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Characteristic of Compact sets in $\boldsymbol{S^*}$-Orlicz Space

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Abstract*: This paper is devoted to a study of the general properties of the compact sets in S^*-Orlicz space, where $S^* = S^*[0,1]$ is the ring of all real measurable functions on* $[0,1]$.

Keywords: converges in measure, compact set, $\boldsymbol{S^*}$-Orlicz space, equiabsolutely continuous norms

1. Introduction

Throughout this paper, m , μ , D, $\|\kappa(x,\theta)\|_F$ and E_F, denote the strictly positive measure, the integral constructed by the measure m ,bounded closed set in S^*, the norm of the characteristic function and the set of closure bounded functions in L_F^*, respectively.

For every $x \in L_F$, we set $\rho(x,F) = \mu(F(x))$, note that $\rho(x,F) \in S^*$. [9]

Recall that uniformly integrable is a subset M of L^1 (Lebesgue space) such that for given $\varepsilon > 0$ there exists $\delta > 0$, therefore $\bigvee(\mu(|f|) : f \in M) < \varepsilon$ whenever $\mu(B) < \delta$. Alternatively M is bounded and uniformly integrable if and only if given $\varepsilon > 0$, there exists $N > 0$ so $\bigvee\{\mu(|f|) :\ |f| > c\ , f \in M\} < \varepsilon$ where $c \geq N$.

2. Basic Concepts

Definition 2.1 : **[1]**

Let $f: [0,\infty) \longrightarrow [0,\infty)$ be a monotone increasing**,** right continuous function with

1. $f(0) = 0$
2. $\lim\limits_{t\to\infty} f(t) = \infty$
3. $f(t) > 0$ whenever $t > 0$, then the function defined by

$$F(u) = \int_0^{|u|} f(t)dt$$

is said to be an N-function.

An alternative view of N-functions gives by the following proposition.

Proposition 2.2 **:[2,11]**

The function F is continuous, even and convex if and only if F is an N-function with

1. $\lim\limits_{u\to 0} \frac{F(u)}{u} = 0$;
2. $\lim\limits_{u\to\infty} \frac{F(u)}{u} = \infty$;
3. $F(u) > 0$ if $u > 0$.

Remark 2.3 **:[6]**

Define $g(s)$ for $s \geq 0$ as $g(s) = \sup\limits_{f(t)\leq s} t$.

Note that $g(0) = 0$, $\lim\limits_{s\to\infty} g(s) = \infty$. Also, we have $g(f(t)) \geq t$ and $f(g(s)) \geq s$.

Now,

$$F(u) = \int_0^{|u|} f(t)dt$$

and

$$G(v) = \int_0^{|v|} g(s)ds$$

are N-functions and one complements each other. Furthermore it is clear that the complement of G is F.[1]

Accordingly, we define an alternative for the complementary N-function G by [1,2]:

$$G(v) = \max\{u|v| - F(u): u \geq 0\}$$

Definition 2.4 **:[7]**

An N-function F is called belong to Δ_2-condition if there exists a constant c such that

$$\mathrm{c}F(2u) \leq cF(u)\ \ \forall\, u > 0.$$

Proposition 2.5:**[4]**

The function $F(u)$ satisfies the Δ_2-condition if and only if

$$F(au) \leq c(a)F(u) \text{ for any } a > 1.$$

Note **[5]:** We shall denote by $C_\infty(Q(\nabla))$ the set of all continuous functions on the Stone compactum $Q(\nabla)$ which can take the values $\pm\infty$ on nowhere dense sets from $Q(\nabla)$. Let $L_1(m)$ be the set of all integrable by the measure m elements from $C_\infty(Q(\nabla))$.

***Definition 2.6 :*[5]**

Assume G is the complementary N-function to $F(u)$.

Set $L_F^* = L_F^*(\nabla, m) = \{x \in C_\infty(Q(\nabla)): xy \in L_1(m)\}$.

for all $y \in L_G$. If $x \in L_F$ and $y \in L_G$, then it follows from the Young inequality $|xy| \leq F(x) + G(y)$, that $xy \in L_1(m)$, it means, that $L_F \subset L_F^*$.

***Proposition 2.7 :*[8]**

For every $x \in L_F^*$, we have

$$\bigvee_{y \in A(G)} |\mu(xy)| < \infty,$$

where $A(G) = \{y \in L_G: \mu(G(y)) \leq 1\}$.

The Orlicz norm on L_F^* defined as

$$\|x\|_F = \bigvee_{y \in A(G)} |\mu(xy)|,$$

for all $x \in L_F^*$.

It follows from the definition of the Orlicz norm that the following usual axioms are satisfied [6]:

1. $\|x\|_F = 0$ if and only if , $x = 0$ almost everywhere ;
2. $\|\alpha x\|_F = |\alpha| \|x\|_F$;
3. $\|x_1 + x_2\|_F \leq \|x_1\|_F + \|x_2\|_F$.

Thus , the set L_F^* becomes a normed linear space which is called an S^*- Orlicz space .

Remark 2.8 :[6]

A collection $\mathcal{K}$ of functions $f(x)$ with equiabsolutely continuous integrals if for given $\varepsilon > 0$ there exists $h > 0$ so that $\underset{\theta}{\mu}(|f(x)|) < \varepsilon$ whenever $m(\theta) < h$.

***Theorem (Vallée Poussin's) 2.9:*[6]**

Let $h(y)$ $(0 \leq x < \infty)$ be a monotonically increasing function which satisfies the condition

$$\lim_{n \to \infty} \frac{h(y)}{y} = \infty.$$

Suppose, for the functions $h(y)$ of some collection $\mathcal{K}$, the integrals of the function $h[|f(x)|]$ are uniformly bounded:

$$\mu[h(|f(x)|)] \leq A < \infty \qquad (f(x) \in \mathcal{K}).$$

Then the collection $\mathcal{K}$ has equiabsolutely continuous integrals.

3. THE IMPORTANT RESULTS

The important results concerning with the characteristic of compact sets in S^*-Orlicz space are investigated.

Firstly, the following information are needs.

***Definition 3.1:*[10]**

A collection $\mathcal{K} \subset L_F^*$ is sad to be has equiabsolutely continuous norms if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\|\kappa(x,\theta)\|_F < \varepsilon$ whenever $m(\theta) < \delta$.

***Remark 3.2:* [6]**

$\mathcal{K} \subset E_F$, If $\mathcal{K}$ is a compact set in E_F, then ithas equi-absolutely continuous norms.

***Definition 3.3:* [6]**

Let $f(x)$ be a function which is summable on D. The function

$$f_r(x) = \frac{1}{\mathcal{Z}_r} \underset{T_r(x)}{\mu}\left(f(t)\right) \qquad (x \in D),$$

Where $T_r(x)$ is the n-dimensional sphere with radius r and center at the point $x \in D$ and $\mathcal{Z}_r$ is the volume of this sphere, is called a Steklov function.

We denote the sphere with radius r and center at the zero of the n-dimensional space by T_0.

***Theorem 3.4:*[3]**

A function $f \in E_F$ if and only if $f \in L_F^*$ has absolutely continuous norm.

***Lemma 3.5:* [3]**

The collection $\{F(x) : x \in \mathcal{K}\}$ is uniformly integrable in L^1. if and only if $F \in \Delta_2$ and $\mathcal{K} \subset L_F^*$ then the set $\mathcal{K}$ has equiabsolutely continuous norms

Lemma 3.6:

A necessary and sufficient condition that a sequence of functions $f_n \in E_F$ ($n = 1,2, \ldots$), which converges in measure, converge in norm is that it have equiabsolutely continuous norms.

Proof:

Let $f_n \in E_F$ be a sequence($n = 1,2, \ldots$), converges in measure and has equiabsolutely continuous norms.

Assume $\varepsilon > 0$ is positive number, we denote by θ_{mn} the $D\{|f_n - f_m| > \alpha\}$, where $\alpha = \varepsilon / \{3m(D) G^{-1}(\frac{1}{m(D)})\}$, and let $\delta > 0$ be a number such that

$$\|f_n \kappa(x,\theta)\|_F < \frac{\varepsilon}{3} \qquad (n = 1,2, \ldots)$$

whenever $m(\theta) < \delta$.

Since the sequence f_n ($n = 1,2, \ldots$) converges in measure, and we can find n_0 such that $m(\theta_{mn}) < \delta$ for $n, m > n_0$.

For $n, m > n_0$, we have

$\|f_n - f_m\|_F \leq \|(f_n - f_m)\kappa(x, \theta_{mn})\|_F + \|(f_n - f_m)\kappa(x, D/\theta_{mn})\|_F \quad \leq \|f_m\kappa(x, \theta_{mn})\|_F + \|f_m\kappa(x, \theta_{mn})\|_F + \alpha\|\kappa(x, D)\|_F$

$$< \varepsilon.$$

This means that the sequence f_n converges in L_F^* .

Theorem 3.7:

If the collection $\mathcal{K} \subset E_F$ is compact in the sense of convergence in measure and has equiabsolutely continuous norms, then the collection $\mathcal{K}$ is compact in L_F^*.

Proof:

From each sequence of the collection $\mathcal{K}$ we can select a subsequence which converges in measure. Also, from (lemma 3.6), we get this subsequence converges with respect to the norm in L_F^*.

Theorem 3.8:

A collection $\mathcal{K}$ of functions in the space E_F is compact if and only if the following properties are satisfied:

i) $\|f\|_F \leq A, f(x) \in \mathcal{K}$;

ii) For arbitrary $\varepsilon > 0$, a $\delta > 0$ such that $d(h, 0) < \delta$ implies that $\|f(x + h) - f(x)\|_F < \varepsilon$ for all $f(x) \in \mathcal{K}$.

Proof:

Suppose $\mathcal{K}$ is a compact collection of functions in E_F.Then, for this collection we can construct an $(\frac{\varepsilon}{3})$net consisting of continuous functions $f_1(x), f_1(x), \dots, f_n(x)$. We denote by c the norm of the characteristic function of the entire set D in the space L_F^*.

Let $\delta > 0$ be a number such that

$$|f_i(x + h) - f_i(x)| < \frac{\varepsilon}{3c} \qquad (i = 1,2, \dots, n)$$

Whenever $d(h, 0) < \delta$.

It is clear that

$$\|f_i(x + h) - f_i(x)\|_F \leq \frac{\varepsilon}{3} \qquad (i = 1,2, \dots, n) \qquad (*)$$

Let $f(x)$ be an arbitrary function in $\mathcal{K}$. We can find a function $f_{i_0}(x)$ such that $\|f - f_{i_0}\|_F < \varepsilon/3$. From (*) and ($\|f(x + s)\|_F = \|f\|_F$), we have

$$\|f(x + h) - f(x)\|_F \leq \|f(x + h) - f_{i_0}(x + h)\|_F + $$

$$+\|f_{i_0}(x + h) - f_{i_0}(x)\|_F + \|f_{i_0}(x) - f(x)\|_F < \varepsilon$$

Conversely, Suppose $f(x) \in \mathcal{K}$ and $f_r(x)$ is the corresponding Steklov function. Then

$$|f(x) - f_r(x)| \leq \frac{1}{\mathcal{Z}_r} \underset{T_{r(x)}}{\mu}(|f(x) - f(t)|)$$

It follows, for $g(x) \in L_G, \rho(g, G) = \mu(G(g)) \leq 1$,then

$$\mu(|f_r(x) - f(x)|g(x)) \leq \frac{1}{\mathcal{Z}_r} \underset{T_{r(x)}}{\mu}[\mu (|f(t) - f(x)|\, g(x)],$$

Since the order of integration and making a change in variables, we obtain

$$\mu(|f(x) - f_r(x)|g(x)) \leq \frac{1}{\mathcal{Z}_r} [\underset{T_0}{\mu}(\mu(|f(x + s) - f(x)|\, g(x)]$$

$$\leq \frac{1}{\mathcal{Z}_r} [\underset{T_0}{\mu}(\|f(x + s) - f(x)\|_F)],$$

It follows from the theorem [7, page 97], then the converse is satisfies.

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