International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 8 Issue 6 June - 2024, Pages: 109-114

Survival Analysis of Lomax Distribution on Type III Censored Data with Maximum Likelihood Estimation and Newton Raphson Methods

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Abstract: *Survival analysis is a statistical technique used to test the durability and reliability of a component. Life time data obtained from a life test experiment is often in the form of type III censored data, which occurs when observations enter at different times and last for varying durations. In survival analysis, data is expected to follow a certain probability distribution. To determine the characteristics of a population, a point estimate of the probability distribution parameters is conducted. This study aims to obtain parameter estimators of the Lomax distribution on type III censored data with the Maximum Likelihood Estimation (MLE) and Newton Raphson methods. Application of parameter estimation results on post-heart surgery survival data in one of the Jakarta hospitals. The result of estimating the parameter* $\hat{\boldsymbol{\alpha}}$ *value in the post-heart surgery patient data is 1.552 and the result of estimating the parameter* $\hat{\boldsymbol{\beta}}$ *in the post-heart surgery patient data is 20.38. Based on these results, it can be concluded that the estimated probability of survival of a post-heart surgery patient for more than 49 days is 14.94% with a risk rate of 2.2%.*

Keywords— Survival Analysis; Lomax Distribution; Type III Censored Data; Maximum Likelihood Estimation; Newton Raphson.

1. INTRODUCTION

In the field of statistics that has developed rapidly with the discovery of analytical tools that can be used to analyze a problem, one of which is survival analysis. Survival analysis is a statistical method that is useful for analyzing data about the time of an event or for analyzing data about survival time (Kleinbaum & Klein, 2012). One of the objectives of survival analysis is to estimate and interpret the survival probability of survival data which is defined as an individual surviving beyond a certain time.

In modeling survival data, data on survival is needed, which can be in the form of complete data (all objects are recorded for survival until all die) or censored data. Censored data is obtained if the survival time of the observed individuals is not known with certainty and there are indications of individuals who remain alive until the specified time period.

The type of censoring that is often used to detect patient survival time is type III censoring. Type III censoring is an observation made if individuals are observed at different times, it is because the patient begins to be detected suffering from a disease at different times and the observation ends at a certain time. Survival data is calculated from the beginning of observation until the individual is declared failed or still alive (Lawless, 1982).

Survival data is expected to follow a certain form of probability distribution that is useful for analysis purposes. One of the distribution functions is The Lomax or Pareto II. it is used to provide a good model in biomedical problems. It is considered as an important model of lifetime models. Also, it has been used in relation with studies of income, size of cities and reliability modeling Also, it has been used in relation with studies of income, size of cities and reliability modeling. It is being widely used for stochastic modeling of decreasing failure rate life components. It also serves as a useful model in the study of queuing theory and biological analysis.

In estimation theory, there are two methods or approaches to estimate population parameters from data distributions, namely the classical approach and the Bayesian approach (Walpole et al., 2002). The classical approach is based on estimating parameters entirely on information obtained from random samples. Some classical approaches that can be used include Least Square, method of moments, and Maximum Likelihood. In this study, the method used to estimate parameters is Maximum Likelihood Estimation (MLE). Performed using a distribution approach by maximizing the likelihood function. In maximizing a function if the parameter value is obtained in implicit and non-linear form, it cannot be solved analytically. So it is necessary to use another method to solve it, namely using the Fisher-Scoring Algorithm. Fisher-Scoring Algorithm is a form of development of the Newton-Raphson method by replacing the hessian matrix with an initialization matrix.

Based on previous research on continuous distribution parameter estimation in survival models is research conducted by Balakrishnan, et al (Balakrishnan & Kateri, 2008). The study discusses the estimation of Weibull distribution parameters and extreme value distribution in type 1 and type 2 censored samples using the Maximum Likelihood Estimation (MLE) and Newton-Raphson methods. Research related to the estimation of continuous distribution parameters in the survival model has also been conducted by Afriani et al (Afriani et al., 2023), the study explains how to estimate the parameters of the exponential distribution in type III censored data with a Bayesian approach using the Linear Exponential Loss Function (LINEX) method based on Jeffrey priors, then the results of the estimation are applied to data on the survival time of patients with chronic renal failure undergoing Hemodialysis therapy at Dr. Sosodoro Djatikoesoemo Bojonegoro Hospital in 2014.

Based on the description of previous studies, researchers are interested in calculating the estimator of the type III censored Lomax distribution parameters using the Maximum Likelihood Estimation (MLE) and Newton Raphson methods and applying the estimation results to actual data. The data used as the application of this research is secondary data containing medical records of 40 patients who experienced death after heart surgery in one of the hospitals in Jakarta.

2. MATERIALS AND METHODS

The data used in this paper is type III censored data on patient deaths after heart surgery in one of the hospitals in Jakarta in 2014 based on Oktaviani's research (Oktaviani, 2015). The dataset consists of 40 patients who experienced failure in the form of death after heart surgery. Patient survival time is measured in days until death. Of the 40 patients, 30 patients had uncensored survival times (died), while the other 10 patients had censored survival times, indicating that they survived more than the observed 15 days. The structure and operational definitions of the variables data to be used in this study are presented in Table 1

Table 1 : Summary of physical parameters.

Variable	Definition	Scale
Times (t_i)	The survival times of the patients after heart surgery	Ratio
Status (δ_i)	Patient condition at the end of observation (dead or censored)	Nominal

This study used a survival analysis method based on the Lomax distribution. The survival function in this study is used to determine the chances of survival of a group of patients or individuals at a certain time after an event begins, such as heart surgery in this case. Meanwhile, the hazard function in this study is used to describe the level of risk or tendency for an event (such as death) to occur at a certain point in time after an event begins. Parameter estimation in the survival function is conducted by Maximum Likelihood Estimation and Newton Raphson methods. The research flowchart is shown in Fig.1 as below.

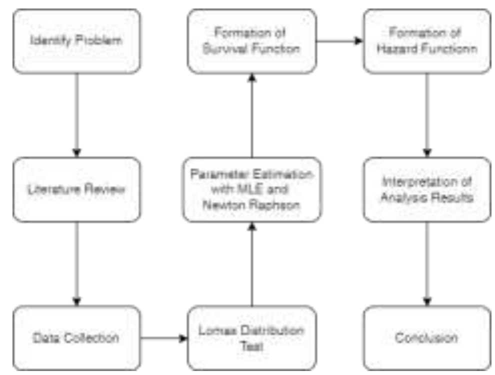

Fig. 1. Research Flowchart

2.1 Lomax Distribution

The Lomax distribution, also known as the Pareto type II distribution, is a probability distribution that has a heavy-tail on its probability density function plot (Lomax, 1954). This distribution was first proposed by K. Lomax and is often used in the fields of actuarial and survival analysis (Salem et al., 2023). The probability density function (PDF) of the Lomax Distribution can be written as follows:.

$$f(t) = \begin{cases} \dfrac{\alpha\beta^{\alpha}}{(t+\beta)^{\alpha+1}}, & t \geq 0, \alpha > 0, \beta > 0 \\ 0, & t < 0 \end{cases} \tag{1}$$

While the Lomax distribution cumulative function can be expressed as below

$$F(t) = 1 - \left(\frac{\beta}{t+\beta}\right)^{\alpha} \tag{2}$$

2.2 Survival Function

The survival function gives the probability of a randomly selected object that survives beyond time $t\ (t > 0)$ (Kleinbaum & Klein, 2012). The survival function can be expressed as below

$$\begin{aligned} S(t) &= P(T \geq t) \\ &= 1 - P(T \leq t) \end{aligned}$$

$$S(t) = 1 - F(t) \tag{3}$$

For the Lomax Distribution, the survival function can be expressed as below:

$$S(t) = \left(\frac{\beta}{t+\beta}\right)^{\alpha} \tag{4}$$

2.3 Hazard Function

The hazard function is a function of the failure rate or risk of an event occurring in a small-time interval during the survival time (Clark et al., 2003). The hazard function can be expressed as below:

$$h(t) = \frac{f(t)}{S(t)} \tag{5}$$

For the Lomax Distribution, the hazard function can be expressed as below:

$$h(t) = \frac{\alpha}{t+\beta} \tag{6}$$

2.4 Maximum Likelihood Estimation

The MLE approach is based on the idea that the most likely parameter value is the value that maximizes the likelihood function of the observed data (Pan & Fang, 2002). Mathematically, the MLE method is expressed as below:

$$(\hat{\alpha}, \hat{\beta}) = \arg\max_{(\alpha,\beta)} \{Log(L(\alpha,\beta))\} \tag{7}$$

2.4.1 Likelihood Function

A likelihood function represents the combined probability density of observed data, treated as a function dependent on the parameters within a statistical model (Cam, 1990). The determination of the likelihood function is the first stage of parameter estimation with the MLE method. The Likelihood function for type III censored data can be expressed as below:

$$L(\alpha,\beta) = \prod_{i=1}^{n} f(t_i)^{\delta_i} S(t_i)^{1-\delta_i}$$

The likelihood function of the Lomax distribution on type III censored data is expressed as below:

$$\begin{aligned} L(\alpha,\beta) &= \prod_{i=1}^{n} f(t_i)^{\delta_i} S(t_i)^{1-\delta_i} \\ &= \prod_{i=1}^{n} \left(\frac{\alpha\beta^{\alpha}}{(t_i+\beta)^{\alpha+1}}\right)^{\delta_i} \left(\left(\frac{\beta}{t_i+\beta}\right)^{\alpha}\right)^{1-\delta_i} \\ &= \prod_{i=1}^{n} \alpha^{\delta_i} \beta^{\alpha} \left(\frac{1}{t_i+\beta}\right)^{\alpha+\delta_i} \end{aligned}$$

$$L(\alpha,\beta) = \alpha^{\sum_{i=1}^{n}\delta_i} \beta^{n\alpha} \prod_{i=1}^{n} \left(\frac{1}{t_i+\beta}\right)^{\alpha+\delta_i} \tag{8}$$

2.4.2 Log-Likelihood Function

The log-likelihood function of the Lomax distribution on type III censored data is expressed as below:

$$\begin{aligned} ln(L(\alpha,\beta)) &= ln\left(\alpha^{\sum_{i=1}^{n}\delta_i} \beta^{n\alpha} \prod_{i=1}^{n} \left(\frac{1}{t_i+\beta}\right)^{\alpha+\delta_i}\right) \\ &= ln\left(\alpha^{\sum_{i=1}^{n}\delta_i}\right) + ln(\beta^{n\alpha}) + ln\left(\prod_{i=1}^{n} \left(\frac{1}{t_i+\beta}\right)^{\alpha+\delta_i}\right) \end{aligned}$$

$$ln(L(\alpha,\beta)) = \sum_{i=1}^{n} \delta_i\, ln(\alpha) + n\alpha\, ln(\beta) - \sum_{i=1}^{n} (\alpha+\delta_i)\, ln(t_i+\beta) \tag{9}$$

2.4.3 Estimation Parameter $\hat{\alpha}$

The estimation of the parameter $\hat{\alpha}$ is done by deriving equation (10) with respect to α and equated to 0. The result of the parameter estimation $\hat{\alpha}$ is expressed as below:

$$\frac{\partial\, ln(L(\alpha,\beta))}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\sum_{i=1}^{n} \delta_i\, ln(\alpha) + n\alpha\, ln(\beta) - \sum_{i=1}^{n} (\alpha+\delta_i)\, ln(t_i+\beta)\right)$$

$$\frac{\partial\, ln(L(\alpha,\beta))}{\partial\alpha} = \sum_{i=1}^{n} \frac{\delta_i}{\alpha} + n\, ln(\beta) - \sum_{i=1}^{n} ln(t_i+\beta)$$

$$0 = \sum_{i=1}^{n} \frac{\delta_i}{\alpha} + n\, ln(\beta) - \sum_{i=1}^{n} ln(t_i+\beta)$$

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} ln(t_i+\beta) - n\, ln(\beta)} \tag{10}$$

2.4.4 Estimation Parameter $\hat{\boldsymbol{\beta}}$

The estimation of the parameter $\hat{\beta}$ is done by deriving equation (10) with respect to α and equated to 0. The result of the parameter estimation $\hat{\beta}$ is expressed as below:

$$\frac{\partial\, ln(L(\alpha,\beta))}{\partial\beta} = \frac{\partial}{\partial\beta}\left(\sum_{i=1}^{n} \delta_i\, ln(\alpha) + n\alpha\, ln(\beta) - \sum_{i=1}^{n} (\alpha+\delta_i)\, ln(t_i+\beta)\right)$$

$$\frac{\partial\, ln(L(\alpha,\beta))}{\partial\beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} \frac{\alpha+\delta_i}{t_i+\beta}$$

$$0 = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} \frac{\alpha+\delta_i}{t_i+\beta}$$

$$\hat{\beta} = \frac{n\alpha}{\sum_{i=1}^{n} \frac{\alpha+\delta_i}{t_i+\beta}} \tag{11}$$

2.5 Newton Raphson

We demonstrate numerical methods for deriving maximum likelihood estimates (MLE) for the parameters in the likelihood function of Lomax distribution based on the Newton-Raphson (NR) algorithm [12]. The NR algorithm is an iterative method for finding the roots of a differentiable function, which produces a sequence of estimates that usually progressively approach the optimal solution [13]. The Newton Raphson equation is mathematically expressed as below:

$$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - \left[H\left(\hat{\theta}^{(i)}\right)\right]^{-1} g\left(\hat{\theta}^{(i)}\right) \tag{12}$$

with:

$$\hat{\theta} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} \tag{13}$$

$$g(\hat{\theta}) = \begin{bmatrix} \dfrac{\partial ln(L(\alpha,\beta))}{\partial\alpha} \\ \dfrac{\partial ln(L(\alpha,\beta))}{\partial\beta} \end{bmatrix} \quad (14)$$

$$H = \begin{bmatrix} \dfrac{\partial^2 ln(L(\alpha,\beta))}{\partial\alpha^2} & \dfrac{\partial^2 ln(L(\alpha,\beta))}{\partial\alpha\partial\beta} \\ \dfrac{\partial^2 ln(L(\alpha,\beta))}{\partial\beta\partial\alpha} & \dfrac{\partial^2 ln(L(\alpha,\beta))}{\partial\beta^2} \end{bmatrix} \quad (15)$$

Based on the calculation results in Eq. (13), Eq. (14), and Eq. (15), the estimators $\hat{\alpha}$ and $\hat{\beta}$ of the survival function with the Lomax distribution are obtained as below:

$$\begin{bmatrix} \hat{\alpha}_{i+1} \\ \hat{\beta}_{i+1} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{bmatrix} - \begin{bmatrix} -\sum_{i=1}^{n} \dfrac{\delta_i}{\alpha^2} & \dfrac{n}{\beta} - \sum_{i=1}^{n} \dfrac{1}{t_i+\beta} \\ \dfrac{n}{\beta} - \sum_{i=1}^{n} \dfrac{1}{t_i+\beta} & \sum_{i=1}^{n} \dfrac{\alpha+\delta_i}{(t_i+\beta)^2} - \dfrac{n\alpha}{\beta^2} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \sum_{i=1}^{n} \frac{\delta_i}{\alpha} + n\, ln(\beta) - \sum_{i=1}^{n} ln(t_i+\beta) \\ \frac{n\alpha}{\beta} - \sum_{i=1}^{n} \frac{\alpha+\delta_i}{t_i+\beta} \end{bmatrix} \quad (16)$$

2.6 Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov test is a model fit test, which means that it examines the degree of similarity between the distribution of a set of values from a sample (observed values) and a certain theoretical distribution (Wilcox, 2011). This test indicates whether the values in the sample can be logically assumed to come from a population with a certain probability distribution.

The test statistic used D, which is the maximum value of $F(X_i) - \frac{i-1}{N}$ or $\frac{i-1}{N} - F(X_i)$. Mathematically, can be expressed as below:

$$D = \max_{1<i<N} \left(F(X_i) - \frac{i-1}{N}, \frac{i-1}{N} - F(X_i) \right) \quad (17)$$

3. Results and Discussion

After obtaining the results of estimating the Lomax distribution parameters using the Maximum Likelihood Estimation (MLE) and Newton Raphson methods, then the application is carried out on real data. The data used to apply the estimation results are secondary data, namely data on medical records of 40 patients after performing heart surgery until death. The survival time of the patient is the time (in days) until death occurs. Of the 40 patients, there are 30 patients with uncensored survival time (died) and 10 patients with censored survival time (patients who still survive until more than 15 days of observation time). The data is presented in the following table.

Table 2: Medical record data of 40 post-heart surgery patients in one of the hospitals in Jakarta.

Observation	Time t_i	Status δ_i	Observation	Time t_i	Status δ_i
1	2	1	21	15	1
2	1	1	22	31	0
3	1	1	23	18	0
4	4	1	24	49	0
5	5	1	25	9	1
6	10	1	26	15	1
7	14	1	27	7	1
8	21	0	28	3	1
9	7	1	29	10	1
10	9	1	30	5	1
11	30	0	31	18	0
12	11	1	32	45	0
13	14	1	33	1	1
14	44	0	34	12	1
15	15	1	35	7	1
16	4	1	36	28	0
17	6	1	37	4	1
18	11	1	38	41	0
19	13	1	39	12	1
20	9	1	40	10	1

According to the table information, status $(\delta_i) = 1$ is the patient who has died and status $(\delta_i) = 0$ is the patient who has not died or the observation is censored. The first step before estimation is testing the distribution of the data, in this study the Lomax (Pareto 2) distribution is used with the following hypothesis:
H_0 = The data follows the Lomax distribution
H_1 = The data does not follow the Lomax distribution.

Using α = 0.05, the decision is H_0 will be rejected if the p-value < α or H_0 will be accepted if the p-value > α.

Table 3: Results of the Lomax distribution test.

Kolmogorov-Smirnov Pareto II					
Sample size Statistic P-Value	40 0.117 0.597				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.165	0.189	0.210	0.235	0.252

With software tools, the output is shown in Table 3 with a p-value of 0.597 for the survival time of post-heart surgery patients. Since the p-value of the distribution fit test for the data is greater than α = 5% (p-value > 0.05), the decision is to fail to reject H_0. Therefore, it can be concluded that the data used, the survival time of post-heart surgery patients, follows the Lomax distribution (Pareto II distribution).

After determining the shape of the data distribution, the values of $\hat{\alpha}$ and $\hat{\beta}$ are then calculated. Based on the calculation using R software, the $\hat{\alpha}$ value in the post-heart surgery patient data is 1.552 and the $\hat{\beta}$ value in the post-heart surgery patient data is 20.38.

To find out the probability of a patient's life within a certain period of time, survival analysis is carried out by substituting the parameter values into the survival function and thus obtaining a survival function for post-heart surgery patient as follows.

$$S(t) = \left(\frac{20.38}{t + 20.38}\right)^{1.552} \qquad (18)$$

Based on the survival function equation above, the survival probability can be presented in the form of Table 4 as follows.

Table 4: Survival probabilities of post-heart surgery patients.

t_i	$S(t)$	t_i	$S(t)$	t_i	$S(t)$
1	0.928	10	0.538	28	0.261
2	0.865	11	0.512	30	0.245
3	0.808	12	0.487	31	0.238
4	0.757	13	0.465	41	0.181
5	0.711	14	0.444	44	0.168
6	0.670	15	0.425	45	0.164
7	0.632	18	0.374	49	0.149
9	0.567	21	0.333		

Then the graph of the survival probabilities of post-heart surgery patients is obtained as follows.

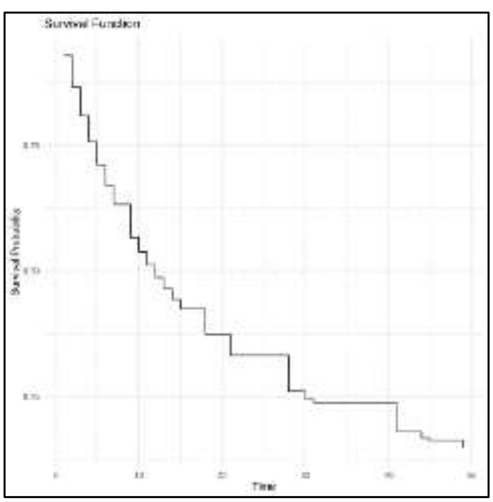

Fig. 2. Graph of the survival probabilities of post-heart surgery patients.

Based on Figure 2, it can be observed that the survival graph decrease as the value of t increases. For example, when $t = 0$, it is observed $S(0) = 1$ signifying a certainty in the probability of an individual surviving before heart disease surgery. However, at $t = 49$ days, $S(49) = 0.149$, indicating that the probability of a post-heart surgery patient surviving for 49 days is approximately 14.94%. Consequently, we can infer that the more time a patient has post-heart surgery, the probability of survive will be smaller.

To determine of the rate of occurrence of an event at a specific point in time, hazard analysis is conducted by substituting the parameter values into the hazard function and thus obtaining a hazard function for post-heart surgery patient as follows.

$$h(t) = \frac{1.552}{t + 20.38} \qquad (19)$$

Based on the hazard function equation above, the hazard probability can be presented in the form of Table 5 as follows.

Table 5: Hazard probabilities of post-heart surgery patients.

t_i	$h(t)$	t_i	$h(t)$	t_i	$h(t)$
1	0.073	10	0.051	28	0.032
2	0.069	11	0.049	30	0.031
3	0.066	12	0.048	31	0.030
4	0.064	13	0.046	41	0.025
5	0.061	14	0.045	44	0.024
6	0.058	15	0.044	45	0.023
7	0.057	18	0.040	49	0.022
9	0.053	21	0.037		

Then the graph of the hazard probabilities of post-heart surgery patients is obtained as follows.

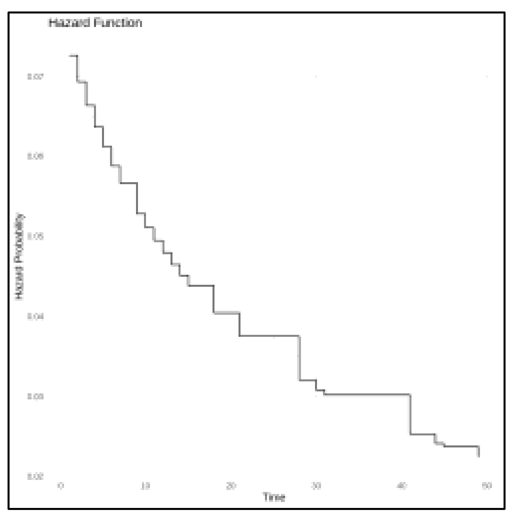

Fig. 3. Graph of the hazard probabilities of post-heart surgery patients.

Based on Figure 3, it can be observed that the hazard graph decreases as the value of t increases. For example, when $t = 0$, it can be observed that $h(0) = 0.076$, signifying the probability of an event (e.g., mortality) occurring at the onset of post-heart surgery is 7.6%. While at $t = 49$ days, $h(9) = 0.022$, indicating that the probability of a post-heart surgery patient experiencing the event (e.g., mortality) within the 49-day period is about 2.2%. Consequently, we can conclude that the longer the time a patient has post-heart surgery, the lower the risk of the event occurring.

4. Conclusion

The application of the estimation results was conducted on secondary data, using data on medical records of 40 post-heart surgery patients at one of the hospitals in Jakarta in 2015. Parameter estimation of $\hat{\alpha}$ and $\hat{\beta}$ using MLE and Newton Raphson methods. The result of estimating the parameter $\hat{\alpha}$ value in the post-heart surgery patient data is 1.552 and the result of estimating the parameter $\hat{\beta}$ in the post-heart surgery patient data is 20.38. Based on these results, it can be concluded that the estimated probability of survival of a post-heart surgery patient for more than 49 days is 14.94% with a risk rate of 2.2%.

5. Acknowledgment

The authors acknowledge the Faculty of Science and Technology, Universitas Airlangga for funding this research through the 2024 research internship program.

6. References

[1] Afriani, Kurniawan, A., & Ana, E. (2023). *Estimator Bayes Parameter Distribusi Eksponensial Data Tersensor Tipe I Dengan Metode Linear Exponential Loss Function Berdasarkan Prior Jeffrey*. Universitas Airlangga.

[2] Balakrishnan, N., & Kateri, M. (2008). On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Statistics & Probability Letters*, *78*(17), 2971–2975. https://doi.org/10.1016/J.SPL.2008.05.019

[3] Cam, L. Le. (1990). Maximum Likelihood: An Introduction. *International Statistical Review*, *58*(2), 153. https://doi.org/10.2307/1403464

[4] Clark, T. G., Bradburn, M. J., Love, S. B., & Altman, D. G. (2003). Survival Analysis Part I: Basic concepts and first analyses. *British Journal of Cancer*, *89*(2), 232. https://doi.org/10.1038/SJ.BJC.6601118

[5] Kleinbaum, D. G., & Klein, M. (2012). *Survival Analysis*. https://doi.org/10.1007/978-1-4419-6646-9

[6] Lawless, J. F. (1982). Statistical models and methods for lifetime data. In *Canadian Journal of Statistics* (Vol. 10, Issue 4). John Wiley & Sons, Ltd. https://doi.org/10.2307/3556199

[7] Lomax, K. S. (1954). Business Failures: Another Example of the Analysis of Failure Data. *Journal of the American Statistical Association*, *49*(268), 847–852. https://doi.org/10.1080/01621459.1954.10501239

[8] Oktaviani, D. (2015). *Analisis Survival Untuk Data Tersensor Menggunakan Model Distribusi Log-Logistik*. Universitas Negeri Jakarta.

[9] Pan, J.-X., & Fang, K.-T. (2002). *Maximum Likelihood Estimation*. 77–158. https://doi.org/10.1007/978-0-387-21812-0_3

[10] Salem, M., Emam, W., Tashkandy, Y., Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. M. (2023). A New Lomax Extension: Properties, Risk Analysis, Censored and Complete Goodness-of-Fit Validation Testing under Left-Skewed Insurance, Reliability and Medical Data. *Symmetry 2023, Vol. 15, Page 1356*, *15*(7), 1356. https://doi.org/10.3390/SYM15071356

[11] Walpole, R. E., Mayers, R. H., Myers, S. L., & Ye, K. (2002). *Probability and statistics for engineers and scientists* (7th ed.). Prentice-Hall.

[12] Wilcox, R. (2011). Introduction to Robust Estimation and Hypothesis Testing, Third Edition. In *Introduction to Robust Estimation and Hypothesis Testing, Third Edition*. Elsevier.