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Some Results of Cone Normed Spaces

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***Abstract**: The notion of cone normed spaces was first presented by M. Gordji, M. Ramezani, H. Khodaei and H. Baghani [1]. In this paper, various results related cone normed space such as intersection, union of cone normed space is a cone normed space is introduced, so product of two complete space of cone normed space is complete. Finally we defined equivalent cone normed space.*

Keywords: cone norm, cone complete, cone continuous, cone equivalent norm

1. Introduction and Preliminaries

The main propositions of complete cone normed spaces proved by [7-8]. Completion of cone normed spaces presented by Sonmez and Cakalli [9]. Moreover, fixed point theorems in cone Banach spaces studied by [3], and some properties of cone Banach spaces studied by [10]. The main properties of cone normed spaces studied by [6]. In this paper, results related cone normed space such as intersection, union of cone normed space is cone normed space and the product of two complete space of cone normed space is complete are proved.

We say a nonempty subset C of a Banach space D is cone if and only if the following properties are fulfilled:

1) C is closed,
2) $ax + by \in C$ for all $x, y \in C$ and $a, b \geq 0$,
3) $C \cap (-C) = \{0\}$.

For a assumed cone $\subseteq D$, we can describe a partial ordering $\leq$ with C as $x \leq y$ if and only if $y - x \in C$. We shall write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$ while $x \ll y$ will stands for $y - x \in intC$, where $intC$ symbolized the interior of .

The cone C is normal if there is number $M > 0$ such that

$$0 \leq x \leq y \iff \|x\| \leq M\|y\| \quad \text{for all } x, y \in D.$$

We say normal constant of C if the least positive number filling the above inequality [4], obvious $M \geq 1$. Assume that C is cone in a real Banach space D with $\leq$ is partial ordering under to C and int$C \neq \varphi$.

Now, We recall the basic definitions and information which are needed in our work.

Definition 1.1 [1]

A cone norm if a mapping $\|.\|_c: C \to D$ where C is a vector space over the field R such that

1) $\|x\|_c \geq 0 \ \forall \ x \in C$,
2) $\|x\|_c = 0$ if and only if $x = 0$
3) $\|\alpha x\|_c = |\alpha|\|x\| \ \forall \ x \in C, \alpha \in R,$
4) $\|x + y\|_c \leq \|x\|_c + \|y\|_c \ \forall x, y \in C$.

Definition 1.2 [1]

Let C be a vector space over the field R and $\|.\|_c$ is a cone norm on C. Then $(C, \|.\|_c)$ is said to be a cone normed space (CNSp). It is obvious every normed space is a cone normed space by setting $E := R, P := [0. \infty)$.

Example 1.3 [6]

Let $E = R^2$ and $P = \{(x, y): x \geq 0, y \geq 0\}$. Then P is the positive cone of R^2 with partial order:

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also it is clear that int$P = \{(x, y): x > 0, y > 0\} \neq \emptyset$. Let $X = R^2$ and define $\|.\|_c: X \to E$ by $\|(x_1, y_1)\|_c = (\alpha|x_1|, \beta|y_1|)$ where $\alpha > 0, \beta > 0$. $(X, \|.\|_c)$ is an CNSp.

Let $(X, \|\cdot\|_c)$ be an CNSp. Set $d(x, y) = \|x - y\|_c$. Then (X, d) is a cone metric space, d is said to be a cone metric of cone norm $\|\cdot\|_c$.[5]

Definition 1.4 [1]

A sequence $\{x_n\}_n$ in C is said to converge to a point $x \in C$ if for every $\epsilon \in D$ with $\epsilon \gg 0$ there is a positive integer n_0 such that $\|x_n - x\|_c \ll \epsilon$, $\forall n \geq n_0$.It will be denoted by $\lim_{n\to\infty} x_n = x \ or \ x_n \to x \ \ as \ n \to \infty$.

Definition 1.5 [1]

A sequence $\{x_n\}_n$ in C is said to be Cauchy sequence if for every $\epsilon \in D$ with $\epsilon \gg 0$ there is a positive integer n_0 such that $\|x_n - x_m\|_c \ll \epsilon$, $\forall\, m, n \geq n_0$.

Definition 1.6 [2]

We say that $(C, \|.\|_c)$ is a cone complete if each Cauchy sequence in C is converges.

Definition 1.7 [1]

The set $B_c(x, r) = \{y \in C: \|y - x\| < r\}$, $r \in R$ is said to be an open ball with center x and radius r.

Definition 1.8 [4]

If $(C, \|.\|_c)$, $(X, \|.\|_c)$ are CNSp and $f: C \to X$ is a function, then f is called cone continuous at $x_0 \in C$, if for each $\epsilon \in E$ with $\epsilon \gg 0, \exists \delta \in E$ with $\delta \gg 0$ whenever $\|x - x_0\|_c \ll \delta$ then $\|f(x) - f(x_0)\|_c \ll \epsilon$.

2. The Main Results

In this section we shall study cone normed space such as intersection, union of cone normed space is a cone normed space, so product of two complete space of cone normed space is complete.

Theorem 2.1

If $(C, \|\cdot\|_c)$ is an CNSp such that $S, T \subseteq C$. Then $(S \cap T, \|\cdot\|_c)$ is an CNSp.

Proof

(i) Let $x \in S, y \in T$ such that $\|x\|_c \geq \theta$ and $\|y\|_c \geq \theta$, then $\|x\|_c \cap \|y\|_c \geq \theta \quad \forall x, y \in X$;

(ii) $\|x\|_c \cap \|y\|_c = \theta \iff \|x\|_c = \theta \ and \ \|y\|_c = \theta \iff x = \theta_X \ and \ y = \theta_X$;

(iii) $\|\alpha x\|_c \cap \|\alpha y\|_c = |\alpha|\|x\|_c \cap |\alpha|\|y\|_c = |\alpha|(\|x\|_c \cap \|y\|_c)$;

(iv) let $x, y \in S, z, w \in T$, then $\|x + y\|_c \cap \|z + w\|_c \leq (\|x\|_c + \|y\|_c) \cap (\|z\|_c + \|w\|_c)$

$$\leq (\|x\|_c \cap \|z\|_c) + (\|y\|_c + \|w\|_c).$$

Therefore $(S \cap T, \|\cdot\|_c)$ is a cone normed space.

Remark 2.2

It is obvious that the union of finite family of subsets of an CNSp is an CNSp.

Theorem 2.3

The product of two cone complete spaces of $(C, \|\cdot\|_c)$ is also cone complete.

Proof

Let $(X, \|\cdot\|_c)$ and $(Y, \|\cdot\|_c)$ be two cone complete space such that every Cauchy sequence in C converges, that is for each $\epsilon \in D$ with $\epsilon \gg 0$ there is a positive integer n_0 whenever $\|x_n - x_v\|_c \ll \epsilon$ and $\|y_m - y_s\|_c \ll \epsilon$ in X and Y respectively, $\forall\, m, n \geq n_0$, then $\|x_n - x\|_c \ll \frac{\epsilon}{2}$ and $\|y_m - y\|_c \ll \frac{\epsilon}{2}$.

Now

$$\|(x_n, y_m) - (x, y)\|_c = \|(x_n - x), (y_m - y)\|_c$$
$$= \|x_n - x\|_c + \|y_m - y\|_c \ll \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This complete the proof.

Definition 2.4

A cone norms $\|\cdot\|_{c_1}$and $\|\cdot\|_{c_2}$ on a vector space X are called equivalent (denoted by $\|\cdot\|_{c_1} \sim \|\cdot\|_{c_2}$) if $\exists a, b > 0$ and $a\|x\|_{c_1} \le \|x\|_{c_2} \le b\|x\|_{c_1} \forall x \in X$.

Example 2.5

Let $\|x\|_{c_1} = \sum_{i=1}^{2} |x_i|$ and $\|x\|_{c_2} = [\sum_{i=1}^{2} |x_i|^2]^{1/2} \forall\, x \in R^2$. Then $\|x\|_{c_1} \sim \|x\|_{c_2}$

Solve

By Cauchy -Schwarz inequality $\sum_{i=1}^{2} |x_i y_i| \le [\sum_{i=1}^{2} |x_i|^2]^{1/2} [\sum_{i=1}^{2} |y_i|^2]^{1/2}$.

Choose $y_i = 1 \;\; \forall\, i = 1,2$, we have $\sum_{i=1}^{2} |x_i| \le [\sum_{i=1}^{2} |x_i|^2]^{1/2} [\sum_{i=1}^{2} 1]^{1/2}$. Then

$\|x\|_{c_1} \le \sqrt{2}\|x\|_{c_2}$, hence $\frac{1}{\sqrt{2}}\|x\|_{c_1} \le \|x\|_{c_2}$. Since $\|x\|_{c_1} \ge \|x\|_{c_2}$, thus

$\frac{1}{\sqrt{2}}\|x\|_{c_1} \le \|x\|_{c_2} \le \|x\|_{c_1}$. Therefore $\|x\|_{c_1} \sim \|x\|_{c_2}$.

Lemma 2.6

Let $(X, \|\cdot\|_{c_1})$ and $(X, \|\cdot\|_c)$ be an CNSp such that $\|\cdot\|_{c_1} \sim \|\cdot\|_{c_2}$. If $\|x_n - x\|_c \to \theta$ in $(X, \|\cdot\|_{c_1})$, then $\|x_n - x\|_c \to \theta$ in $(X, \|\cdot\|_{c_2})$

Proof

The proof is clear.

Lemma 2.7

Let $x \in C$. Then $B_c(x, r) = rB_c(x, 1)$

Proof

Let $y \in B_c(x, r)$. Assume $-x = z$, hence $y - x = r\frac{1}{r}z \Rightarrow y - x = rw \; for \; w \in B_c(0,1)$. So $\|y - x\| = |r|\|w\|$, since $\|y - x\| < r$. We have $|r|\|w\| < r$, consequently $\|w\| < 1$ $(w \in B_c(0,1))$. Therefore $y = rw \in rB_c(x, 1)$.

Conversely, $y \in rB_c(x, 1)$. Hence $y = rz \;\; for \; z \in B_c(x, 1)$, we have $y = x + rz \Longrightarrow y - x = rz \Longrightarrow \|y - x\| = |r|\|z\|$, so $\|y - x\| < r$. Then $y \in B_c(x, r)$.

Theorem 2.8

If $x_n \to x$, $y_n \to y$ in an CNSp, then

(1) $\|x_n\|_c \to \|x\|_c$

(2) $\|x_n - y_n\|_c \to \|x - y\|_c$

Proof

(1) $|\|x_n\|_c - \|x\|_c| \le \|x_n - x\|_c$

Since $x_n \to x$ then $\|x_n - x\|_c \to \theta$. So $\|x_n\|_c - \|x\|_c \to \theta$, thus $\|x_n\|_c \to \|x\|_c$.

(2) $|\|x_n - y_n\|_c - \|x - y\|_c| \le \|(x_n - y_n) - (x - y)\|_c \le \|x_n - x\|_c + \|y_n - y\|_c$

Since $x_n \to x$, $y_n \to y$, then $\|x_n - x\|_c \to \theta, \|y_n - y\|_c \to \theta$. We have $|\|x_n - y_n\|_c - \|x - y\|_c| \to \theta$, then $\|x_n - y_n\|_c \to \|x - y\|_c$.

Proposition 2.9

The cone norm is cone continuous function.

Proof

Let $\epsilon \in E$ with $\epsilon \gg 0$ there exists $\delta \in E$ with $\delta \gg 0$ such that $\|x - x_0\|_c \ll \delta = \epsilon$.

$|f(x) - f(x_0)| = |\|x\|_c - \|x_0\|_c| \ll \|x - x_0\|_c \ll \epsilon$. Then f is a cone continuous.

Proposition 2.10

If $(X, \|\cdot\|_c)$ is an CNSp

1) The mapping f: X × X → X which defined by $f(x, y) = x + y$ is continuous.

2) The mapping f: X × X → X which defined by $g(\alpha, x) = \alpha x$ is continuous.

Proof

1) Since $\|(x - x_0)\|_c \ll \epsilon$ *and* $\|(y - y_0)\|_c \ll \epsilon$ $(\epsilon \gg 0)$, then

$$\begin{aligned}\|f(x, y) - f(x_0, y_0)\|_c &= \|(x + y) - (x_0 + y_0)\|_c \\ &= \|(x - x_0) + (y - y_0)\|_c \\ &\leq \|(x - x_0)\|_c + \|(y - y_0)\|_c \ll 2\epsilon\end{aligned}$$

Thus, f is continuous.

2) Let $\|(x - x_0)\|_c \ll \epsilon$ *and* $|\alpha - \alpha_0| \ll \epsilon$ $(\epsilon \gg 0)$. Then

$$\begin{aligned}\|g(\alpha, x) - g(\alpha_0, x_0)\|_c &= \|\alpha x - \alpha_0 x_0\|_c \\ &= \|\alpha x - \alpha x_0 + \alpha x_0 - \alpha_0 x_0\|_c \\ &\leq |\alpha|\|(x - x_0)\|_c + |\alpha - \alpha_0|\|x_0\|_c \ll |\alpha|\epsilon + \epsilon\|x_0\|_c\end{aligned}$$

Hence, g is continuous.

Theorem 2.11

If $f: X \to Y$ and $g: Y \to Z$ are two a cone continuous map. Then the composition function $g \circ f: X \to Z$ is also cone continuous.

Proof

Let $\epsilon \in E$ with $\epsilon \gg 0$. Since g is continuous at $f(x_0) \in Y$, there exists $\delta \in E$ with $\delta \gg 0$ such that $\|x - x_0\|_c \ll \delta \Longrightarrow \|g(y) - g(f(x_0))\|_c \ll \epsilon$.

Since f is continuous, there exists $\delta \in E$ with $\delta \gg 0$ such that $\|x - x_0\|_c \ll \delta$, thus $\|f(x) - f(x_0)\|_c \ll \epsilon$. Then $\|g(f(x)) - g(f(x_0))\|_c \ll \epsilon$, so $g \circ f$ is cone continuous.

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