International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 8 Issue 5 May - 2024, Pages: 17-23

Inner Ideals Of The Real Five-Dimensional Lie Algebras With Two-Dimensional Derived $L^{'}$ That Is Not In The Center But It Has A Common Element

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Abstract: *Inner ideal of the five-dimensional non-commutative Lie alge- bras over the real fields with two-dimensional derived were classified. It is proved that one, two, three and four-dimensional inner ideals are exist in every five-dimensional Lie algebra. It is also proved that five-dimensional Lie algebras contain inner ideals which are neither ideals nor sub-algebras.*

Keywords: Inner ideal, Lie algebra, five-dimensional Lie algebra

1. INTRODUCTION

Georgia Benkart, an American scientist, introduced the notion of inner ideals (see [3]) in 1976. According to Benkart's definition, one can say that an inner ideal is a subspace V of a Lie algebra L, with the property $[V, [V, L]] \subseteq V$, where $[V, [V, L]] = \text{span}\ \{[v_1, [v_2, \ell]] : v_1, v_2 \in V, \ell \in L\}$

An inner ideal V is called commutative if $[V, V] = 0$. She demonstrated that a strong correlation exist between elements in Lie algebras that are ad-nilpotents with inner ideal [4]. An accomplishment of Benkart's results is done in [5] by Benkart and Fernandiz Lopes and a generalization of there results is done in [6] by Brox, Fernandez Lopez and Gomez Lozano in 2016 to the case of centrally closed prime rings with involution of characteristic not 2, 3 or 5.

It can be seen from [9] and [11] that in Lie algebra the role of inner ideals is equivalent to that of the one sided ideals in associative algebras. Therefore, Artin's theory can be generalized if one takes into account the inner ideals of Lie algebras. It was proved in [8] that an Artinian Lie algebra is a Lie algebra that has the property that every decreasing inner ideals chain must be terminates. In [1, Proposition 2], it was proved that every one-sided ideal of the finite dimensional associative algebra A admits Levi decomposition and can be generated by an idempotent if some minimal conditions are met. The same results was obtained for inner ideals by Baranov and Shlaka in [2], where they showed that every inner ideal of the Lie algebra [A, A] admits Levi decomposition and can be generated by idempotent pair (if satisfied some minimal conditions). These results were recently Generalized in [14] for the case of a sub-algebra of finite dimensional algebra. Further generalization is done in [10] and [16] for the infinite dimensional Lie sub-algebras of associative algebras. Abelian non-Jordan Lie inner ideals we also been studied in 2022 (see [15] for more details). Further motivation for studying inner ideals comes from [9], where Fern´andez L´opez et al showed that when L is an arbitrary non- degenerate Lie algebras over an abelian ring F together with two and three convertible, then for every nonzero commutative inner ideal V of finite length of L is complemented by an commutative inner ideal [9].

The classification of the real five-dimensional Lie algebras is given by Schöbel in [13]. He classified them in terms of the derived sub-algebra of these Lie algebras. Keep in mind that a derived sub-algebra $L^{'}$ of L is the set $L^{'} = [L, L] = \text{span}\ \{[\ell_1, \ell_2] \mid \ell_1, \ell_2 \in L\}$.

In [12] Saeed and Shlaka studied inner ideals of the four-dimensional Lie algebras over the real fields with two-dimensional derived. They proved that one, two and three- dimensional non-trivial inner ideals exist in every four-dimensional Lie algebra with 2-dimensional derived. Prior to that (see [17]) they classify inner ideals of the two and three-dimensional Lie algebras.

In this paper, we use techniques similar to [12] to study inner ideals of the real five-dimensional Lie algebras with 2-dimensional derived. Suppose that L is a five- dimensional Lie algebra over the real field with 1-dimensional derived. If L commu- tative, then it is easy to see that every 1, 2, 3 and 4-dimensional subspace of L is an inner ideal. Suppose now that L is non-commutative, then we get the following results, which is one of our main results:

Theorem 1.1: Let L be a five dimensional Lie algebra over the real field R with 2- dimensional derived $L^{'}$. Then L contains a commutative and non-commutative I-ideal.

Recall that if L is 5-dimensional with 2-dimensional derived, then by Theorem2.6 and Theorem2.7 L is either L_1 or L_2. Thus, to prove the theorem we need to consider all of the cases.

2. PRELIMINARIES

Definition 2.1 [7]: Let L be a vector space over any field F with a bilinear form $L \times L \to L$, where $(\ell_1, \ell_2) \to [\ell_1, \ell_2]$, for all $\ell_1, \ell_2 \in L$. Then L is called a Lie algebra over F , if the following conditions are satisfied:

(1) $[\ell_1, \ell_2] = 0$ for all $\ell_1, \ell_2 \in L$.

(2) $[\ell_1, [\ell_2, \ell_3]] + [\ell_2, [\ell_3, \ell_1]] + [\ell_3, [\ell_1, \ell_2]] = 0$ for all $\ell_1, \ell_2, \ell_3 \in L$.

Definition 2.2 [7]: The subspace B of a Lie algebra L is said to be a Lie sub-algebra of L, if $[b_1, b_2] \in B$ for all $b_1, b_2 \in B$.

Definition 2.3 [7]: The derived of a Lie algebra L is the set $L' = \text{span}\{[a, b] \mid a, b \in L\}$, where L' is a Lie sub algebra of L.

Definition 2.4 [7]: The center of L is the set $Z = \{x \in L \mid [x, y] = 0, \forall y \in L\}$.

Definition 2.5 [2]: Let V be a subspace of L. Then V is said to be an inner ideal of L when $[V, [V, L]] \subseteq V$. We denote by I-ideal to be an inner ideal of L. The inner ideal V is said to be commutative if $[V, V] = 0$.

Note that in every Lie algebra L, we have L, $\{0\}$ are inner ideals of L called the trivial inner ideals. Recall that every ideal I of L is inner ideal, because $[I, [I, L]] = [I, I] \subseteq I$, but the inverse is not true. Since L is five-dimensional, the dimension of L' may be 1, 2, 3, 4 or 5. In [13] Schöbel classified the real n-dimensional Lie algebras by relating the dimensional of L'. For the dimensional of L' is 2, we have the following result, for the proof see [13, Theorem 1].

Theorem 2.6 [13]: Suppose that L is a real n-dimensional Lie algebra with a two- dimensional derived L', such that $L' \not\subseteq Z$ Then $L = L_4 \oplus Z_1$, where L_4 is a 4-dimensional real Lie algebra with 2-dimensional derived algebra and $L' \not\subseteq Z_4$, Z_n is the n-dimensional center of L.

Theorem 2.7 [13]: Suppose that L is a real 4-dimensional Lie algebra with a two- dimensional derived L', such that $\dim (L' \cap Z) = 1$, and let $\{x_1, x_2, x_3, x_4\}$ be a basis of L. Then L is one of the following two standard forms.

$L_1 : [x_2, x_4] = x_1, [x_3, x_4] = x_2$, and otherwise is zero.

$L_2 : [x_2, x_4] = x_2, [x_3, x_4] = x_1$, and otherwise is zero.

3. Inner Ideals of The Five Dimensional Lie Algebra

Throughout this section, we prove some results related to inner ideal of the 5- dimensional real Lie algebra with 2-dimensional derived. Our aim is to prove the following theorem.

Proposition 3.1: Suppose that $L = L_1$ and $L' \subseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal which is not ideal.
2. L contains a 2-dimensional commutative I-ideal which is not ideal .
3. L contains a 3-dimensional commutative I-ideal which is not ideal.
4. L contains a 3-dimensional non-commutative I-ideal which is not ideal.
5. L contains a 3-dimensional non-commutative I-ideal which is not sub-algebra.
6. L contains a 4-dimensional commutative I-ideal.
7. L contains a 4-dimensional non-commutative I-ideal which is not ideal.
8. L contains a 4-dimensional non-commutative I-ideal which is not sub-algebra.

Proof: By Theorems 2.6 and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_1 with the Lie multiplication $[x_2, x_4] = x_1$, $[x_3, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2$, $y = \alpha_2 x_2$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]] = [\alpha_1 x_2, \alpha_2\beta_4 x_1] = 0 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V,$
V is not ideal of L, as required.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 z$, $y = \alpha_3 x_2 + \alpha_4 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_2 + \alpha_4 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 z, \alpha_3\beta_4 x_1] = 0 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_2 + \alpha_2 z, \alpha_3 x_2 + \alpha_4 z] = 0$ Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_2 + \alpha_2 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V$,

Therefore, V is not ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_5\beta_4 x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z] = 0$

Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_2\beta_4 x_2 \notin V$,

Therefore, V is not ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_5\beta_4 x_2 - \alpha_6\beta_2 x_1 - \alpha_6\beta_3 x_2]$

$= (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_3)x_1 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is a non-commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4]$

$= (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$

$= (\alpha_2\beta_4 - \alpha_3\beta_3)x_2 - \alpha_3\beta_2 x_1 \notin V$,

Therefore, V is not ideal of L.

5) By (4) we note that for all $x, y \in V$, then $[x, y] \notin V$. So V is not sub-algebra of L.

6) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_6\beta_4 x_1 + \alpha_7\beta_4 x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z] = 0$

Therefore, V is a commutative I-ideal of L, as required.

7) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_3, x_4, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z$, $y = \alpha_5 x_1 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_6\beta_4 x_2 - \alpha_7\beta_2 x_1 - \alpha_7\beta_3 x_2]$

$= (-\alpha_3\alpha_6\beta_4 + \alpha_3\alpha_7\beta_3)x_1 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5 x_1 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z]$

$= (\alpha_2\alpha_7 - \alpha_3\alpha_6)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$

$= (\alpha_2\beta_4 - \alpha_3\beta_3)x_2 - \alpha_3\beta_2 x_1 \notin V$

Therefore, V is not ideal of L.

8) By (7) we note that for all $x, y \in V$, then $[x, y] \notin V$. So V is not sub-algebra of L. □

Remark 3.2: Theorem 3.1, is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because $L = L_1$ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

Example 3.3: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_1 with the Lie mul- tiplication $[x_2, x_4] = x_1$, $[x_3, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_4$, $y = \alpha_2 x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_4, -\alpha_2\beta_2 x_1 - \alpha_2\beta_3 x_2] = \alpha_1\alpha_2\beta_3 x_1 \notin V,$
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_3 + \alpha_2 x_4$, $y = \alpha_3 x_3 + \alpha_4 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_3 + \alpha_4 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_3 + \alpha_2 x_4, \alpha_3\beta_4 x_2 - \alpha_4\beta_2 x_1 - \alpha_4\beta_3 x_2] = (-\alpha_2\alpha_3\beta_4 + \alpha_2\alpha_4\beta_3)x_1 \notin V,$
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_2, x_4, z\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_4 + \alpha_3 z$, $y = \alpha_4 x_2 + \alpha_5 x_4 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_2 + \alpha_5 x_4 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 x_4 + \alpha_3 z, \alpha_4\beta_4 x_1 - \alpha_5\beta_2 x_1 - \alpha_5\beta_3 x_2] = \alpha_2\alpha_5\beta_3 x_1 \notin V,$
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace $V = \text{span}\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z$, $y = \alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5\beta_4 x_1 + \alpha_6\beta_4 x_2 - \alpha_7\beta_2 x_1 - \alpha_7\beta_3 x_2]$
$= (-\alpha_3\alpha_6\beta_4 + \alpha_3\alpha_7\beta_3)x_1 \notin V,$
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

Proposition 3.4: Suppose that $L = L_2$ and $L' \subseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal which is not ideal.
2. L contains a 2-dimensional commutative I-ideal which is not ideal .
3. L contains a 2-dimensional non-commutative I-ideal which is not ideal.
4. L contains a 3-dimensional commutative I-ideal which is not ideal.
5. L contains a 3-dimensional non-commutative I-ideal which is not ideal.
6. L contains a 3-dimensional non-commutative I-ideal which is not sub-algebra.
7. L contains a 4-dimensional commutative I-ideal.
8. L contains a 4-dimensional non-commutative I-ideal which is not ideal.
9. L contains a 4-dimensional non-commutative I-ideal which is not sub-algebra.

Proof: By Theorems 2.6 and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_2 with the Lie multiplication $[x_2, x_4] = x_2$,$[x_3, x_4] = x_1$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_3\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_3$, $y = \alpha_2 x_3$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]] = [\alpha_1 x_3, \alpha_2\beta_4 x_1] = 0 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V,$
V is not ideal of L, as required.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_3 + \alpha_2 z$, $y = \alpha_3 x_3 + \alpha_4 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_3 + \alpha_4 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_3 + \alpha_2 z, \alpha_3\beta_4 x_1] = 0 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is commutative. Since
$[x, y] = [\alpha_1 x_3 + \alpha_2 z, \alpha_3 x_3 + \alpha_4 z] = 0$. Therefore, V is a commutative I-ideal of L.
It remains to us show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_3 + \alpha_2 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V,$
Therefore, V is not ideal of L.

3) We claim that the 2-dimensional subspace $V = \text{span}\{x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_4$, $y = \alpha_3 x_2 + \alpha_4 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_2 + \alpha_4 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 x_4, \alpha_3\beta_4 x_2 - \alpha_4\beta_3 x_1] = -\alpha_2\alpha_3\beta_4 x_2 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is a non-commutative. Since
$[x, y] = [\alpha_1 x_2 + \alpha_2 x_4, \alpha_3 x_2 + \alpha_4 x_4] = (\alpha_1\alpha_4 - \alpha_2\alpha_3)x_2 \neq 0$ Therefore, V is a non-commutative I-ideal of L.
It remains to us show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_2 + \alpha_2 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$
$= (\alpha_1\beta_4 - \alpha_2\beta_2)x_2 - \alpha_2\beta_3 x_1 \notin V,$
Therefore, V is not ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 z$, $y = \alpha_4 x_2 + \alpha_5 x_3 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_2 + \alpha_5 x_3 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 z, \alpha_4\beta_4 x_2 + \alpha_5\beta_4 x_1] = 0 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is commutative. Since
$[x, y] = [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 z, \alpha_4 x_2 + \alpha_5 x_3 + \alpha_6 z] = 0$
Therefore, V is a commutative I-ideal of L.
It remains to us show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$
$= \alpha_1\beta_4 x_2 + \alpha_2\beta_4 x_1 \notin V,$
Therefore, V is not ideal of L.

5) We claim that the 3-dimensional subspace $V = \text{span}\{x_2, x_3, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4$, $y = \alpha_4 x_2 + \alpha_5 x_3 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_2 + \alpha_5 x_3 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_4\beta_4 x_2 + \alpha_5\beta_4 x_1 - \alpha_6\beta_2 x_2 - \alpha_6\beta_3 x_1]$
$= (-\alpha_3\alpha_4\beta_4 + \alpha_3\alpha_6\beta_2)x_2 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is a non-commutative. Since
$[x, y] = [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_4 x_2 + \alpha_5 x_3 + \alpha_6 x_4]$
$= (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_1 + (\alpha_1\alpha_6 - \alpha_3\alpha_4)x_2 \neq 0$
Therefore, V is a non-commutative I-ideal of L, as required.
It remains to us show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$
$= (\alpha_1\beta_4 - \alpha_3\beta_2)x_2 + (\alpha_2\beta_4 - \alpha_3\beta_3)x_1 \neq V,$
Therefore, V is not ideal of L.

6) By (5) we note that for all $x, y \in V$, then $[x, y] \notin V$. So V is not sub-algebra of L.

7) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_6\beta_4 x_2 + \alpha_7\beta_4 x_1] = 0 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is commutative. Since
$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z] = 0$
Therefore, V is a commutative I-ideal of L, as required.

8) We claim that the 4-dimensional subspace $V = \text{span}\{x_2, x_3, x_4, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_2 x_1 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z$, $y = \alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$
$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5\beta_4 x_2 + \alpha_6\beta_4 x_1 - \alpha_7\beta_2 x_2 - \alpha_7\beta_3 x_1]$
$= (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_7\beta_2)x_2 \in V,$
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
It remains to show that $[x, y] \neq 0$ Since
$[x, y] = [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z]$
$= (\alpha_2\alpha_7 - \alpha_3\alpha_6)x_1 + (\alpha_1\alpha_7 - \alpha_3\alpha_5)x_2 \neq 0$
Therefore, V is a non-commutative I-ideal of L, as required.
It remains to us show that V is not ideal. Since
$[x, \ell] = [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$

$= (-\alpha_3\beta_3 + \alpha_2\beta_4)x_1 + (\alpha_1\beta_4 - \alpha_3\beta_2)x_2 \notin V$

Therefore, V is not ideal of L.

9) By (8) we note that for all $x, y \in V$, then $[x, y] \notin V$. So V is not sub-algebra of L. □

Remark 3.5: Theorem 3.4, is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because $L = L_2$ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

Example 3.6. Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_2 with the Lie mul- tiplication $[x_2, x_4] = x_2, [x_3, x_4] = x_1$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace$V = \text{span}\ \{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_4$, $y = \alpha_2x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_4, -\alpha_2\beta_2x_2 - \alpha_2\beta_3x_1] = \alpha_1\alpha_2\beta_2x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace $V = \text{span}\ \{x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_3 + \alpha_2x_4$, $y = \alpha_3x_3 + \alpha_4x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3x_3 + \alpha_4x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_3 + \alpha_2x_4, \alpha_3\beta_4x_1 - \alpha_4\beta_2x_2 - \alpha_4\beta_3x_1] = \alpha_2\alpha_4\beta_2x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\ \{x_1, x_4, z\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_1 + \alpha_2x_4 + \alpha_3z$, $y = \alpha_4x_1 + \alpha_5x_4 + \alpha_6z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_4 + \alpha_6z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_4 + \alpha_3z, -\alpha_5\beta_2x_2 - \alpha_5\beta_3x_1] = \alpha_2\alpha_5\beta_2x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace $V = \text{span}\ \{x_1, x_3, x_4, z\}$ is not an I- ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_1 +\alpha_2x_3 +\alpha_3x_4 +\alpha_4z$, $y = \alpha_5x_1 +\alpha_6x_3 +\alpha_7x_4 +\alpha_8z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5x_1 + \alpha_6x_3 + \alpha_7x_4 + \alpha_8z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_3 + \alpha_3x_4 + \alpha_4z, \alpha_6\beta_4x_1 - \alpha_7\beta_2x_2 - \alpha_7\beta_3x_1] = \alpha_3\alpha_7\beta_2x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

Now we are ready to prove Theorem 1.1. Recall that L is either L_1 or L_2. We need to show that L contains a commutative and non-commutative I-ideal.

Proof for Theorem 1.1:

If $L = L_1$, by the Proposition 3.1, L contains a commutative and non-commutative I-ideal.

If $L = L_2$, by the Proposition 3.4, L contains a commutative and non-commutative I-ideal.

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