ i \in I$. Then for each $\mathfrak{i} \in \cup_i \breve{E}(H_i)$ imply that $\exists\ i_o \in I$ such that $\mathfrak{i} \in \breve{E}(H_{i_o}) \subseteq \cup_i \breve{E}(H_i)$, hence $\text{INVE}(\breve{\varphi}) \subseteq \breve{E}(H_{i_o}) \subseteq \cup_i \breve{E}(H_i)$, that is $\cup_i \breve{E}(H_i) \in \dagger_i\ \ \forall\ i \in I$. We prove with the same technique $\dagger_n$ is topology on $\breve{E}(\mathfrak{G})$.

Theorem 2.14: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space , if $\mathfrak{G}$ is simple connected graph then $\dagger_i$ (resp. $\dagger_n$) forms an indiscrete topology on $\breve{E}(\mathfrak{G})$.

Proof : Assume there is a proper subgraph H of $\mathfrak{G}$. Such that $H \in \dagger_i$ by definition (2.12)

$$\dagger_i = \left\{ \begin{matrix} H \subseteq \mathfrak{G};\ \forall \mathfrak{i} \in \breve{E}(H) \text{ where } \breve{\varphi} \in V(H) \\ \text{incidence on } \mathfrak{i} \text{ such that INVE}(\breve{\varphi}) \subseteq \breve{E}(H) \end{matrix} \right\}$$

Let’s take $\mathfrak{i} = (\breve{\varphi}_0, \breve{\varphi}_1) \in \breve{E}(H)$ must $\text{INVE}(\breve{\varphi}_0) \subseteq \breve{E}(H)$ such that $\text{INVE}(\breve{\varphi}_0)$ contains at least one edge let it be $\mathfrak{i}$ and $\text{INVE}(\breve{\varphi}_1) \subseteq \breve{E}(H)$ such that $\text{INVE}(\breve{\varphi}_1)$ contains at least one edge different from $\mathfrak{i}$ let it be σ because $\mathfrak{G}$ is connected graph thus must $\sigma = (\breve{\varphi}_1, \breve{\varphi}_2) \in \breve{E}(H)$ also $\text{INVE}(\breve{\varphi}_1) \subseteq \breve{E}(H)$ such that $\text{INVE}(\breve{\varphi}_1)$ contains at least one edge let it be σ and $\text{INVE}(\breve{\varphi}_2) \subseteq \breve{E}(H)$ such that $\text{INVE}(\breve{\varphi}_2)$ contains at least one edge different from σ let it be ς because $\mathfrak{G}$ is connected graph, and by repeating this process we find that $H = \mathfrak{G}$, thus H is not proper subgraph of $\mathfrak{G}$ and $\dagger_i = \{\emptyset, \mathfrak{G}\}$ this indiscrete topology on"$\mathfrak{G}$. Assume there is a proper subgraph H of $\mathfrak{G}$. Such that $\mathrm{H} \in \dagger_n$ by definition (2.12)

$$\dagger_n = \left\{ \begin{array}{c} \mathrm{H} \subseteq \mathfrak{G};\ \forall\, \breve{\iota} \in \breve{E}(\mathrm{H}) \text{ where } \breve{\omega} \in V(\mathrm{H}) \\ \text{incidence on } \breve{\iota} \text{ such that } \mathrm{NINVE}(\breve{\omega}) \subseteq \breve{E}(\mathrm{H}) \end{array} \right\}.$$

which implies that H must include more and more edges from $\mathfrak{G}$ that are not incidence to vertices in H . Since $\mathfrak{G}$ is connected, every vertex in H is connected to other vertices by edges in $\mathfrak{G}$, so H would eventually need to include these edges and corresponding vertices. If $\mathrm{NINVE}(\breve{\omega}) \subseteq \breve{E}(\mathrm{H})$ where $\breve{\omega} \in V(\mathrm{H})$ incidence on $\breve{\iota}$ for each $\breve{\iota} \in \breve{E}(\mathrm{H})$, then H must include all edges and vertices of $\mathfrak{G}$ to satisfy the condition.This contradicts the assumption that H is a proper subgraph because H would ultimately become $\mathrm{H} = \mathfrak{G}$,thus H is not proper subgraph of $\mathfrak{G}$ and $\dagger_n = \{ \emptyset , \mathfrak{G} \}$ this indiscrete topology on"$\mathfrak{G}$.

Note 2.15: The families $H_1 -$incidence composed (resp. $H_1 -$ non incidence composed) it is generalization to $H_2 -$ incidence composed (resp. $H_2 -$ non incidence composed) i.e :(every $\dagger_i$ (resp. $\dagger_n$) is $\mathcal{T}_i$ (resp. $\mathcal{T}_n$)) .

In this research, we confirm that the study will be limited to the generalized approximation space using the relation $\mathcal{T}_i$ (resp. $\mathcal{T}_n$) only, which represents a supra topology. The relation $\mathcal{T}_i$ (resp. $\mathcal{T}_n$) is a generalization of the relation $\dagger_i$ (resp. $\dagger_n$) , which represents an ordinary topology. However, the focus will be entirely on $\mathcal{T}_i$ (resp. $\mathcal{T}_n$) due to its flexibility and ability to provide more comprehensive results. The relation $\dagger_i$ (resp. $\dagger_n$) will not be used in the analysis, and our attention will be directed solely to studying the properties, approximations (lower and upper), and criteria associated with $\mathcal{T}_i$ (resp. $\mathcal{T}_n$)) . to avoid overlap or confusion between the different relations.

Definition 2.16: Let $ß = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space and $\mho_{\mathfrak{h}_i}$, $\mho_{\mathfrak{h}_n}$ and $\mho_{\mathfrak{h}_a}$ be the supra topologies induced by ß and let $\mathrm{H} \subseteq \mathfrak{G}$. Then:

(1) The i-lower and i-upper approximations of H are defined respectively by:

$$L_i(\breve{E}(\mathrm{H})) = \mathrm{Int}_i(\breve{E}(\mathrm{H})).$$
$$U_i(\breve{E}(\mathrm{H})) = \mathrm{Cl}_i(\breve{E}(\mathrm{H})).$$

(2) The n-lower and n-upper approximations of H are defined respectively by:

$$L_n(\breve{E}(\mathrm{H})) = \mathrm{Int}_n(\breve{E}(\mathrm{H})).$$
$$U_n(\breve{E}(\mathrm{H})) = \mathrm{Cl}_n(\breve{E}(\mathrm{H})).$$

(3) The a-lower and a-upper approximations of H are defined respectively by:

$$L_a(\breve{E}(\mathrm{H})) = \mathrm{Int}_a(\breve{E}(\mathrm{H})).$$
$$U_a(\breve{E}(\mathrm{H})) = \mathrm{Cl}_a(\breve{E}(\mathrm{H})).$$

Example 2.17: Let $\mathfrak{G} = (\mathbb{W}(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ such that $\mathbb{W}(\mathfrak{G}) = \{\breve{\omega}_1, \breve{\omega}_2, \breve{\omega}_3, \breve{\omega}_4\}$, $\breve{E}(\mathfrak{G}) = \{\breve{\iota}_1 = (\breve{\omega}_1, \breve{\omega}_1)$, $\breve{\iota}_2 = (\breve{\omega}_1, \breve{\omega}_4), \breve{\iota}_3 = (\breve{\omega}_2, \breve{\omega}_3), \breve{\iota}_4 = (\breve{\omega}_2, \breve{\omega}_4), \breve{\iota}_5 = (\breve{\omega}_3, \breve{\omega}_4)\}$.

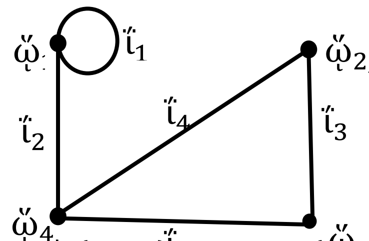

Figure 2.1: graph $\mathfrak{G}$ given in Example (2.17).

Then $\mathfrak{h}_i(\breve{\omega}_1) = \{\{\breve{\iota}_1, \breve{\iota}_2\}\}$, $\mathfrak{h}_i(\breve{\omega}_2) = \{\{\breve{\iota}_3, \breve{\iota}_4\}\}$, $\mathfrak{h}_i(\breve{\omega}_3) = \{\{\breve{\iota}_3, \breve{\iota}_5\}\}$, $\mathfrak{h}_i(\breve{\omega}_4) = \{\{\breve{\iota}_2, \breve{\iota}_4, \breve{\iota}_5\}\}$. And

$\mathfrak{h}_n(\breve{\omega}_1) = \{\{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}\}$, $\mathfrak{h}_n(\breve{\omega}_2) = \{\{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_5\}\}$, $\mathfrak{h}_n(\breve{\omega}_3) = \{\{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_4\}\}$, $\mathfrak{h}_n(\breve{\omega}_4) = \{\{\breve{\iota}_1, \breve{\iota}_3\}\}$. And

$\mathfrak{h}_a(\breve{\omega}_1) = \{\{\breve{\iota}_1, \breve{\iota}_2\}, \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}\}$, $\mathfrak{h}_a(\breve{\omega}_2) = \{\{\breve{\iota}_3, \breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_5\}\}$, $\mathfrak{h}_a(\breve{\omega}_3) = \{\{\breve{\iota}_3, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_4\}\}$, $\mathfrak{h}_a(\breve{\omega}_4) = \{\{\breve{\iota}_2, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_3\}\}$. And

$$\mho_{\mathfrak{h}_i} = \left\{ \begin{array}{c} \breve{E}(\mathfrak{G}), \emptyset, \{\breve{\iota}_1, \breve{\iota}_2\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_3, \breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_3, \breve{\iota}_5\}, \\ \{\breve{\iota}_3, \breve{\iota}_4\}, \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_2, \breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_3, \breve{\iota}_5\} \\ , \{\breve{\iota}_2, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_4, \breve{\iota}_5\} \\ . \end{array} \right\}.$$

$$\mho_{\mathfrak{h}_n} = \{\breve{E}(\mathfrak{G}) , \emptyset \} .$$

$$\mho_{\mathfrak{h}_a} = \left\{ \begin{array}{c} \breve{E}(\mathfrak{G}), \emptyset, \{\breve{\iota}_1, \breve{\iota}_2\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_3, \breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_3, \breve{\iota}_5\}, \\ \{\breve{\iota}_3, \breve{\iota}_4\}, \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\} , \{\breve{\iota}_2, \breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\} \\ , \{\breve{\iota}_3, \breve{\iota}_5\}, \{\breve{\iota}_2, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_4, \breve{\iota}_5\} \end{array} \right\}.$$

Also we note that the i-closed , n-closed and a-closed sets in this example are :

$$\dot{F}_{\mathfrak{h}_i} = \left\{ \begin{array}{c} \breve{E}(\mathfrak{G}), \emptyset, \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_5\}, \{\breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_2\}, , \\ \{\breve{\iota}_1\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_3\}, \{\breve{\iota}_3\} \\ . \end{array} \right\}.$$

$$\dot{F}_{\mathfrak{h}_n} = \{\breve{E}(\mathfrak{G}) , \emptyset \} .$$

$$\dot{F}_{\mathfrak{h}_a} = \left\{ \begin{array}{c} \breve{E}(\mathfrak{G}), \emptyset, \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}, \{\breve{\iota}_5\}, \{\breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_5\}, \{\breve{\iota}_1, \breve{\iota}_2\}, , \\ \{\breve{\iota}_1\}, \{\breve{\iota}_2\}, \{\breve{\iota}_1, \breve{\iota}_2, \breve{\iota}_4\}, \{\breve{\iota}_1, \breve{\iota}_3\}, \{\breve{\iota}_3\} \end{array} \right\}.$$

We can get the following four tables:

Table2.1:$L_i(\breve{E}(\mathrm{H}))$,$L_n(\breve{E}(\mathrm{H}))$ and $L_a(\breve{E}(\mathrm{H}))$ for all $\mathrm{H} \subseteq \mathfrak{G}$.

$\breve{E}(\mathrm{H})$	$L_i(\breve{E}(\mathrm{H}))$	$L_n(\breve{E}(\mathrm{H}))$	$L_a(\breve{E}(\mathrm{H}))$
$\{\breve{\iota}_1\}$	ϕ	ϕ	ϕ

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$\{\breve{\iota}_2\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_3\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_4\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_5\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_1,\breve{\iota}_2\}$	$\{\breve{\iota}_1,\breve{\iota}_2\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2\}$
$\{\breve{\iota}_1,\breve{\iota}_3\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_1,\breve{\iota}_4\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_1,\breve{\iota}_5\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_2,\breve{\iota}_3\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_2,\breve{\iota}_4\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_2,\breve{\iota}_5\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_3,\breve{\iota}_4\}$	$\{\breve{\iota}_3,\breve{\iota}_4\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_4\}$
$\{\breve{\iota}_3,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_5\}$
$\{\breve{\iota}_4,\breve{\iota}_5\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3\}$	$\{\breve{\iota}_1,\breve{\iota}_2\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$	$\{\breve{\iota}_1,\breve{\iota}_2\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$	$\{\breve{\iota}_1,\breve{\iota}_2\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2\}$
$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4\}$	$\{\breve{\iota}_3,\breve{\iota}_4\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_4\}$
$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_5\}$
$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_1\}$	$\{\breve{\iota}_3,\breve{\iota}_4\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_4\}$
$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$
$\{\breve{\iota}_4,\breve{\iota}_5,\breve{\iota}_1\}$	ϕ	ϕ	ϕ
$\{\breve{\iota}_4,\breve{\iota}_5,\breve{\iota}_2\}$	$\{\breve{\iota}_4,\breve{\iota}_5,\breve{\iota}_2\}$	ϕ	$\{\breve{\iota}_4,\breve{\iota}_5,\breve{\iota}_2\}$
$\{\breve{\iota}_1,\breve{\iota}_3,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_3,\breve{\iota}_5\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_5\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_5\}$
$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$
$\{\breve{\iota}_1,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4,\breve{\iota}_5\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4,\breve{\iota}_5\}$	ϕ	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4,\breve{\iota}_5\}$
$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
ϕ	ϕ	ϕ	ϕ

Table2.2: $U_i\left(\breve{E}(Ħ)\right)$, $U_n\left(\breve{E}(Ħ)\right)$ and $U_a\left(\breve{E}(Ħ)\right)$ for all $Ħ \subseteq \mathfrak{G}$.

$\breve{E}(Ħ)$	$U_i\left(\breve{E}(Ħ)\right)$	$U_n\left(\breve{E}(Ħ)\right)$	$U_a\left(\breve{E}(Ħ)\right)$
$\{\breve{\iota}_1\}$	$\{\breve{\iota}_1\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1\}$
$\{\breve{\iota}_2\}$	$\{\breve{\iota}_1,\breve{\iota}_2\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_2\}$
$\{\breve{\iota}_3\}$	$\{\breve{\iota}_3\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_3\}$
$\{\breve{\iota}_4\}$	$\{\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_4\}$
$\{\breve{\iota}_5\}$	$\{\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_5\}$
$\{\breve{\iota}_1,\breve{\iota}_2\}$	$\{\breve{\iota}_1,\breve{\iota}_2\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2\}$
$\{\breve{\iota}_1,\breve{\iota}_3\}$	$\{\breve{\iota}_1,\breve{\iota}_3\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_3\}$
$\{\breve{\iota}_1,\breve{\iota}_4\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$
$\{\breve{\iota}_1,\breve{\iota}_5\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$
$\{\breve{\iota}_2,\breve{\iota}_3\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_2,\breve{\iota}_4\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$
$\{\breve{\iota}_2,\breve{\iota}_5\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$
$\{\breve{\iota}_3,\breve{\iota}_4\}$	$\{\breve{\iota}_3,\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_3,\breve{\iota}_4\}$
$\{\breve{\iota}_3,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$
$\{\breve{\iota}_4,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4\}$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_5\}$
$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_1\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\{\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$
$\{\breve{\iota}_4,\breve{\iota}_5,\breve{\iota}_1\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_4,\breve{\iota}_5,\breve{\iota}_2\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_1,\breve{\iota}_3,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_2,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_1,\breve{\iota}_3,\breve{\iota}_4,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\{\breve{\iota}_1,\breve{\iota}_2,\breve{\iota}_4,\breve{\iota}_5\}$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$	$\breve{E}(\mathfrak{G})$
ϕ	ϕ	ϕ	ϕ

Proposition 2.18: Let $ß = (V(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space, and $\dot{F}_{\mathfrak{h}_i}$, $\dot{F}_{\mathfrak{h}_n}$ and $\dot{F}_{\mathfrak{h}_a}$ be the families of i-closed, n-closed"and a-closed graphs induced by ß. Then any i-closed (or n-closed) graph is a-closed.

Proof : Let $Ħ \subseteq \mathfrak{G}$ be an i-closed graph then $\left[\breve{E}(Ħ)\right]_i' \subseteq \breve{E}(Ħ)$. From definition (2. 2(1)) :

$$\left[\breve{E}(Ħ)\right]_i' = \left\{\begin{matrix} \breve{\iota} \in \breve{E}(\mathfrak{G}); INVE(\ddot{\wp}) \cap \left(\breve{E}(Ħ) - \{\breve{\iota}\}\right) \neq \emptyset, \\ \text{where } \ddot{\wp} \text{ incidence on } \breve{\iota} \end{matrix}\right\}$$

and from introduction:

$$\left[\breve{E}(Ħ)\right]_a' = \left\{\begin{matrix} \breve{\iota} \in \breve{E}(\mathfrak{G}); AVE(\ddot{\wp}) \cap \left(\breve{E}(Ħ) - \{\breve{\iota}\}\right) \neq \emptyset, \\ \text{where } \ddot{\wp} \text{ incidence on } \breve{\iota} \end{matrix}\right\}$$

Consequently, $[\breve{E}(Ħ)]_a' \subseteq [\breve{E}(Ħ)]_i'$ and since Ħ is i-closed so $[\breve{E}(Ħ)]_i' \subseteq \breve{E}(Ħ)$ thus $[\breve{E}(Ħ)]_a' \subseteq \breve{E}(Ħ)$ which implies Ħ is a-closed. Therefore any i-closed graph is a-closed. Similarly, we can prove that any n-closed is a-closed .

Proposition 2.19: Let $ß = (V(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space and $\mathcal{T}_{\mathfrak{h}_i}$, $\mathcal{T}_{\mathfrak{h}_n}$ and $\mathcal{T}_{\mathfrak{h}_a}$ be the supra topologies induced by ß. Then any i-open (or n-open) graph is a-open.

Proof : Let $K \subseteq \mathfrak{G}$ be an i-open graph and $Ħ = \mathfrak{G} - K$. So Ħ is i-closed graph and by using Proposition (2.18) Ħ is a-closed. Hence $K = \mathfrak{G} - Ħ$ is a-open. Accordingly, any i-open graph is a-open. By the same manner we can prove that any n-open graph is a-open .

Proposition 2.20: Let $ß = (V(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space and $Ħ \subseteq \mathfrak{G}$. Then:

(1) $L_i\left(\breve{E}(Ħ)\right) \cup L_n\left(\breve{E}(Ħ)\right) \subseteq L_a\left(\breve{E}(Ħ)\right)$.

(2) $U_a\left(\breve{E}(Ħ)\right) \subseteq U_i\left(\breve{E}(Ħ)\right) \cap U_n\left(\breve{E}(Ħ)\right)$.

Proof: (1) Since

$L_i(\breve{E}(H)) = \cup \{\breve{E}(S) \in \mho_{\mathfrak{h}_i}; \breve{E}(S) \subseteq \breve{E}(H)\}$. Hence $L_i(\breve{E}(H)) \subseteq \breve{E}(H)$ and $L_i(\breve{E}(H))$ is i-open since the union of any family of i-open graph is i-open. Since $L_n(\breve{E}(H)) = \cup \{\breve{E}(S) \in \mho_{\mathfrak{h}_n}; \breve{E}(S) \subseteq \breve{E}(H)\}$. So $L_n(\breve{E}(H)) \subseteq \breve{E}(H)$ and $L_n(\breve{E}(H))$ is n-open since the union of any family of n-open graph is n-open. Since $L_i(\breve{E}(H))$ is i-open, then by Proposition (2.19) it is a-open and since $L_n(\breve{E}(H))$ is n-open, then by Proposition (2.19) it is also a-open. Hence $L_i(\breve{E}(H)) \cup L_n(\breve{E}(H))$ is a-open and $L_i(\breve{E}(H)) \cup L_n(\breve{E}(H)) \subseteq \breve{E}(H)$. But,
$L_a(\breve{E}(H)) = \cup \{\breve{E}(S) \in \mho_{\mathfrak{h}_a}; \breve{E}(S) \subseteq \breve{E}(H)\}$.

Consequently,
$L_i(\breve{E}(H)) \cup L_n(\breve{E}(H)) \subseteq L_a(\breve{E}(H))$.

(2) Since
$U_i(\breve{E}(H)) = \cap \{\breve{E}(k) \in \dot{F}_{\mathfrak{h}_i}; \breve{E}(H) \subseteq \breve{E}(k)\}$. Hence $\breve{E}(H) \subseteq U_i(\breve{E}(H))$ and $U_i(\breve{E}(H))$ is i-closed since the intersection of any family of i-closed graph is i-closed. Since
$U_n(\breve{E}(H)) = \cap \{\breve{E}(k) \in \dot{F}_{\mathfrak{h}_n}; \breve{E}(H) \subseteq \breve{E}(k)\}$. thus $\breve{E}(H) \subseteq U_n(\breve{E}(H))$ and $U_n(\breve{E}(H))$ is n-closed since the intersection of any family of n-closed graph is n-closed. Since $U_i(\breve{E}(H))$ is i-closed then, by Proposition (2.18) it is a-closed and since $U_n(\breve{E}(H))$ is n-closed then, by Proposition 2.18) it is also a-closed. Hence $U_i(\breve{E}(H)) \cap U_n(\breve{E}(H))$ is a-closed and $\breve{E}(H) \subseteq U_i(\breve{E}(H)) \cap U_n(\breve{E}(H))$ .But $U_a(\breve{E}(H)) = \cap \{\breve{E}(k) \in \dot{F}_{\mathfrak{h}_a}; \breve{E}(H) \subseteq \breve{E}(k)\}$.
According. $U_a(\breve{E}(H)) \subseteq U_i(\breve{E}(H)) \cap U_n(\breve{E}(H))$.

Remark 2.21: Let $\beta = (V(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space and $H \subseteq \mathfrak{G}$. Then the following statements are not necessarily true:

(1) $L_a(\breve{E}(H)) = L_i(\breve{E}(H)) \cup L_n(\breve{E}(H))$.

(2) $U_a(\breve{E}(H)) = U_i(\breve{E}(H)) \cap U_n(\breve{E}(H))$.
The next example shows pervious remark.

Example 2. 22: According to example" (2.17):

(1) Let $H = (V(H), \breve{E}(H))$ such that $V(H) = \{\breve{\varphi}_1, \breve{\varphi}_2, \breve{\varphi}_3, \breve{\varphi}_4\}$ and $\breve{E}(H) = \{\breve{\iota}_1, \breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}$. Then $L_a(\breve{E}(H)) = \{\breve{\iota}_1, \breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}$, $L_i(\breve{E}(H)) = \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}$ and $L_n(\breve{E}(H)) = \emptyset$ such that
$L_i(\breve{E}(H)) \cup L_n(\breve{E}(H)) = \{\breve{\iota}_3, \breve{\iota}_4, \breve{\iota}_5\}$ and so
$L_a(\breve{E}(H)) \neq L_i(\breve{E}(H)) \cup L_n(\breve{E}(H))$.

(2) Let $H = (V(H), \breve{E}(H))$ such that $V(H) = \{\breve{\varphi}_1, \breve{\varphi}_2\}$ and $\breve{E}(H) = \{\breve{\iota}_2\}$. Then $U_a(\breve{E}(H)) = \{\breve{\iota}_2\}$, $U_i(\breve{E}(H)) = \{\breve{\iota}_1, \breve{\iota}_2\}$ and $U_n(\breve{E}(H)) = \breve{E}(\mathfrak{G})$, such that $U_i(\breve{E}(H)) \cap U_n(\breve{E}(H)) = \{\breve{\iota}_1, \breve{\iota}_2\}$ and so $U_a(\breve{E}(H)) \neq U_i(\breve{E}(H)) \cap U_n(\breve{E}(H))$.

Proposition 2.23: Let $\beta = (V(\mathfrak{G}), \breve{E}(\mathfrak{G}))$ be a generalized approximation space and $H \subseteq \mathfrak{G}$. If the relation $\mho_{\mathfrak{h}_a} = \mho_{\mathfrak{h}_i} \cup \mho_{\mathfrak{h}_n}$ holds, then the following relations will be valid :
(1) $L_a(\breve{E}(H)) = L_i(\breve{E}(H)) \cup L_n(\breve{E}(H))$.
(2) $U_a(\breve{E}(H)) = U_i(\breve{E}(H)) \cap U_n(\breve{E}(H))$.

Proof : (1) Let $\breve{\iota} \in \mathbf{L_a}(\breve{E}(H)) \Leftrightarrow \breve{\iota} \in \mathbf{Int_a}(\breve{E}(H))$
$\Leftrightarrow \breve{\iota} \in \cup\{\breve{E}(\mathbf{S}) \in \mho_{\mathfrak{h}_a}; \breve{E}(\mathbf{S}) \subseteq \breve{E}(H)\}$
$\Leftrightarrow \breve{\iota} \in \cup\{\breve{E}(\mathbf{S}) \in \mho_{\mathfrak{h}_i} \cup \mho_{\mathfrak{h}_n}; \breve{E}(\mathbf{S}) \subseteq \breve{E}(H)\}$
$\Leftrightarrow \breve{\iota} \in \cup\{\breve{E}(\mathbf{S}) \in \mho_{\mathfrak{h}_i}; \breve{E}(\mathbf{S}) \subseteq \breve{E}(H)\}$
$\bigcup \breve{\iota} \in \cup\{\breve{E}(\mathbf{S}) \in \mho_{\mathfrak{h}_n}; \breve{E}(\mathbf{S}) \subseteq \breve{E}(H)\}$
$\Leftrightarrow \breve{\iota} \in \mathbf{Int_i}(\breve{E}(H)) \cup \breve{\iota} \in \mathbf{Int_n}(\breve{E}(H))$
$\Leftrightarrow \breve{\iota} \in \mathbf{L_i}(\breve{E}(H)) \cup \breve{\iota} \in \mathbf{L_n}(\breve{E}(H))$
$\Leftrightarrow \breve{\iota} \in \mathbf{L_i}(\breve{E}(H)) \cup \mathbf{L_n}(\breve{E}(H))$
Therefore $L_a(\breve{E}(H)) = \mathbf{L_i}(\breve{E}(H)) \cup \mathbf{L_n}(\breve{E}(H))$.
(2) Let $\breve{\iota} \in \mathbf{U_a}(\breve{E}(H)) \Leftrightarrow \breve{\iota} \in Cl_{\mathbf{a}}(\breve{E}(H))$
$\Leftrightarrow \breve{\iota} \in \cap\{\breve{E}(K) \in \dot{F}_{\mathfrak{h}_a}; \breve{E}(H) \subseteq \breve{E}(K)\}$
$\Leftrightarrow \breve{\iota} \in \cap\{\breve{E}(K) \in \dot{F}_{\mathfrak{h}_i} \cup \dot{F}_{\mathfrak{h}_n}; \breve{E}(H) \subseteq \breve{E}(K)\}$
$\Leftrightarrow \breve{\iota} \in \cap\{\breve{E}(K) \in \dot{F}_{\mathfrak{h}_i}; \breve{E}(H) \subseteq \breve{E}(K)\}$
$\cap\ \breve{\iota} \in \cap\{\breve{E}(K) \in \dot{F}_{\mathfrak{h}_n}; \breve{E}(H) \subseteq \breve{E}(K)\}$
$\Leftrightarrow \breve{\iota} \in Cl_{\mathbf{i}}(\breve{E}(H)) \cap \breve{\iota} \in Cl_{\mathbf{n}}(\breve{E}(H))$
$\Leftrightarrow \breve{\iota} \in \mathbf{U_i}(\breve{E}(H)) \cap \breve{\iota} \in \mathbf{U_n}(\breve{E}(H))$
$\Leftrightarrow \breve{\iota} \in \mathbf{U_i}(\breve{E}(H)) \cap \mathbf{U_n}(\breve{E}(H))$
Therefore $U_a(\breve{E}(H)) = \mathbf{U_i}(\breve{E}(H)) \cap \mathbf{U_n}(\breve{E}(H))$.

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