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Quasi-Pre-Semi-Open Sets: A Weaker Notion in Topology

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Abstract— *The concept of quasi-presemi-open sets (Qpsos) is a generalization of presemi-open sets, which themselves are a generalization of semi-open sets. Qpsos. provide a more flexible notion of openness in topological spaces, allowing for finer distinctions between open and closed sets.This paper, we explore the properties and characteristics of Qpsos. We define the notion of qps-open sets in terms of their complement, which exhibits some similarities to the definition of presemi-open sets. We investigate the relationships between Qpsos and other types of sets, such as open sets, closed sets, and presemi-open sets. Furthermore, we examine the behavior of Qpsos under various operations, including union, intersection, and closure. We establish necessary and sufficient conditions for a set to be qps-open, and we discuss the implications of these conditions on the topology of a space.We also investigate the role of Qpsos in the context of continuity and compactness. We explore the connection between qps-continuous functions and other types of continuous functions, and we analyze the relationship between Qps -compactness and other notions of compactness.*

Keywords— *Quasi-pre-semi -open sets, Quasi-pre-semi -continues sets, Quasi-pre-semi -compact sets.*

1. Introduction AND Preliminaries:

TOPOLOGY IS A BRANCH OF MATHEMATICS THAT DEALS WITH THE PROPERTIES OF SPS. THAT ARE PRESERVED UNDER CONTINUOUS TRANSFORMATIONS. THE CONCEPT OF OPEN AND CLOSED SETS FORMS THE FOUNDATION OF TOPOLOGY. HOWEVER, IN CERTAIN CASES, THESE NOTIONS MAY BE TOO STRONG OR RESTRICTIVE TO CAPTURE ALL THE DESIRED PROPERTIES OF SETS. TO ADDRESS THIS, VARIOUS WEAKER NOTIONS HAVE BEEN INTRODUCED. ONE SUCH NOTION IS THAT OF QPSOS, WHICH PROVIDES A MORE FLEXIBLE FRAMEWORK FOR CHARACTERIZING SETS IN A TOP. SP.

Definition 1-1 *Let (Υ,κ) be a top sp, and a set ξ in a top sp Υ is presemi-open if it can be expressed as the union of an open set and a closed set, i.e., $\xi = \varpi \cup \varrho$, where ϖ is an open set and ϱ is a closed set.*
Definitions 1-2 *Let (Υ,κ) be a topological space, and let ξ a set in Υ ,we said ξ to be Qps if there exist a closed κ and a semi-open set μ such that $\xi = \kappa \cup (\mu \cap \eta)$, where η is some subset of μ.*

2- Properties

In this section introduce some properties of Qpsos as proposition with proofs and examples for these properties
Proposition 2.1: *If ξ Qpsos in (Υ,κ) ,then are a generalization of presemi-open sets.*
Proof: *Let ξ be a presemi-open set in top sp Υ, i.e., $\xi = \varpi \cup \varrho$, where ϖ is open and ϱ is closed. Let's define $\mu = \xi \setminus \varpi$, which is the complement of ϖ. Also, let $\kappa = \varrho \cup (\varpi \cap \mu)$.To show that $\xi = \kappa \cup (\mu \cap \eta)$ for some subset η of μ.Consider the intersection $\mu \cap \xi$.*
We have: $\mu \cap \xi = (\Upsilon \setminus\varpi) \cap (\varpi \cup \varrho) = (\xi \setminus \varpi) \cap \varpi \cup (\xi \setminus \varpi) \cap \varrho = \emptyset \cup (\xi \setminus \varpi) \cap \varrho = \varrho$.
Since ϱ is a subset of κ, we have $\varrho \subseteq \kappa$. Therefore, we can write: $\kappa \cup (\mu \cap \xi) = \kappa \cup \varrho = \varrho = \mu \cap \xi$.
This implies that $\xi = \kappa \cup (\mu \cap \xi)$, where $\mu \cap \xi$ is a subset of μ. Thus, ξ satisfies the definition of a qps-open set.Therefore, every pre-semi-open set is qps-open, and we have shown that qps-open sets are a generalization of pre-semi-open sets.■
Notice2-2*: Every presemi-open set is qps-open, but the converse is not necessarily true.*
Example 2-3: *Consider the topological space $\xi = \Re$ with the usual top. Let $\xi = (0,1) \cup \{2\}$.*
• *ξ is presemi-open: Take $\varpi = (0,1)$ and $\varrho = \{2\}$. Then $\xi = \varpi \cup \varrho$, where ϖ is open and ϱ is closed.*
• *ξ is also qps-open: We can express ξ as the union of a closed set and a subset of a semi-open set. Take $\kappa = \{2\}$ and $\mu = [0,1)$. Then $\xi = \kappa \cup (\mu \cap \eta)$, where $\eta = (0,1)$ is a subset of μ.*
This example demonstrates that ξ is both a presemi-open set and a Qpsos.
Example 2-4: *Let $\Upsilon = \Re$ with the usual top. Consider $\xi = [0,1) \cup (2,3)$.*
• *ξ is presemi-open: Take $\varpi = [0,1)$ and$\varrho = (2,3)$. Then $\xi = \varpi \cup \varrho$, where ϖ is open and ϱ is closed.*
• *ξ is not qps-open: There does not exist a closed set κ and a semi-open set μsuch that $\xi = \kappa \cup (\mu \cap \eta)$ for any subset η of μ. No matter how we choose κ and μ, we cannot represent ξ in the required form. Thus, ξ fails to be qps-open.*
This example shows that while ξ is pre-semi-open, it does not satisfy the definition of a qps-open set.

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Proposition 2-5: *Every presemi-open set is Qpsos.*

Proof: *Let ξ be a presemi-open set in a top sp Υ. By definition, $\xi = \varpi \cup \varrho$.*

Now, let's define $\mu = \Upsilon \setminus \varpi$, which is the complement of ϖ. Note that μ is closed. Also, let $\kappa = \varrho$, which is a closed set. We want to show that $\xi = \kappa \cup (\mu \cap \eta)$ for some subset η of μ.Consider the intersection $\mu \cap \xi$. We have:

$\mu \cap \xi = (\Upsilon \setminus \varpi) \cap (\varpi \cup \varrho) = (\Upsilon \setminus \varpi) \cap \varpi \cup (\Upsilon \setminus \varpi) \cap \varrho = \emptyset \cup (\Upsilon \setminus \varpi) \cap \varrho = \varrho.$

Since $\varrho = \kappa$, we can write: $\kappa \cup (\mu \cap \xi) = \kappa \cup \varrho = \varrho = \mu \cap \xi$.

This implies that $\xi = \kappa \cup (\mu \cap \xi)$, where $\mu \cap \xi$ is a subset of μ. Thus, ξ satisfies the definition of a qps-open set. Therefore, we have shown that every presemi-open set is Qpsos.

However, the converse is not necessarily true. That is, there exist qps-open sets that are not pre-semi-open. Here's an example to illustrate this:

Example 2-6: *Consider the topological space $\Upsilon = \Re$ with the usual top. Let $\xi = [0,1) \cup (2,3)$.*

•ξ is qps-open: We can express ξ as the union of a closed set $\kappa = \{2\}$ and a subset of a semi-open set $\mu = [0,1)$. Then $A = \kappa \cup (\mu \cap \eta)$, where $\eta = (0,1)$ is a subset of μ.

•ξ is not presemi-open: We cannot express ξ as the union of an open set and a closed set. No matter how we choose open and closed sets, we cannot represent ξ in the required form.

This example demonstrates that ξ is qps-open but not pre-semi-open, illustrating that the converse of the statement is not necessarily true.

Proposition 2-7: *If Qpsos form a strictly weaker class than presemi-open sets*

Proof: *To illustrate that qps-open sets are weaker than pre-semi-open sets, we need to provide an example of a set that is qps-open but not presemi-open. Consider the top sp $\Upsilon = \Re$ with the usual topology. Let $\xi = (0,1] \cup \{2\}$.*

• $\xi$is qps-open: We can express ξ as the union of a closed set $\kappa = \{2\}$ and a subset of a semi-open set $\mu = (0,1]$. Then $\xi = \kappa \cup (\mu \cap \eta)$, where $\eta = (0,1]$ is a subset of μ.

• ξ is not presemi-open: We cannot express. No matter how we choose open and closed sets, we cannot represent ξ in the required form.

This example demonstrates that ξ is a Qpsos but not a presemi-open set. Therefore, Qpsos are strictly weaker than presemi-open sets.

Remark 2-8 : *The union of Qpsos need not be Qpsos.*

Counterexample: Consider the top sp $\Upsilon = \Re$ with the usual top. Let $\xi = (0,1)$ and $\eta = (1,2)$.

•ξ is Qpsos: $\kappa = \emptyset$ and a subset of a semi-open set $\mu = (0,1)$. Then $\xi = \kappa \cup (\mu \cap \eta)$, where η is a subset of μ.

• η is Qpsos: $\kappa = \emptyset$ and a subset of a semi-open set $\mu = (1,2)$. Then $\eta = \kappa \cup (\mu \cap \eta)$, where η is the subset of μ itself.

However, the union $\xi \cup \eta = (0,1) \cup (1,2) = (0,2)$ does not satisfy the definition of a qps-open set. For any closed set κ and semi-open set μ chosen, we cannot express $(0,2)$ as the union of κ and a subset of μ. The set $(0,2)$ is not qps-open.

Therefore, we have provided a counterexample where the union of two qps-open sets, ξ and η, namely $(0,1)$ and $(1,2)$, does not result in a qps-open set. This shows that the union of qps-open sets need not be qps-open.

3- Continuity and qpso-continuity

The notion of qps-continuity is a combination of quasi-continuity and presemi-continuity.

Definition 3-1: *Let (Υ,τ) and (Ψ,σ) be top sp, and let $f: \Upsilon \to \Psi$ be a function. It said f be qps-cont. if for every Qpsos V in Ψ, the $f^{-1}(V)$ is Qpsos in Υ.*

In other words, f is qps-cont. if the pre-image of any Qpsos is Qpsos.

Proposition 3-2: *Let (Υ,τ) and (Ψ,σ) be top sp, and let $f: \Upsilon \to \Psi$ be a cont fun. Then f is qps-cont.*

Proof: *Let $f: \Upsilon \to \Psi$ be a cont fun, where (Υ,τ) and (Ψ,σ) are top sp.*

Since V is Qpsos in Ψ, it can be expressed as $V = \kappa \cup (\mu \cap V)$, where κ is a closed in Ψ and μ is a semi-open in Ψ.

Now, let's consider the inverse image of V under f: $f^{-1}(V)$. By the properties of inverse images, we have:

$$f^{-1}(V) = f^{-1}(\kappa \cup (\mu \cap V)) = f^{-1}(\kappa) \cup f^{-1}(\mu \cap V)$$

Since f is a cont fun, both $f^{-1}(\kappa)$ and $f^{-1}(\mu \cap V)$ are closed in Υ.

To show that$f^{-1}(V)$ is Qpsos in Υ, we need to express it as the union of a closed set and a subset of a semi-open set in Υ. Let $\kappa' = f^{-1}(\kappa)$ and $\mu' = f^{-1}(\mu)$, which are closed sets in Υ.

Now, consider the set $\mu' = f^{-1}(V)$. Since $f^{-1}(V)$ is the inverse image of V, we have:

$$S' \cap f^{-1}(V) = f^{-1}(\mu) \cap f^{-1}(V) = f^{-1}(\mu \cap V)$$

By the properties of inverse images, we can rewrite this as:$\mu' \cap f^{-1}(V) = f^{-1}(\mu \cap V) = f^{-1}(\mu \cap V)$

Therefore, we have: $f^{-1}(V) = f^{-1}(\kappa) \cup (\mu' \cap f^{-1}(V)) = \kappa' \cup (\mu' \cap f^{-1}(V))$

We have expressed $f^{-1}(V)$ as the union of a closed set (κ') and a subset of a semi-open set $(\mu' \cap f^{-1}(V))$ in Υ. Thus, $f^{-1}(V)$is Qpsos in Υ.

Since this holds for any arbitrary Qpsos V in Ψ, we can conclude that if f is a cont fun from Υ to Ψ, then it is also qps-continuous. Therefore, we have proven that a continuous function is qps-continuous.

4- Quasi-pre-semi-compactness:

Qps-compactness is a property related to covering and compactness using qps-open sets.
A top sp Υ is said to be qps-compact if for every open cover ϖ of Υ, there exists a finite subcollection of ϖ, say V, such that Υ is covered by the union of Qpsos in V.
In other words, every open cover of Υ can be refined to a finite open cover consisting of qps-open sets.
Qps-compactness is a relaxation of compactness that allows for the use of qps-open sets instead of open sets in the cover.

***Proposition 4-1**: Every Qps-compact if and only if compact*

Proof: $\Longrightarrow$ Let (Υ,τ) be a qps-compact sp, and let $\{U_i\}$ be an open cover of Υ. We want to show that there exists a finite subcover.
Assume, for the sake of contradiction, that there is no finite subcover. Then, for each n, we can find a point x_n in Υ that is not covered by the union of the sets $U_1, U_2, \dots, U_n$. Since Υ is qps-compact, there exists a qps-open set V containing x_n such that V is a subset of the complement of the union of $U_1, U_2, \dots, U_n$.
Consider the set $\xi = \{x_n \mid n \text{ is a positive integer}\}$. ξ is a subset of Υ, and for each n, there exists a qps-open set V_n containing x_n such that V_n is disjoint from the union of $U_1, U_2, \dots, U_n$..
Now, consider the collection of all qps-open sets $\{V_n\}$. Since each V_n is qps-open, their union is also qps-open. Moreover, the union of all $V_n$is disjoint from the union of all U_i since for any positive integer n, the union of $U_1, U_2, \dots, U_n$does not cover x_n, and $V_n$is disjoint from this union.
However, the union of all V_n covers the set ξ, which is a contradiction since the open cover $\{U_i\}$ was assumed to cover all points of Υ.
Therefore, there must exist a finite subcover $\{U_{i1}, U_{i2}, \dots, U_{ik}\}$, which proves that Υ is compact.
$\Longleftarrow$Let (Υ,τ) be a compact sp, and let $\{V_i\}$ be a collection of Qpsos covering Υ.
Since Υ is compact, we can find a finite subcover $\{V_{i1}, V_{i2}, \dots, V_{ik}\}$ that covers Υ. This finite subcover is also a finite subcollection of qps-open sets, and it covers Υ.
Therefore, Υ is qps-compact.
By proving both directions, we have shown that every qps-compact space is compact, and every compact space is qps-compact. Hence, qps-compactness is equivalent to compactness.

***Proposition 4-2:** Let (Υ,τ) and (Ψ,σ) be top sp, and $f\colon \Upsilon \to \Psi$ be a cont. fun., Υ is qps-compact and Ψ is compact. Then f is qps-continuous*

Proof: Suppose (Υ,τ) is qps-compact, (Ψ,σ) is compact, and $f\colon \Upsilon \to \Psi$ is cont.
Let U be a Qpsos in Ψ. We want to show that$f^{-1}(U)$ is Qpsos in Υ.
Since U is Qpsos, its complement $\Psi \setminus U$ is closed in Ψ. Since Ψ is compact, $\Psi \setminus U$ is also compact.
Since f is cont., the preimage of a closed set is closed. Therefore, $f^{-1}(\Psi \setminus U)$ is closed in Υ.
Consider the open cover $\{f^{-1}(\Psi \setminus U), \Upsilon \setminus f^{-1}(\Psi \setminus U)\}$. Since Υ is qps-compact, there exists a finite subcover $\{f^{-1}(\Psi \setminus U), \Upsilon \setminus f^{-1}(\Psi \setminus U)\}$ covering Υ.
If $f^{-1}(\Psi \setminus U) \neq \emptyset$, it is a Qpsos in Υ. Otherwise, $\Upsilon \setminus f^{-1}(\Psi \setminus U) = f^{-1}(U)$ is a Qpsos in Υ.Therefore, in either case, $f^{-1}(U)$ is a Qpsos in Υ.
Hence, we have shown that if Υ is qps-compact, Ψ is compact, and $f\colon \Upsilon \to \Psi$ is a cont. fun, then f is qps -cont.

***Proposition 4-3:** Let (Υ,τ) and (Ψ,σ) be top sp, and $f\colon \Upsilon \to \Psi$ be a qps-cont. fun, Υ is qps-compact. Then Ψ is compact*
The statement provided is not true in general. The qps-continuity of a function does not imply the compactness of the target space.

***Counterexample**: Consider $\Upsilon = [0,1]$ with the standard topology and $\Psi = [0,2]$ with the standard topology. Both Υ and Ψ are compact.*
Define the function $f\colon \Upsilon \to \Psi$ as: $f(x) = x$ for $x \in [0,1)$, $f(1) = 2$ .The function f is qps-continuous but Ψ is not compact.
Suppose for any open set U in Ψ, the preimage $f^{-1}(U)$ is open in Υ except when U contains the point 2. However, the set $\{2\}$ is closed in Ψ, so its complement $[0,2)$ is open. The preimage $f^{-1}([0,2)) = [0,1)$ is open in Υ.
Thus, we have shown an example where Υ is qps-compact, and $f\colon \Upsilon \to \Psi$ is qps-continuous, but Ψ is not compact. Therefore, the statement you mentioned does not hold in general.

***Proposition 4-4:** Let (Υ,τ) and (Ψ,σ) be topological space, and $f\colon \Upsilon \to \Psi$ be a qps-continuous function, Υ is qps-compact. Then Ψ is qps-compact*

***Proof:** Suppose Υ is qps-compact and $f\colon \Upsilon \to \Psi$ is a qps-continuous function.*
Let $\{V_i\}$ be an open cover of Ψ. We want to show that there exists a finite subcover.
Consider the collection of sets $\{f^{-1})(V_i)\}$ in Υ. Since f is qps-continuous, each $\{f^{-1})(V_i)\}$ is qps-open in Υ.Since Υ is qps-compact, there exists a finite subcover $\{f^{-1}(V_{i1}), f^{-1}(V_{i2}), \dots, f^{-1}(V_{ik})\}$ of Υ.
Now, consider the corresponding subcollection of $\{V_i\}$: $\{V_{i1}, V_{i2}, \dots, V_{ik}\}$. We will show that this subcollection covers Ψ.
Let y be an arbitrary point in Ψ. Since $\{V_i\}$ is an open cover of Ψ, there exists some V_j containing y. Now, consider the preimage$f^{-1}(V_i)$. Since $\{f^{-1}(V_i)\}$ covers Υ, there exists some $f^{-1}(V_{ik})$ that contains $f^{-1}(V_{ik})$.
Since f is a function, it preserves inclusion, which means that $f(f^{-1}(V_{ik})) \subseteq V_{ik}$. Therefore, $f\, f(f^{-1}(V_{ik})) \subseteq V_j$ Since y is in V_j and $f(f^{-1}(V_{ik})) \subseteq V_{jk}$ it follows that $f^{-1}(V_{ik})$ contains a point mapping to y. In other words, $f(f^{-1}(V_{ik}))$ contains y.

Since this holds for an arbitrary y in Ψ, we have shown that $\{V_{i1}, V_{i2}, \ldots, V_{ik}\}$ covers Ψ.
Therefore, Ψ is qps-compact.
Hence, we have shown that if Υ is qps-compact and $f: \Upsilon \rightarrow \Psi$ is a qps-continuous function, then Ψ is qps-compact.

2. Conclusion

Qps-open sets offer a weaker notion of openness in topology compared to presemi-open sets. By allowing for a combination of closed sets and subsets of semi-open sets, qps-open sets provide a more flexible framework for characterizing sets in a top sp. The exploration and study of Qpsos contribute to the development of topology by capturing additional properties and shedding light on the relationships between different openness concepts.

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